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Nils Kürbis

A BINARY QUANTIFIER FOR DEFINITE DESCRIPTIONS IN INTUITIONIST NEGATIVE FREE LOGIC: NATURAL DEDUCTION AND NORMALISATION

Abstract

This paper presents a way of formalising definite descriptions with a binary quantifier ι, where $\iota x[F, G]$ is read as 'The F is G'. Introduction and elimination rules for ι in a system of intuitionist negative free logic are formulated. Procedures for removing maximal formulas of the form $\iota x[F, G]$ are given, and it is shown that deductions in the system can be brought into normal form.

Keywords: definite descriptions, negative intuitionist free logic, natural deduction, normalization.

1. Introduction

The definite description operator ι, the formal analogue of the definite article 'the', is usually taken to be a term forming operator: if A is a predicate, then ιxA is a term denoting the sole A, if there is one, or nothing or an arbitrary object if there is no or more than one A. This paper follows a different approach to definite descriptions by formalising them instead with a primitive binary quantifier: ι forms a formula from two predicates, and 'The F is G' is formalised as $\iota x[Fx, Gx]$. The notation, and the way of treating definite descriptions that comes with it, was suggested by Dummett [2, p.162].[1]

[1]Bostock considers a similar approach and explains definite descriptions as a special case of restricted quantification, where the restriction is to a single object. [1, Sec. 8.4] Bostock writes $(Ix\colon Fx)\ Gx$ for 'The F is G', but prefers to treat definite descriptions

The current paper treats definite descriptions purely proof theoretically. The proof theory of a term forming ι operator has been investigated in the context of sequent calculi for classical free logic by Indrzejczak [3, 4]. Tennant gives rules for such an operator in natural deduction [7, p.110].[2] The approach followed here may be new to the literature.

In this paper, I investigate the binary quantifier ι in the context of a system of natural deduction for intuitionist negative free logic. The application of the present treatment of definite descriptions to other systems of logic and their comparisons to systems known from the literature are left for further papers. To anticipate, using a negative free logic, the approach proposed here lends itself to a natural formalisation of a Russellian theory of definite descriptions, while it provides a natural formalisation of Lambert's minimal theory of definite descriptions when the logic is positive and free.

First, notation. I will use A^x_t to denote the result of replacing all free occurrences of the variable x in the formula A by the term t or the result of substituting t for the free variable x in A. t is free for x in A means that no (free) occurrences of a variable in t become bound by a quantifier in A after substitution. In using the notation A^x_t I assume that t is free for x in A or that the bound variables of A have been renamed to allow for substitution without 'clashes' of variables, but for clarity I also often mention the condition that t is free for x in A explicitly. I also use the notation Ax to indicate that x is free in A, and At for the result of substituting t for x in A.

2. Natural Deduction for ι in Intuitionist Logic

The introduction and elimination rules for the propositional logical constants of intuitionist logic **I** are:

$$\wedge I\colon \quad \frac{A \qquad B}{A \wedge B} \qquad\qquad \wedge E\colon \quad \frac{A \wedge B}{A} \qquad\qquad \frac{A \wedge B}{B}$$

with a term forming operator. I owe the reference to Bostock to a referee for this journal, who also pointed me to the paper by Scott to be referred to in footnote 4 and made valuable comments on this paper.

[2]Tennant is not explicit whether the logic in this paper is classical or intuitionist. However, as he is partial to anti-realism and constructive mathematics, we are justified in assuming that his preferred route is to add these rules to a system of intuitionist free logic. The rules are also in [8, Ch. 7.10], where the logic is classical.

$$\rightarrow I: \quad \dfrac{\begin{array}{c}\overline{A}^{\,i} \\ \Pi \\ B\end{array}}{A \rightarrow B}\,i \qquad\qquad \rightarrow E: \quad \dfrac{A \rightarrow B \qquad A}{B}$$

$$\vee I: \quad \dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B} \qquad\qquad \vee E: \quad \dfrac{A \vee B \qquad \begin{array}{c}\overline{A}^{\,i} \\ \Pi \\ C\end{array} \qquad \begin{array}{c}\overline{B}^{\,i} \\ \Sigma \\ C\end{array}}{C}\,i$$

$$\bot E: \quad \dfrac{\bot}{B}$$

where the conclusion B of $\bot E$ is restricted to atomic formulas.

The introduction and elimination rules for the quantifiers of **I** are:

$$\forall I: \quad \dfrac{A^x_y}{\forall x A} \qquad\qquad \forall E: \quad \dfrac{\forall x A}{A^x_t}$$

where in $\forall I$, y is not free in any undischarged assumptions that A^x_y depends on, and either y is the same as x or y is not free in A; and in $\forall E$, t is free for x in A.

$$\exists I: \quad \dfrac{A^x_t}{\exists x A} \qquad\qquad \exists E: \quad \dfrac{\exists x A \qquad \begin{array}{c}\overline{A^x_y}^{\,i} \\ \Pi \\ C\end{array}}{C}\,i$$

where in $\exists I$, t is free for x in A; and in $\exists E$, y is not free in C nor any undischarged assumptions it depends on in Π except A^x_y, and either y is the same as x or it is not free in A.

The introduction and elimination rules for identity are:

$$= I: \quad \dfrac{}{t = t} \qquad\qquad = E: \quad \dfrac{t_1 = t_2 \qquad A^x_{t_1}}{A^x_{t_2}}$$

where A is an atomic formula. To exclude vacuous applications of $= E$, we can require that x is free in A and that t_1 and t_2 are different. An induction over the complexity of formulas shows that the rule holds for formulas of any complexity.

To formalise definite descriptions, one could add the binary quantifier ι to **I**. Its introduction and elimination rules would be:

$$\iota I: \quad \dfrac{F^x_t \qquad G^x_t \qquad \begin{array}{c}\overline{F^x_z}^{\,i}\\ \Pi\\ z=t\end{array}}{\iota x[F,G]}\,i$$

where t is free for x in F and in G, and z is different from x, not free in t and does not occur free in any undischarged assumptions in Π except F^x_z.[3]

$$\iota E^1: \quad \dfrac{\iota x[F,G] \qquad \begin{array}{c}\underbrace{\overline{F^x_z}^{\,i}, \ \overline{G^x_z}^{\,i}}\\ \Pi\\ C\end{array}}{C}\,i$$

where z is not free in C nor any undischarged assumptions it depends on except F^x_z and G^x_z, and either z is the same as x or it is not free in F nor in G.

$$\iota E^2: \quad \dfrac{\iota x[F,G] \qquad F^x_{t_1} \qquad F^x_{t_2}}{t_1 = t_2}$$

where t_1 and t_2 are free for x in F.

For simplicity we could require that x occurs free in F and G. If we don't, the truth or falsity of $\iota x[F,G]$ may depend on properties of the domain of quantification: if F is true and does not contain x free, then $\iota x[F,G]$ is false if there is more than one thing in the domain of quantification, and it is true if there is only one thing and G is true (of the one thing, if x is free in G).

$\iota x[F,G]$ and $\exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$ are interderivable. Notice that the rules for identity are not applied in the two deductions to follow.

---

[3] A more precise and general statement of the introduction rule for ι would result if we were to require Π to be a deduction of $(y = t)^y_z$ from $(F^x_y)^y_z$, where y is different from x and not free in t, and either z is the same as y or z is not free in F^x_y nor in $y = t$.

1. $\iota x[F, G] \vdash \exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$

Let y be different from x and not free in F or G:

$$
\begin{prooftree}
\AxiomC{$\iota x[F, G]$}
\AxiomC{$\iota x[F, G]$}
\AxiomC{$\overline{F^x_y}^{\,1}$}
\AxiomC{$\overline{F}^{\,2}$}
\RightLabel{ιE^2}
\TrinaryInfC{$y = x$}
\RightLabel{1}
\UnaryInfC{$(F^x_y \to y = x)$}
\UnaryInfC{$\forall y(F^x_y \to y = x)$}
\AxiomC{$\overline{F}^{\,2}$}
\BinaryInfC{$F \wedge \forall y(F^x_y \to y = x)$}
\AxiomC{$\overline{G}^{\,2}$}
\BinaryInfC{$(F \wedge \forall y(F^x_y \to y = x) \wedge G)$}
\UnaryInfC{$\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$}
\RightLabel{2 ιE^1}
\BinaryInfC{$\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G)$}
\end{prooftree}
$$

2. $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G) \vdash \iota x[F, G]$

Let y be different from x and not free in F or G, and let $\circledast$ be the formula $(F \wedge \forall y(F^x_y \to y = x) \wedge G)$:

$$
\begin{prooftree}
\AxiomC{$\exists x \circledast$}
\AxiomC{$\overline{\circledast}^{\,2}$}
\UnaryInfC{F}
\AxiomC{$\overline{\circledast}^{\,2}$}
\UnaryInfC{G}
\AxiomC{$\overline{\circledast}^{\,2}$}
\UnaryInfC{$\forall y(F^x_y \to y = x)$}
\UnaryInfC{$F^x_y \to y = x$}
\AxiomC{$\overline{F^x_y}^{\,1}$}
\BinaryInfC{$y = x$}
\RightLabel{1 ιI}
\TrinaryInfC{$\iota x[F, G]$}
\RightLabel{2}
\BinaryInfC{$\iota x[F, G]$}
\end{prooftree}
$$

3. Intuitionist Free Logic

It is more interesting to add the ι quantifier to a free logic. I will use formalisations of intuitionist free logic with a primitive predicate $\exists!$, to be interpreted as 'x exists' or 'x refers' or 'x denotes'. The introduction and elimination rules for the quantifiers are:

$$\forall I: \quad \dfrac{\begin{array}{c}\overline{\exists! y}^{\,i}\\ \Pi \\ A^x_y\end{array}}{\forall x A}\,i \qquad\qquad \forall E: \quad \dfrac{\forall x A \qquad \exists! t}{A^x_t}$$

where in $\forall I$, y does not occur free in any undischarged assumptions of Π except $\exists! y$, and either y is the same as x or y is not free in A; and in $\forall E$, t is free for x in A.

$$\exists I: \quad \dfrac{A^x_t \qquad \exists! t}{\exists x A} \qquad\qquad \exists E: \quad \dfrac{\exists x A \qquad \begin{array}{c}\underbrace{\overline{A^x_y}^{\,i}, \ \overline{\exists! y}^{\,i}}\\ \Pi \\ C\end{array}}{C}\,i$$

where in $\exists I$, t is free for x in A; and in $\exists E$, y is not free in C nor any undischarged assumptions of Π, except A^x_y and $\exists! y$, and either y is the same as x or it is not free in A.

The elimination rule for identity in intuitionist free logic is the same as in **I**.

In *intuitionist positive free logic* **IPF**, identity has the same introduction rule as in intuitionist logic, i.e. $\vdash t = t$, for any term t. Semantically speaking, in positive free logic any statement of self-identity is true, irrespective of whether a term refers or not.

In *intuitionist negative free logic* **INF** the introduction rule for identity is weakened and requires an existential premise:

$$=I^n: \quad \dfrac{\exists! t}{t = t}$$

In **INF** the existence of t_i may be inferred if t_i occurs in an atomic formula:

$$AD: \quad \dfrac{At_1 \ldots t_n}{\exists! t_i}$$

where A is an n-place predicate letter (including identity) and $1 \leqslant i \leqslant n$. Speaking semantically, for an atomic sentence, including identities, to be true, all terms in it must refer. If the language has function symbols, there is also the rule of functional denotation:

$$FD: \quad \frac{\exists! ft_1 \ldots t_n}{\exists! t_i}$$

where f is an n-place function letter and $1 \leqslant i \leqslant n$. Speaking semantically, for the value of a function to exist, all of its arguments must exist. $= I^n$, AD and FD are called the rules of strictness.[4]

Hintikka's Law $\exists! t \leftrightarrow \exists x\ x = t$, where x not in t, is provable in **INF** and **IPF**. In **IPF**, it suffices to observe the following:

$$\frac{\overline{t = t} \qquad \exists! t}{\exists x\ x = t} \qquad\qquad \frac{\exists x\ x = t \qquad \dfrac{\overline{x = t}^{\,1} \qquad \overline{\exists! x}^{\,1}}{\exists! t}}{\exists! t}\,1$$

In **INF**, conclude $t = t$ from $\exists! t$.

The degree of a formula is the number of connectives occurring in it. $\bot$, being a connective, is of degree 1. This excludes the superfluous case in which $\bot$ is inferred from $\bot$ by $\bot E$. $\exists! t$ is an atomic formula of degree 0.

The *major premise* of an elimination rule is the premise with the connective that the rule governs. The other premises are minor premises. A *maximal formula* is one that is the conclusion of an introduction rule and the major premise of an elimination rule for its main connective. A *segment* is a sequence of formulas of the same shape, all minor premises and conclusions of $\vee E$ or $\exists E$, except the first and the last one; the first is only a minor premise, the last only a conclusion. A segment is maximal if its first formula has been derived by an application of an introduction rule for its main connective, and its last formula is the major premise of an elimination rule. A deduction is *in normal form* if it contains neither maximal formulas nor maximal segments. A normalisation theorem establishes that any deduction can be brought into normal form by applying *reduction procedures* for the removal of maximal formulas from deductions and *permutative reduction procedures* for reducing maximal segments to maximal formulas.

Notice that the conditions imposed on applications of $= E$ have the consequence that there are no maximal formulas of the form $t_1 = t_2$.

---

[4]**INF** is the system introduced by Scott [6] and called **Nie** by Troelstra and Schwichtenberg [9, 200] but with a simpler theory of identity. It is the system that results if classical *reductio ad absurdum*, the rule that licenses the derivation of A if $\neg A$ entails a contradiction, is not taken to form part of the system Tennant presents in [8, Ch. 7.10].

AD and FD have the characteristics of introduction rules for $\exists!$, and $=I^m$ has the characteristics of an elimination rule for it. In a sense $\forall E$ and $\exists I$ of free logic also eliminate formulas of the form $\exists!t$. I will, however, not count these rules as introduction and elimination rules for $\exists!$, as there is no general way of removing formulas of the form $\exists!t$ that have been concluded by AD or FD and are premises of $=I^n$, $\forall E$ or $\exists I$.

Proofs of the normalisation theorem for intuitionist logic, such as those given by Prawitz [5, Ch. IV.1] and Troelstra and Schwichtenberg [9, Ch. 6.1], can be modified to carry over to the intuitionist free logics considered here.

A normalisation theorem for intuitionist negative free logic with a term forming ι operator can be reconstructed from material Tennant provides in [8]. In particular, as in the case of **I**, we can assume that every application of $\forall I$ and $\exists E$ has its own variable, that is, the free variable y of an application of such a rule occurs only in the hypotheses discharged by the rule and formulas concluded from them and, for $\forall I$, in the premise of that rule and the formulas it has been derived from. This way we avoid 'clashes' between the restrictions on the variables of different application of these rules when reduction procedures are applied to a deduction containing maximal formulas. Applying the reduction procedures for quantifiers of free logic can only introduce maximal formulas of lower degree than the one removed. I leave the details to the reader.

4. Natural Deduction for ι in INF

The interderivability of $\iota x[F, G]$ and $\exists x(F \wedge \forall y(F^x_y \to x = y) \wedge G)$ is the hall mark of a Russellian theory of definite descriptions, in which any statement of the form 'The F is G' is false if there is no F or if there is more than one. It is the generally accepted treatment of definite descriptions in negative free logic. To establish how to modify the rules for ι given in Section 2 to yield a Russellian theory of definite descriptions when the logic is intuitionist negative free logic, we analyse the deductions establishing the interderivability of $\iota x[F, G]$ and $\exists x(F \wedge \forall y(F^x_y \to x = y) \wedge G)$ in **I** given at the end of that section.

Looking at the derivation of $\exists x(F \wedge \forall y(F^x_y \to x = y) \wedge G)$ from $\iota x[F, G]$, had the application of the universal quantifier introduction rule be one of free logic, it would have allowed the discharge of an assumption $\exists!y$, and had the existential quantifier introduction rule been one of free logic, a further assumption $\exists!x$ would have been required. Both lend themselves as additional premises of ιE^2, as premises analogous to the existence assumptions in the rules of the quantifiers of free logic. $\exists!y$ would be discharged by the application of the universal quantifier introduction rule of free logic, so in order for the conclusion of the deduction not to depend on $\exists!x$, it would have to be discharged, and the only option here is that it is discharged by the application of ιE^1. This is also a natural option, corresponding, as it does, to the discharge of existence assumptions by the quantifier rules of free logic.

Generalising the first observation, we add the premises $\exists!t_1$ and $\exists!t_2$ to ιE^2:

$$\iota E^2 : \quad \frac{\iota x[F, G] \qquad \exists!t_1 \qquad \exists!t_2 \qquad F^x_{t_1} \qquad F^x_{t_2}}{t_1 = t_2}$$

where t_1 and t_2 are free for x in F.

To implement the second observation, we add $\exists!z$ as an additional discharged assumption to ιE^1:

$$\iota E^1 : \quad \frac{\iota x[F, G] \qquad \begin{matrix} \underbrace{\overline{F^x_z}^{\,i}, \ \overline{G^x_z}^{\,i}, \ \overline{\exists!z}^{\,i}} \\ \Pi \\ C \end{matrix}}{C}\, i$$

where is z not free in C nor any undischarged assumptions it depends on except F^x_z, G^x_z and $\exists!z$, and either z is the same as x or it is not free in F nor in G.

To find suitable modifications of the introduction rule for ι, we look at the derivation of $\iota x[F, G]$ from $\exists x(F \wedge \forall y(F^x_y \rightarrow x = y) \wedge G)$ in **I**. Had the application of the universal quantifier elimination rule been one of free logic, a further assumption $\exists!y$ would have been required, and had the existential quantifier elimination rule been one of free logic, it would have allowed the discharge of an assumption $\exists!x$. The latter lends itself as an additional premise of ιI, the former as an additional assumption discharged by that rule, which is again analogous to the existence assumptions required and discharged in applications of the rules for the quantifiers of free logic.

Generalising the second observation, we add $\exists! t$ as a further premise, and to implement the first observation we add $\exists! z$ as a further discharged assumption to ιI:

$$\iota I: \qquad \dfrac{F^x_t \qquad G^x_t \qquad \exists! t \qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\,i},\ \overline{\exists! z}^{\,i}} \\ \Pi \\ z = t\end{matrix}}{\iota x[F, G]}\, i$$

where t is free for x in F and in G, and z is different from x, not free in t and does not occur free in any undischarged assumptions in Π except F^x_z and $\exists! z$.[5]

It is obvious that $\iota x[F, G]$ and $\exists x(F \wedge \forall y(F^x_y \rightarrow x = y) \wedge G)$ are interderivable in **INF** when ι is governed by the modified rules, but we give the deductions for convenience.

1. $\iota x[F, G] \vdash \exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)$

Let x and y be different variables, where y is not free in F nor in G:

$$\dfrac{\iota x[F,G] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\iota x[F,G]} \quad \overline{\exists! y}^{\,2} \quad \overline{\exists! x}^{\,3} \quad \overline{F^x_y}^{\,1} \quad \overline{F}^{\,3}}{y = x}\,\iota E^2}{(F^x_y \rightarrow y = x)}\,1}{\forall y(F^x_y \rightarrow y = x)}\,2 \qquad \overline{F}^{\,3}}{F \wedge \forall y(F^x_y \rightarrow y = x)} \qquad \overline{G}^{\,3}}{F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G} \qquad \overline{\exists! x}^{\,3}}{\exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)}}{\exists x(F \wedge \forall y(F^x_y \rightarrow y = x) \wedge G)}\, 3\ \iota E^1$$

[5] A more precise and general statement of the introduction rule for ι would result if we were to require Π to be a deduction of $(y = t)^y_z$ from $(F^x_y)^y_z$ and $\exists! z$, where y is different from x and not free in t, and either z is the same as y or z is not free in F^x_y nor in $y = t$.

2. $\exists x(F \wedge \forall y(F^x_y \to y = x) \wedge G) \vdash \iota x[F, G]$

Let $\circledast$ be the formula $(F \wedge \forall y(F^x_y \to y = x) \wedge G)$, where y is different from x and not free in F or G:

$$\dfrac{\exists x \circledast \qquad \dfrac{\dfrac{\overline{\circledast}^{\,2}}{F} \quad \dfrac{\overline{\circledast}^{\,2}}{G} \quad \overline{\exists! x}^{\,2} \quad \dfrac{\dfrac{\dfrac{\overline{\circledast}^{\,2}}{\forall y(F^x_y \to y = x)} \quad \overline{\exists! y}^{\,1}}{F^x_y \to y = x} \quad \overline{F^x_y}^{\,1}}{y = x}}{\iota x[F, G]}\;1\ \iota I}{\iota x[F, G]}\;2$$

Let $\mathbf{INF}^\iota$ denote the systems of intuitionist negative free logic augmented with the rules for ι given in this section.

In order to prove a normalisation theorem for $\mathbf{INF}^\iota$, we first observe that $\bot E$ can be restricted to atomic conclusions in this system:

1. Instead of inferring $\forall x A$ from $\bot$, infer A^x_y, for some y not occurring in any assumption that $\bot$ depends on, and apply $\forall I$, discharging vacuously.
2. Instead of inferring $\exists x A$ from $\bot$, infer A^x_t, for some t that is free for x in A, infer $\exists! t$, and apply $\exists I$.
3. Instead of inferring $\iota x[F, G]$ from $\bot$, infer F^x_t, G^x_t, $\exists! t$ and $z = t$, for some t that is free for x in F and in G and some z that is not free in any assumption that $\bot$ depends on, and apply ιI, discharging vacuously.

Next, $= E$ can be restricted to atomic formulas in $\mathbf{INF}^\iota$. Consider an application of this rule with premise $\iota x[F, G]^y_{t_1}$:

$$\dfrac{t_1 = t_2 \qquad \iota x[F, G]^y_{t_1}}{\iota x[F, G]^y_{t_2}}$$

where t_1 and t_2 are free for y in $\iota x[F, G]$. The exclusion of vacuous applications of $= E$ means that y must be different from x, and so $\iota x[F, G]^y_{t_1}$ is $\iota x[F^y_{t_1}, G^y_{t_1}]$. Let v and z be different variables not occurring in F, G, t_1, t_2. The induction step applying $= E$ to subformulas of $\iota x[F, G]^y_{t_1}$ is the following:

$$\cfrac{\iota x[F^y_{t_1}, G^y_{t_1}] \qquad \cfrac{\mathbf{A} \qquad \mathbf{B} \qquad \mathbf{C} \qquad \mathbf{D}}{\iota x[F^y_{t_2}, G^y_{t_2}]}}{\iota x[F^y_{t_2}, G^y_{t_2}]}\ {}_{2\ \iota E^1},$$

where

$$\mathbf{A} = \cfrac{t_1 = t_2 \qquad \overline{(F^y_{t_1})^x_z}^{\,2}}{(F^y_{t_2})^x_z} \qquad \mathbf{B} = \cfrac{t_1 = t_2 \qquad \overline{(G^y_{t_1})^x_z}^{\,2}}{(G^y_{t_2})^x_z} \qquad \mathbf{C} = \overline{\exists! z}^{\,2}$$

$$\mathbf{D} = \cfrac{\iota x[F^y_{t_1}, G^y_{t_1}] \qquad \overline{\exists! v}^{\,1} \qquad \overline{\exists! z}^{\,2} \qquad \overline{(F^y_{t_1})^x_v}^{\,1} \qquad \overline{(F^y_{t_1})^x_z}^{\,2}}{v = z}\ {}_{\iota E^2}.$$

As for applications of $\forall I$ and $\exists E$, we can assume that every application of ιI and ιE^1 has its own free variable, i.e. the variable z of an application of ιI or ιE^2 occurs only in the premises discharged by the rule and formulas derived from the discharged premises, and nowhere else in the deduction.

I will now give the reduction procedures for maximal formulas of the form $\iota x[F, G]$ and the permutative reduction procedures for maximal segments consisting of a formula of that form.

There are two cases of reduction procedures for maximal formulas of the form $\iota x[F, G]$ to be considered. First, the conclusion of ιI is the major premise of ιE^1:

$$\cfrac{\cfrac{\begin{matrix}\Sigma_1 \\ F^x_t\end{matrix} \qquad \begin{matrix}\Sigma_2 \\ G^x_t\end{matrix} \qquad \begin{matrix}\Sigma_3 \\ \exists! t\end{matrix} \qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\,i}, \ \overline{\exists! z}^{\,i}} \\ \Pi \\ z = t\end{matrix}}{\iota x[F, G]}\ {}_i \qquad \begin{matrix}\underbrace{\overline{F^x_v}^{\,j}, \ \overline{G^x_v}^{\,j}, \ \overline{\exists! v}^{\,j}} \\ \Xi \\ C\end{matrix}}{C}\ {}_j$$

Transform such steps in a deduction into the following, where Ξ^v_t is the deduction resulting from Ξ by replacing the variable v everywhere with the term t:

$$\begin{matrix}\underbrace{\begin{matrix}\Sigma_1 \\ F^x_t\end{matrix}, \ \begin{matrix}\Sigma_2 \\ G^x_t\end{matrix}, \ \begin{matrix}\Sigma_3 \\ \exists! t\end{matrix}} \\ \Xi^v_t \\ C\end{matrix}$$

The conditions on variables ensure that no clashes arise from the replacement.

Second, the conclusion of ιI is the major premise of ιE^2:

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ F^x_{t_1}\end{matrix}\qquad \begin{matrix}\Sigma_2\\ G^x_{t_1}\end{matrix}\qquad \begin{matrix}\Sigma_3\\ \exists! t_1\end{matrix}\qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\,i},\ \overline{\exists! z}^{\,i}}\\ \Pi\\ z=t_1\end{matrix}}{\iota x[F,G]}\,{\scriptstyle i}\qquad \begin{matrix}\Xi_1\\ \exists! t_2\end{matrix}\qquad \begin{matrix}\Xi_2\\ \exists! t_3\end{matrix}\qquad \begin{matrix}\Xi_3\\ F^x_{t_2}\end{matrix}\qquad \begin{matrix}\Xi_4\\ F^x_{t_3}\end{matrix}}{t_2=t_3}$$

Transform such steps in a deduction into the following, where $\Pi^z_{t_2}$ and $\Pi^z_{t_3}$ are the deductions resulting from Π by replacing z with t_2 and t_3, respectively, and the last rule is an application of $=E$:

$$\dfrac{\begin{matrix}\underbrace{\begin{matrix}\Xi_4\\ F^x_{t_3}\end{matrix},\begin{matrix}\Xi_2\\ \exists! t_3\end{matrix}}\\ \Pi^z_{t_3}\\ t_3=t_1\end{matrix}\qquad \begin{matrix}\underbrace{\begin{matrix}\Xi_3\\ F^x_{t_2}\end{matrix},\begin{matrix}\Xi_1\\ \exists! t_2\end{matrix}}\\ \Pi^z_{t_2}\\ t_2=t_1\end{matrix}}{t_2=t_3}$$

The conditions on variables ensure that no clashes arise from the replacements.

The second reduction procedure for maximal formulas of the form $\iota x[F,G]$ is slightly unusual, as it appeals to a rule for another logical constant, i.e. identity. However, as the conclusion of ιE^2 is an identity, it is to be expected that its rules may have to be appealed to in the workings of the rules for ι.

I only give two examples of permutative reduction procedures for formulas of the form $\iota x[F,G]$ that are the conclusion of $\vee E$, $\exists E$ or ιE^1 and the major premise of ιE^1 or ιE^2. As in previous cases, clashes between variables are avoidable by choosing different variables for the applications of $\exists E$ and the elimination rules for ι.

First example. The major premise of ιE^1 is concluded by $\exists E$:

$$\dfrac{\dfrac{\exists v A \qquad \begin{matrix}\overline{A^v_y}^{\,i}\\ \Sigma \\ \iota x[F,G]\end{matrix}}{\iota x[F,G]}\,i \qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\,j},\ \overline{G^x_z}^{\,j},\ \overline{\exists! z}^{\,j}}\\ \Pi \\ C\end{matrix}}{C}\,j$$

Replace such steps in a deduction by:

$$\dfrac{\exists v A \qquad \dfrac{\begin{matrix}\overline{A^v_y}^{\,i}\\ \Sigma \\ \iota x[F,G]\end{matrix} \qquad \begin{matrix}\underbrace{\overline{F^x_z}^{\,j},\ \overline{G^x_z}^{\,j},\ \overline{\exists! z}^{\,j}}\\ \Pi \\ C\end{matrix}}{C}\,j}{C}\,i$$

Second example. The major premise of ιE^2 is the conclusion of $\exists E$:

$$\dfrac{\dfrac{\exists v A \qquad \begin{matrix}\overline{A^v_y}^{\,i}\\ \Sigma \\ \iota x[F,G]\end{matrix}}{\iota x[F,G]}\,i \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2}}{t_1 = t_2}$$

Replace such steps in a deduction by:

$$\dfrac{\exists v A \qquad \dfrac{\begin{matrix}\overline{A^v_y}^{\,i}\\ \Sigma \\ \iota x[F,G]\end{matrix} \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2}}{t_1 = t_2}}{t_1 = t_2}\,i$$

The remaining cases are similar.

I am not counting ιE^2 as an introduction rule for $=$. There is no general way of removing formulas $t_1 = t_2$ concluded by ιE^2 and eliminated by $= E$, as the following illustrates:

$$\cfrac{\cfrac{\iota x[F,G] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2}}{t_1 = t_2} \qquad A^x_{t_1}}{A^x_{t_2}}$$

Thus there are no further maximal formulas to be considered in $\mathbf{INF}^\iota$. After the theorem, I will give an alternative second elimination rule for ι that avoids this problem.

We have the following:

Theorem 1. *For any deduction* Π *of* A *from* Γ *in* $\mathbf{INF}^\iota$ *there is a deduction of the same conclusion from some of the formulas in* Γ *that is in normal form.*

Proof: By induction over the rank of proofs. The length of a segment is the number of formulas it consists of and its degree the number of logical constants in that formula. Let a maximal formula be a maximal segment of length 1. The *rank* of a deduction is the pair $\langle d, l\rangle$, where d is the highest degree of a maximal segment or 0 if there is none, and l is the sum of the lengths of maximal segments of highest degree. $\langle d, l\rangle < \langle d', l'\rangle$ iff either (i) $d < d'$ or (ii) $d = d'$ and $l < l'$. Applying the reduction procedures to a suitably chosen maximal segment of highest degree and longest length reduces the rank of a deduction. □

We can reformulate the second elimination rule for ι to incorporate an application of Leibniz' Law instead of concluding with an identity:

$$\iota E^{2A}: \quad \frac{\iota x[F,G] \qquad \exists! t_1 \qquad \exists! t_2 \qquad F^x_{t_1} \qquad F^x_{t_2} \qquad A^x_{t_1}}{A^x_{t_2}}$$

A can be restricted to atomic formulas, an induction over the complexity of formulas showing that the general version with A a formula of any degree is admissible. Call the system resulting from $\mathbf{INF}^\iota$ by replacing ιE^2 with ιE^{2A} $\mathbf{INF}^{\iota\prime}$.

ιE^2 and ιE^{2A} are interderivable in virtue of the rules for identity:

1. To derive ιE^{2A}, given premises $\iota x[F,G]$, $\exists! t_1$, $\exists! t_2$, $F^x_{t_1}$ and $F^x_{t_2}$, derive $t_1 = t_2$ by ιE^2 and apply $= E$ to it and the premise $A^x_{t_1}$ to derive $A^x_{t_2}$.
2. To derive ιE^2, let A be $t_1 = x$, so that $A^x_{t_1}$ is $t_1 = t_1$: derive it from $\exists! t_1$ by $= I^n$, apply ιE^{2A} to derive $A^x_{t_2}$, i.e. $t_1 = t_2$.

Thus $\mathbf{INF}^\iota$ and $\mathbf{INF}^{\iota\prime}$ are equivalent.

In $\mathbf{INF}^{\iota\prime}$, steps in a deduction that conclude $t_1 = t_2$ by ιE^{2A} (with $t_1 = t_1$ as $A^x_{t_1}$) and using it as the identity in Leibniz' Law are redundant: ιE^{2A} can instead be applied with the premise and conclusion of Leibniz' Law. Such identities can therefore be removed from deductions, and we are now at liberty to count them amongst the maximal formulas.

If a maximal formula arises from introducing $\iota x[F, G]$ by ιI and eliminating it by ιE^{2A}, we have the following situation:

$$\cfrac{\cfrac{\begin{array}{c}\Sigma_1\\ F^x_{t_1}\end{array} \quad \begin{array}{c}\Sigma_2\\ G^x_{t_1}\end{array} \quad \begin{array}{c}\Sigma_3\\ \exists! t_1\end{array} \quad \begin{array}{c}\underbrace{\overline{F^x_z}^{\,i}, \ \overline{\exists! z}^{\,i}}\\ \Pi\\ z = t_1\end{array}}{\iota x[F, G]}{\scriptstyle i} \quad \begin{array}{c}\Xi_1\\ \exists! t_2\end{array} \quad \begin{array}{c}\Xi_2\\ \exists! t_3\end{array} \quad \begin{array}{c}\Xi_3\\ F^x_{t_2}\end{array} \quad \begin{array}{c}\Xi_4\\ F^x_{t_3}\end{array} \quad \begin{array}{c}\Xi_5\\ A^x_{t_2}\end{array}}{A^x_{t_3}}$$

We now have two options for removing the maximal formula. We can proceed as previously: conclude $t_2 = t_3$ by an application of Leibniz' Law to the conclusions $t_2 = t_1$ of $\Pi^z_{t_1}$ and $t_3 = t_1$ of $\Pi^z_{t_2}$, and then apply Leibniz' Law once more with $A^x_{t_2}$ as further premise and $A^x_{t_3}$ as conclusion. Alternatively, we can first conclude $A^x_{t_1}$ from the conclusion $t_2 = t_1$ of $\Pi^z_{t_1}$ and $A^x_{t_2}$, and then conclude $A^x_{t_3}$ from $A^x_{t_2}$ and the conclusion $t_3 = t_1$ of $\Pi^z_{t_2}$. Thus deductions in the system resulting by replacing ιE^2 by ιE^{2A} also normalise, and it has the additional advantage of avoiding identities concluded by ιE^2 and eliminated by Leibniz' Law.

Thus we have the following:

THEOREM 2. *For any deduction Π of A from Γ in $\mathbf{INF}^{\iota\prime}$ there is a deduction of the same conclusion from some formulas in Γ that is in normal form.*

Deductions in $\mathbf{INF}^{\iota\prime}$ have slightly neater proof-theoretic properties than those in $\mathbf{INF}^{\iota}$, as deductions in normal form in $\mathbf{INF}^{\iota\prime}$ do not contain redundant identities introduced by ιE^2 and eliminated by $= E$. Deductions in $\mathbf{INF}^{\iota}$ are, however, slightly simpler if we are interested in establishing identities, and this will be the case if we are interested in comparing the present system with the standard treatment of ι as a term forming operator: axioms and rules for the latter invariably appeal to identity.

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THE METHOD OF SOCRATIC PROOFS MEETS CORRESPONDENCE ANALYSIS

Abstract

The goal of this paper is to propose correspondence analysis as a technique for generating the so-called erotetic (*i.e.* pertaining to the logic of questions) calculi which constitute the method of Socratic proofs by Andrzej Wiśniewski. As we explain in the paper, in order to successfully design an erotetic calculus one needs invertible sequent-calculus-style rules. For this reason, the proposed correspondence analysis resulting in invertible rules can constitute a new foundation for the method of Socratic proofs.

Correspondence analysis is Kooi and Tamminga's technique for designing proof systems. In this paper it is used to consider sequent calculi with non-branching (the only exception being the rule of cut), invertible rules for the negation fragment of classical propositional logic and its extensions by binary Boolean functions.

Keywords: Socratic proofs, correspondence analysis, invertible rule, inferential erotetic logic, classical propositional logic, sequent calculus.

The authors kindly devote this paper
to Andrzej Wiśniewski.

1. Introduction

1.1. The method of Socratic proofs

The method of Socratic proofs is a proof method grounded in the logic of questions called *inferential erotetic logic* (**IEL**, for short).[1] Developed mainly in the nineties by Andrzej Wiśniewski[2], the logic focuses its attention on the analysis of inferential relations between questions, distinguishes some classes of inferences with questions involved (henceforth called *erotetic inferences*), and, finally, develops criteria of validity of such erotetic inferences.

Undertaking the task to formally model erotetic inferences has led **IEL** to distinguishing some proof methods, especially the method of Socratic proofs.[3] The core of the method is the idea of answering questions by questioning, that is, by transforming the structure of the initially posed question. When the questions concern, for example, validity in a logic **L**, then the method of Socratic proofs constitutes a proof method for **L**. However, the general goal is more ambitious: it is to capture and provide a formal model for a kind of cognitive phenomenon, when an agent tends to solve a problem by consecutive questions.

The fact that we do perform such reasoning is incontestable. The erotetic calculi designed so far may be claimed to successfully model erotetic reasoning such as:

- *Is A a tautology? Well, is $\neg A$ satisfiable?*

But they also shed the light of explanation on such more general examples as:

- From [11, p. 47]: *Let me rephrase my question; what I am really asking is ...*

whereas the very notion of erotetic implication may be adjusted to provide an account of the following:

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[1] The word *erotetic* comes from the Greek $\epsilon\rho\omega\tau\eta\mu\alpha$ which means *question*.

[2] Tadeusz Kubiński, one of the pioneers in the logic of questions, has already focused on some relations between questions [17], [16]. For **IEL**, see [38] or [35] for a concise introduction. See also [36] for the most recent account of **IEL**.

[3] Introduced in [37] for the case of classical propositional logic it has been later adjusted to the first-order case (see [40]) as well as various non-classical cases (*int.al.* modal [19, 21], and paraconsistent [41, 6]). The most recent developments of the method are discussed in the monographs [5, 20].

Q_1 : Is $\sqrt{2}^{\sqrt{2}}$ a rational number?

A_1 : (Gelfond-Schneider Theorem) If x and y are algebraic numbers, $x \neq 0$, $x \neq 1$, and y is irrational, then x^y is a transcendental number.

A_2 : Every (real) transcendental number is irrational.

Q_2 : Is $\sqrt{2}$ an algebraic number?

There are two conditions defining the notion of erotetic implication. First, if question Q_1 erotetically implies question Q_2 (on the basis of X), then ***soundness of the first question***, Q_1, ***warrants soundness of the second question***, Q_2. This means that if in a given situation it is reasonable to ask Q_1, then it is also reasonable to ask Q_2. (More precisely, soundness of a question under a valuation amounts to the existence of an answer to the question which is true under the valuation. We postpone the technical details to Section 3.) For example, *if* a question:

- *Is Sabrina in the bedroom or in the living room?*

is sound in a given situation, *and if* one can hear Sabrina's voice, then the following question is also sound:

- *Does her voice come from the bedroom or from the living room?*

The second condition defining the notion of erotetic implication amounts to the fact that Q_2 is *asked for a purpose*: ***every answer to*** Q_2 ***must bring one closer to answering*** Q_1. It is the case in our example with Sabrina: every answer to the second question (providing the information that Sabrina's voice comes from the bedroom or from the living room) entails the answer to the first question.

The requirement "***every answer***" is a very strong one, and for this reason it is often weakened: when weak erotetic implication is considered, it is enough that at least ***one answer*** to the second question is useful in resolving the first one. For example, the affirmative answer to the above question Q_2: "Yes, $\sqrt{2}$ is an algebraic number",[4] makes theorem A_1 applicable, and so, together with A_2, yields the negative answer to Q_1: the number is not rational. However, in this case the requirement ***every answer*** is not satisfied, as the negative answer to Q_2 does not entail any solution to the problem expressed by Q_1. Yet, with the weaker variants of erotetic implication **IEL** can easily deal with that.

[4]It is the true answer, as the polynomial '$x^2 - 2$' witnesses.

What properties should a formal model possess in order to capture at least some of the cognitive phenomena described above? Well, the fundamental properties of the model designed in the framework of **IEL**, that is, of the method of Socratic proofs, are the following:

1. Syntactic, quasi-reductionist approach to questions (see [39] or [36, Chapter 2] for this and other approaches). It means, *int.al.*, that questions are distinguished as separate expressions in the language, expressions differing with respect to statements; and that a question is *identified* (understood) by knowing what counts as an answer. (*Cf.* the so-called Hamblin's postulates, [10].)

2. The rules transforming questions, *i.e.* erotetic rules, are built on a proof-theoretical skeleton of sequent calculus. As we shall see, questions transform certain units composed of sequents.

3. The crucial property: the construction of erotetic rules warrants that they retain the relation of erotetic implication between the question-premise and the question-conclusion.

For the last property to hold, the rules must be *semantically invertible*, that is, semantic correctness of the conclusion of a rule must warrant semantic correctness of its premise. This property is used in proving soundness of the method. However, regardless of their invertibility, the order of the application of erotetic rules is settled.

Each rule of an erotetic calculus transforms a question, but it focuses on a single *constituent* of a question, which is a sequent. When viewed as a rule acting on a sequent, an erotetic rule is a sequent-calculus rule *inverted*, so the derivation process as defined by the rules reflects the backward proof-search in sequent calculi—from the final conclusion in the root to the leaves.

From a purely proof-theoretical point of view, erotetic rules need not be sound in the sense of preserving semantic correctness top-down. Let us observe that the situation is similar in the case of sequent calculi, where, in general, the rules need not be semantically invertible in order to obtain the adequateness result. However, from the erotetic point of view, both soundness (top-down) and invertibility (bottom-up) of erotetic rules are necessary to obtain *erotetic correctness* of the rules. Hence comes the idea to examine the potential of correspondence analysis in the version introduced in the paper [22].

Both directions of applications of the rules open up more opportunities to search for proofs; however, it is probably more important that the two directions give more possibilities in modelling erotetic reasoning. Moreover, invertibility of the rules is essentially used in the completeness proof of the calculi presented in [22].

1.2. The notion of correspondence analysis

Correspondence analysis is Kooi and Tamminga's [15] proof-theoretic approach which, originally, was developed in order to axiomatize via natural deduction systems all the truth-functional unary and binary extensions of three-valued logic **LP** (**L**ogic of **P**aradox) [1, 31]. Later, Tamminga [33], using correspondence analysis, presented natural deduction systems for all the unary and binary extensions of Kleene's strong three-valued logic $\mathbf{K_3}$ [14, 13].

Further, Petrukhin [23] formulated via correspondence analysis natural deduction systems for all the unary and binary extensions of Belnap-Dunn's four-valued logic **FDE** (**F**irst **D**egree **E**ntailment) [2, 3, 7] supplied with Boolean negation. Petrukhin and Shangin have recently applied correspondence analysis and a proof-searching procedure for **FDE** itself [29]. Petrukhin and Shangin [26] developed a proof-searching algorithm for natural deduction systems for all the binary extensions of **LP**. In [27], the authors extended their proof searching technique to the case of all the binary extensions of $\mathbf{K_3}$. Petrukhin [24] presented via correspondence analysis natural deduction systems for all the unary and binary extensions of Kubyshkina and Zaitsev's [18] four-valued logic **LRA** (**L**ogic of **R**ational **A**gent). Besides, he generalized Kooi and Tamminga's ([15], [33]) results for a wider class of three-valued logics [25]. Petrukhin and Shangin [30] used correspondence analysis to syntactically characterize Tomova's natural logics [34, 12]. Petrukhin and Shangin [28] presented correspondence analysis for **PWK** (**P**araconsistent **W**eak **K**leene logic) [9, 4] which is Kleene's weak logic $\mathbf{K_3^w}$ [14, 13] with two designated values.

Finally, in [22], the authors showed how to use the framework to obtain sequent calculi with the following properties: all the rules are semantically invertible (understood as before, see also explanations below) *and actually inverted*, that is, used in both directions; the rules for connectives (the *logical* rules) are linear, the only branching rule is the rule of the cut, and the rule is not eliminable.

2. Sequent calculi obtained *via* correspondence analysis

We start with a summary of [22]. Some details, that may be found there, are skipped.

Notation. We use $\mathscr{P}$ for a countably infinite set $\{p, q, \ldots\}$ of propositional variables and $\mathscr{B} = \{\circ_\bot, \wedge, \not\to, \circ_1, \not\leftarrow, \circ_2, \veebar, \vee, \downarrow, \equiv, \circ_{\neg 2}, \leftarrow, \circ_{\neg 1}, \to, \uparrow, \circ_\top\}$ for a set of binary operators, where:

A	B	$\circ_\bot$	$\wedge$	$\not\to$	$\circ_1$	$\not\leftarrow$	$\circ_2$	$\veebar$	$\vee$
1	1	0	1	0	1	0	1	0	1
1	0	0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0

A	B	$\downarrow$	$\equiv$	$\circ_{\neg 2}$	$\leftarrow$	$\circ_{\neg 1}$	$\to$	$\uparrow$	$\circ_\top$
1	1	0	1	0	1	0	1	0	1
1	0	0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1	1	1
0	0	1	1	1	1	1	1	1	1

Let $\mathscr{L}^\circ_\neg$ be propositional language with the alphabet $\langle \mathscr{P}, \mathscr{B}, \neg, (,) \rangle$; the set $\mathscr{F}^\circ_\neg$ of all $\mathscr{L}^\circ_\neg$'s formulas is defined as usually.

Sequents are introduced as expressions of language $\mathscr{L}^\circ_{\neg\Rightarrow}$ which is built upon $\mathscr{L}^\circ_\neg$ by adding '$\Rightarrow$' (the sequent arrow) and the comma ',' to the alphabet. The only category of a well-formed expression of $\mathscr{L}^\circ_{\neg\Rightarrow}$ is that of a *sequent of* $\mathscr{L}^\circ_{\neg\Rightarrow}$, which is an expression of the form:

$$\Gamma \Rightarrow \Delta \tag{2.1}$$

where Γ and Δ are finite, possibly empty multisets of formulas of $\mathscr{L}^\circ_\neg$. We use comma in the antecedent and in the succedent both as a separator of the elements of a multiset and as the sum of multisets (the context makes it clear).

The sequent calculi introduced in [22] are built upon the rules (Ax), $(\Rightarrow \neg)$, $(\neg \Rightarrow)$, for the negation fragment of **CPL**, together with the structural rule (cut) which is the only branching rule of the system.

$$(\text{Ax})\quad A, \Gamma \Rightarrow \Delta, A \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{cut})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\Rightarrow \neg) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow)$$

The following rules, presented in [22], have been found by correspondence analysis:

$$R_\circ^{(01)}\ \frac{A, \Gamma \Rightarrow \Delta, B}{A \circ B, \Gamma \Rightarrow \Delta, B} \qquad R_\circ^{(02)}\ \frac{B, \Gamma \Rightarrow \Delta, \neg(A \circ B)}{A \circ B, \Gamma \Rightarrow \Delta, A}$$

$$R_\circ^{(03)}\ \frac{B, \Gamma \Rightarrow \Delta, A}{A \circ B, \Gamma \Rightarrow \Delta, A} \qquad R_\circ^{(04)}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\neg B, \Gamma \Rightarrow \Delta, A \circ B}$$

$$R_\circ^{(05)}\ \frac{B, \Gamma \Rightarrow \Delta, A \circ B}{\neg A, \Gamma \Rightarrow \Delta, A \circ B} \qquad R_\circ^{(06)}\ \frac{A \circ B, \Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta, B}$$

$$R_\circ^{(07)}\ \frac{B, \Gamma \Rightarrow \Delta, A \circ B}{B, \Gamma \Rightarrow \Delta, A} \qquad R_\circ^{(08)}\ \frac{A \circ B, \Gamma \Rightarrow \Delta, B}{A \circ B, \Gamma \Rightarrow \Delta, A}$$

$$R_\circ^{(09)}\ \frac{B, \Gamma \Rightarrow \Delta, \neg(A \circ B)}{B, \Gamma \Rightarrow \Delta, A} \qquad R_\circ^{(10)}\ \frac{\neg B, \Gamma \Rightarrow \Delta, A \circ B}{\neg A, \Gamma \Rightarrow \Delta, A \circ B}$$

$$R_\circ^{(11)}\ \frac{A, \Gamma \Rightarrow \Delta, A \circ B}{A, \Gamma \Rightarrow \Delta, B} \qquad R_\circ^{(12)}\ \frac{A, \Gamma \Rightarrow \Delta, \neg(A \circ B)}{A, \Gamma \Rightarrow \Delta, B}$$

$$A_{\circ\uparrow}^{(\mathrm{I})}\ A, B, \Gamma \Rightarrow \Delta, A \circ B \qquad A_{\circ\downarrow}^{(\mathrm{I})}\ A \circ B, \Delta \Rightarrow \Gamma, A, B$$

$$A_{\circ\uparrow}^{(\mathrm{II})}\ \neg A, \neg B, \Gamma \Rightarrow \Delta, A \circ B \qquad A_{\circ\downarrow}^{(\mathrm{II})}\ A \circ B, \Delta \Rightarrow \Gamma, \neg A, \neg B$$

$$A_{\circ\uparrow}^{(\mathrm{III})}\ A, \neg B, \Gamma \Rightarrow \Delta, A \circ B \qquad A_{\circ\downarrow}^{(\mathrm{III})}\ A \circ B, \Delta \Rightarrow \Gamma, A, \neg B$$

$$A_{\circ\uparrow}^{(\mathrm{IV})}\ \neg A, B, \Gamma \Rightarrow \Delta, A \circ B \qquad A_{\circ\downarrow}^{(\mathrm{IV})}\ A \circ B, \Delta \Rightarrow \Gamma, \neg A, B$$

$$R_\circ^{(\mathrm{I})}\ \frac{A, B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma, \Rightarrow \Delta, A, B} \qquad R_\circ^{(\mathrm{II})}\ \frac{\neg A, \neg B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, \neg A, \neg B}$$

$$R_\circ^{(\mathrm{III})}\ \frac{A, \neg B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, A, \neg B} \qquad R_\circ^{(\mathrm{IV})}\ \frac{\neg A, B, \Gamma \Rightarrow \Delta, A \circ B}{A \circ B, \Gamma \Rightarrow \Delta, \neg A, B}$$

The tables 1 and 2 (below) summarize the construction of our sequent calculi for various fragments of **CPL** expressed in language $\langle \mathscr{P}, \{\circ\}, \neg, (,) \rangle$.

To increase readability, under a connective $\circ$ we indicate in the table the numbers of the $\circ$-specific rules instead of their names. In the case of the primary connectives[5], we define two types of sequent calculi: one can add an axiom or a rule with the respective Roman numeral.

Table 1. Rules for non-primary connectives

$A \circ_{\perp} B$	$A \circ_1 B$	$A \circ_2 B$	$A \veebar B$
(02), (08)	(01), (07)	(03), (11)	(01), (09) (02), (10) (03), (12)

$A \equiv B$	$A \circ_{\neg 2} B$	$A \circ_{\neg 1} B$	$A \circ_{\top} B$
(04), (07) (05), (08) (06), (11)	(06), (12)	(04), (09)	(05), (10)

Table 2. Rules for primary connectives

(I) and (08)	$A \circ B = A \wedge B$
(I) and (10)	$A \circ B = A \vee B$
(II) and (08)	$A \circ B = A \downarrow B$
(II) and (10)	$A \circ B = A \uparrow B$
(III) and (02)	$A \circ B = A \not\to B$
(III) and (05)	$A \circ B = A \leftarrow B$
(IV) and (02)	$A \circ B = A \not\leftarrow B$
(IV) and (05)	$A \circ B = A \to B$

Semantics for $\mathscr{L}^{\circ}_{\neg\Rightarrow}$. As in [22], we will use the symbol '$\models$' for entailment in both languages: $\mathscr{L}^{\circ}_{\neg\Rightarrow}$ and $\mathscr{L}^{\circ}_{\neg}$. If v is a valuation, then we say that sequent (2.1) is *true under* v iff if every element of Γ is true under v, then some element of Δ is true under v as well. For example, every sequent of the form (Ax): $A, \Gamma \Rightarrow \Delta, A$ is true under every valuation.

[5]A binary connective $\circ$ is called *primary*, if $\{\neg, \circ\}$ is functionally complete (see [8, p. 13]). In [22] we show the difference between the primary and the non-primary connectives via correspondence analysis.

Finally, by:

$$\Gamma \Rightarrow \Delta \models \Theta \Rightarrow \Lambda$$

we mean that for every valuation v, if sequent $\Gamma \Rightarrow \Delta$ is true under v, then sequent $\Theta \Rightarrow \Lambda$ is true under v.

The notions of a derivation and a proof are defined in a standard manner. By $\vdash_{\mathscr{C}} \Gamma \Rightarrow \Delta$ we mean that sequent $\Gamma \Rightarrow \Delta$ has a proof in sequent calculus $\mathscr{C}$. In [22] we proved that:

THEOREM 1 (Soundness and Completeness of $\mathscr{C}$). For each formula $A \in \mathscr{L}^{\circ}_{\neg}$, $\models A$ iff $\vdash_{\mathscr{C}} \Rightarrow A$.

3. Erotetic calculi

3.1. Language

Erotetic calculi are worded in languages containing questions; the declarative expressions are, first of all, sequents.

We enrich language $\mathscr{L}^{\circ}_{\neg\Rightarrow}$ with the question forming operator: '?', the semicolon ';', the signs for negation: $\underline{ng}$ and conjunction: & (in order to build complex declarative formulas from sequents). The resulting language will be called $\mathscr{QL}$. Atomic declarative formulas of $\mathscr{QL}$ are, simply, sequents of $\mathscr{L}^{\circ}_{\neg\Rightarrow}$. The remaining declarative formulas are built from the atomic ones by the use of $\underline{ng}$ and/or & in a usual way. *Questions of* $\mathscr{QL}$ are expressions of the form:

$$?(\Gamma_1 \Rightarrow \Delta_1; \ldots; \Gamma_n \Rightarrow \Delta_n) \tag{3.1}$$

where $\Gamma_i \Rightarrow \Delta_i$ is a sequent, also called a *constituent of question* (3.1).

Erotetic calculus is a set of erotetic rules, that is, rules transforming a question into a question. In the original account, each erotetic step is supposed to simplify the logical structure of the analysed problem by elimination of a logical constant (or better – due to the use of the unified notation[6] – by *decomposition* of complex α-, β- formulas into their components). The use of correspondence analysis changes this picture since the simplification is sometimes lost.

[6] As far, erotetic calculi have been usually formed with the use of the unified notation: see [36], [5], [20]. However, in [20] the author considers also erotetic calculi where this convention is dropped.

3.2. Rules

One of the characteristic proof-theoretic features of the method of Socratic proofs is that its rules are designed with the aim to capture erotetic implication. For this reason one needs invertibility on the level of declaratives of $\mathscr{QL}$ (that is, sequents). Hence follows the choice of calculi $\mathscr{C}$ introduced in [22] and recalled above as the basis of erotetic calculi $\mathcal{E}_{\mathscr{C}}$.

Greek letters Φ and Ψ are used for finite, possibly empty sequences of sequents. For simplicity, the semicolon is used both as a separator between sequents and as a concatenation symbol between sequences of sequents.

$\mathcal{E}_{\mathscr{C}}$ is any set of rules containing erotetic version of cut, the rules for negation:

$$ER_{cut} \ \frac{?(\Phi\ ;\ \Gamma \Rightarrow \Delta\ ;\ \Psi)}{?(\Phi\ ;\ \Gamma \Rightarrow \Delta, A\ ;\ A, \Gamma \Rightarrow \Delta\ ;\ \Psi)}$$

$$ER_{\neg\Rightarrow} \ \frac{?(\Phi\ ;\ \neg A, \Gamma \Rightarrow \Delta\ ;\ \Psi)}{?(\Phi\ ;\ \Gamma \Rightarrow \Delta, A\ ;\ \Psi)} \qquad ER_{\Rightarrow\neg} \ \frac{?(\Phi\ ;\ \Gamma \Rightarrow \Delta, \neg A\ ;\ \Psi)}{?(\Phi\ ;\ A, \Gamma \Rightarrow \Delta\ ;\ \Psi)}$$

and a combination of the $\circ$-specific rules. To save space, these may be given by the following general scheme: if $R = \phi/\psi$ is a $\circ$-specific rule of $\mathscr{C}$, that is, R is one of $R_\circ^{(01)}$-$R_\circ^{(12)}$ or $R_\circ^{(\mathrm{I})}$-$R_\circ^{(\mathrm{IV})}$, then the following:

$$ER \ \frac{?(\Phi\ ;\ \psi\ ;\ \Psi)}{\overline{?(\Phi\ ;\ \phi\ ;\ \Psi)}}$$

is a rule of $\mathcal{E}_{\mathscr{C}}$. For example, if $R_\circ^{(01)}$ belongs to $\mathscr{C}$, then $ER_\circ^{(01)}$ belongs to $\mathcal{E}_{\mathscr{C}}$, where:

$$ER_\circ^{(01)} \ \frac{?(\Phi\ ;\ A \circ B, \Gamma \Rightarrow \Delta, B\ ;\ \Psi)}{\overline{?(\Phi\ ;\ A, \Gamma \Rightarrow \Delta, B\ ;\ \Psi)}}$$

As above, the double line indicates that the rules are not only semantically invertible, but may be applied *in both directions*.

As we can see, calculi $\mathscr{C}$ form the proof-theoretical skeleton of $\mathcal{E}_{\mathscr{C}}$.

Definition 1 (Socratic proof in $\mathcal{E}_{\mathscr{C}}$ v.1). Let $\Gamma \Rightarrow \Delta$ be a sequent of language $\mathscr{Q}\mathscr{L}$, and assume that $\mathscr{C}$ does not contain any of axioms: $A_\circ^{(\mathrm{I})}$, $A_\circ^{(\mathrm{II})}$, $A_\circ^{(\mathrm{III})}$, $A_\circ^{(\mathrm{IV})}$. A *Socratic proof of* $\Gamma \Rightarrow \Delta$ *in* $\mathcal{E}_{\mathscr{C}}$ is a finite sequence of questions $\langle Q_1, \ldots, Q_n\rangle$ such that:

1. $Q_1 = ?(\Gamma \Rightarrow \Delta)$,
2. for each $i, n \geq i > 1$: Q_i results from Q_{i-1} by a rule of $\mathcal{E}_{\mathscr{C}}$, and
3. each constituent of Q_n is of the form (Ax): $A, \Gamma \Rightarrow \Delta, A$.

If there exists a Socratic proof of a sequent in $\mathcal{E}_{\mathscr{C}}$, then we say that the sequent is *provable in* $\mathcal{E}_{\mathscr{C}}$.

Further, in the case of $\mathcal{E}_{\mathscr{C}}$, where $\mathscr{C}$ contains axioms:

Definition 2 (Socratic proof in $\mathcal{E}_{\mathscr{C}}$ v.2). Let $\Gamma \Rightarrow \Delta$ be a sequent of language $\mathscr{Q}\mathscr{L}$, where $\mathscr{C}$ contains at least one of axioms: $A_\circ^{(\mathrm{I})}$, $A_\circ^{(\mathrm{II})}$, $A_\circ^{(\mathrm{III})}$, $A_\circ^{(\mathrm{IV})}$. A *Socratic proof of* $\Gamma \Rightarrow \Delta$ *in* $\mathcal{E}_{\mathscr{C}}$ is a finite sequence of questions $\langle Q_1, \ldots, Q_n\rangle$ such that:

1. $Q_1 = ?(\Gamma \Rightarrow \Delta)$,
2. for each $i, n \geq i > 1$: Q_i results from Q_{i-1} by a rule of $\mathcal{E}_{\mathscr{C}}$, and
3. each constituent of Q_n is of the form (Ax): $A, \Gamma \Rightarrow \Delta, A$, or of the form $A_\circ^x$, where $A_\circ^x$ belongs to $\mathscr{C}$.

If there exists a Socratic proof of a sequent in $\mathcal{E}_{\mathscr{C}}$, then we say that the sequent is *provable in* $\mathcal{E}_{\mathscr{C}}$.

Here is an example of a Socratic proof in $\mathcal{E}_{\mathscr{C}}$. To save some space, $A \equiv B$ stands for $(p \equiv q) \equiv (q \equiv p)$; as soon as a sequent of the form (Ax*) is arrived at, it is represented as Ax_i.

$$
\begin{array}{rc}
 & ?(\Rightarrow (p \equiv q) \equiv (q \equiv p)) \\ \cline{2-2}
ER_{cut} & ?(\Rightarrow A \equiv B, \neg(q \equiv p) \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER_{\Rightarrow\neg} & ?(q \equiv p \Rightarrow A \equiv B \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER^{(07)} \uparrow & ?(q \equiv p \Rightarrow p \equiv q \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER_{cut} & ?(B \Rightarrow A, \neg q \ ; \ \neg q, q \equiv p \Rightarrow p \equiv q \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER_{\Rightarrow\neg} & ?(q, B \Rightarrow A \ ; \ \neg q, B \Rightarrow A \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER^{(07)} \uparrow & ?(q, B \Rightarrow p \ ; \ \neg q, B \Rightarrow A \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER^{(08)} \uparrow & ?(q, B \Rightarrow q \ ; \ \neg q, B \Rightarrow A \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER^{(04)} \downarrow & ?(Ax_1 \ ; \ p, q \equiv p \Rightarrow q \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER^{(08)} \downarrow & ?(Ax_1 \ ; \ p, q \equiv p \Rightarrow p \ ; \ \neg(q \equiv p) \Rightarrow A \equiv B) \\ \cline{2-2}
ER^{(04)} \downarrow & ?(Ax_1 \ ; \ Ax_2 \ ; \ p \equiv q \Rightarrow q \equiv p) \\ \cline{2-2}
ER_{cut} & ?(Ax_1 \ ; \ Ax_2 \ ; \ p \equiv q \Rightarrow q \equiv p, \neg p \ ; \ \neg p, p \equiv q \Rightarrow q \equiv p) \\ \cline{2-2}
ER_{\Rightarrow\neg} & ?(Ax_1 \ ; \ Ax_2 \ ; \ p, p \equiv q \Rightarrow q \equiv p \ ; \ \neg p, p \equiv q \Rightarrow q \equiv p) \\ \cline{2-2}
ER^{(07)} \uparrow & ?(Ax_1 \ ; \ Ax_2 \ ; \ p, p \equiv q \Rightarrow q \ ; \ \neg p, p \equiv q \Rightarrow q \equiv p) \\ \cline{2-2}
ER^{(08)} \uparrow & ?(Ax_1 \ ; \ Ax_2 \ ; \ p, p \equiv q \Rightarrow p \ ; \ \neg p, p \equiv q \Rightarrow q \equiv p) \\ \cline{2-2}
ER^{(04)} \downarrow & ?(Ax_1 \ ; \ Ax_2 \ ; \ Ax_3 \ ; \ q, p \equiv q \Rightarrow p) \\ \cline{2-2}
ER^{(08)} \downarrow & ?(Ax_1 \ ; \ Ax_2 \ ; \ Ax_3 \ ; \ q, p \equiv q \Rightarrow q)
\end{array}
$$

The main differences between $\mathscr{C}$ and $\mathcal{E}_{\mathscr{C}}$ are:

- direction: in $\mathscr{C}$, as in all standard sequent calculi, the direction of *proving* as defined by the rules and the direction of *proof-search* as performed by a logician are opposite; in erotetic calculi it is the same direction; obviously, here it holds only for the rules of cut and negation;

- sequent calculi define derivations as trees, in erotetic calculi derivations (called Socratic transformations) are defined as sequences of questions, and questions are based on sequences of sequents; as one can see, the external context Φ, Ψ is rewritten every time a rule is applied, the result is such that all the semantic information is saved in the last question; in the account of implementation it means that no backtracking is needed.

3.3. MiES

This section shows the importance of invertibility of rules in the erotetic context.

Let us start with:

Definition 3. An erotetic rule $?(\Phi)/?(\Psi)$ is:

- *sound* iff, for each valuation v, the truth of each constituent of Φ under v warrants the truth of each constituent of Ψ under v,
- *invertible* iff, for each valuation v, the truth of each constituent of Ψ under v warrants the truth of each constituent of Φ under v.

Corollary 1. If a rule $\Gamma_1 \Rightarrow \Delta_1/\Gamma_2 \Rightarrow \Delta_2$ of $\mathscr{C}$ is sound and invertible, then an erotetic rule of the form:

$$\frac{?(\Phi\, ;\ \Gamma_2 \Rightarrow \Delta_2\ ;\ \Psi)}{?(\Phi\, ;\ \Gamma_1 \Rightarrow \Delta_1\ ;\ \Psi)}$$

is sound and invertible.

It is easy to see that ER_{cut} is sound and invertible, hence:

Corollary 2. Each rule of $\mathcal{E}_{\mathscr{C}}$ is sound and invertible.

From the fact that axioms (Ax) are true under every valuation, and from the fact that the rules of $\mathcal{E}_{\mathscr{C}}$ are invertible, it follows that:

Theorem 2 (soundness of $\mathcal{E}_{\mathscr{C}}$). Let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathscr{QL}$. If $\Gamma \Rightarrow \Delta$ has a Socratic proof in $\mathcal{E}_{\mathscr{C}}$, then $\Gamma \Rightarrow \Delta$ is true under every valuation.

Similarly, completeness of $\mathcal{E}_{\mathscr{C}}$ follows from completeness of $\mathscr{C}$.

Theorem 3 (completeness of $\mathcal{E}_{\mathscr{C}}$). If a sequent of $\mathscr{QL}$ is true under every valuation, then it has a Socratic proof in $\mathcal{E}_{\mathscr{C}}$.

Erotetic implication

As we explained in the first section, the construction of erotetic calculi should warrant that the relation of erotetic implication, a central notion for inferential erotetic logic, holds between a question-premise and a question-conclusion. Now we define the notion.

Suppose that we deal with a language rich enough to distinguish between declaratives and questions. Let Q, Q^* stand for questions and dQ, dQ^* for the respective sets of direct answers to these questions. We adjust the definition from [36, p. 67]:

Definition 4 (erotetic implication). A *question Q implies a question Q^* on the basis of a set of declaratives X* (in symbols: $\mathsf{Im}(Q, X, Q^*)$) iff:

1. for each $A \in dQ$, for each valuation v, if each formula in $X \cup \{A\}$ is true under v, then *some* formula in dQ^* is true under v, and
2. for each $B \in dQ^*$, there exists a non-empty proper subset Y of dQ such that, for each valuation v, if each formula in $X \cup \{B\}$ is true under v, then *some* formula in Y is true under v.

Definition 4 is based upon the semantic notion of Boolean valuation; in the case of $\mathscr{QL}$ we need something more general. The notions introduced below are central tools of the so-called *Minimal Erotetic Semantics* (MiES), a very general framework for a semantic analysis of both declaratives and questions developed by Andrzej Wiśniewski.[7] The primary notion is that of a partition of a language, which comes from [32].

Definition 5 (partition of language $\mathscr{QL}$). Let $\mathsf{D}_{\mathscr{QL}}$ be the set of sequents of language $\mathscr{QL}$. By a *partition of* $\mathsf{D}_{\mathscr{QL}}$ (or a *partition of language* $\mathscr{QL}$) we mean an ordered pair $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ such that $\mathsf{T_P} \cup \mathsf{U_P} = \mathsf{D}_{\mathscr{QL}}$ and $\mathsf{T_P} \cap \mathsf{U_P} = \varnothing$.

In the case of complex languages with questions, the counterpart of the semantic notion of Boolean valuation is that of an *admissible partition.*

Definition 6 (admissible partition of language $\mathscr{QL}$). Let $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ be a partition of language $\mathscr{QL}$. We say that P is *admissible for $\mathscr{QL}$* iff the following conditions hold:

1. '$\Gamma \Rightarrow \Delta$' $\in \mathsf{T_P}$ iff both '$\Gamma \Rightarrow \Delta, A$' $\in \mathsf{T_P}$ and '$A, \Gamma \Rightarrow \Delta$' $\in \mathsf{T_P}$, for each formula A;
2. '$\Gamma, \neg A, \Delta \Rightarrow \Theta, \Lambda$' $\in \mathsf{T_P}$ iff '$\Gamma, \Delta \Rightarrow \Theta, A, \Lambda$' $\in \mathsf{T_P}$;
3. '$\Gamma, \Delta \Rightarrow \Theta, \neg A, \Lambda$' $\in \mathsf{T_P}$ iff '$\Gamma, A, \Delta \Rightarrow \Theta, \Lambda$' $\in \mathsf{T_P}$;
4. '$\Gamma, A, \Delta \Rightarrow \Theta, B, \Lambda$' $\in \mathsf{T_P}$ iff '$\Gamma, \neg B, \Delta \Rightarrow \Theta, A \equiv B, \Lambda$' $\in \mathsf{T_P}$;
5. '$\Gamma, B, \Delta \Rightarrow \Theta, A \equiv B, \Lambda$' $\in \mathsf{T_P}$ iff '$\Gamma, B, \Delta \Rightarrow \Theta, A, \Lambda$' $\in \mathsf{T_P}$.

To save some space, in the above definition we have specified only the machinery for '$\circ$' = '$\equiv$' corresponding to rules $R_\circ^{(04)}$ and $R_\circ^{(07)}$. It is analogous in the remaining cases. Let us also observe that the above definition does not take into account the axioms that may be present in $\mathscr{C}$.

[7] For the details see [36] or [39].

The reason for their absence is that on the level of language $\mathscr{QL}$ we are interested in entailment only, not in validity.

Definition 7 (entailment in $\mathscr{QL}$). Suppose that $\mathfrak{X}$ is a set of sequents of language $\mathscr{QL}$ and $\mathfrak{t}$ is a single sequent. We say that *set $\mathfrak{X}$ entails formula $\mathfrak{t}$ in language $\mathscr{QL}$*, symbolically:

$$\mathfrak{X} \models_{\mathscr{QL}} \mathfrak{t}$$

iff there is no admissible partition P for language $\mathscr{QL}$ such that $\mathfrak{X} \subseteq \mathsf{T}_\mathsf{P}$ and $\mathfrak{t} \notin \mathsf{T}_\mathsf{P}$.

As before, dQ stands for the set of direct answers to Q. In the case of questions of $\mathscr{QL}$ of the form (3.1) (see page 107), the set is composed of two declarative formulas of the language: the affirmative answer (3.2) and the negative answer (3.3).

$$(\Gamma_1 \Rightarrow \Delta_1) \ \& \ (\ldots \& \ ((\Gamma_{n-1} \Rightarrow \Delta_{n-1}) \ \& \ (\Gamma_n \Rightarrow \Delta_n) \ldots) \tag{3.2}$$

$$\underline{ng}((\Gamma_1 \Rightarrow \Delta_1) \ \& \ (\ldots \& \ ((\Gamma_{n-1} \Rightarrow \Delta_{n-1}) \ \& \ (\Gamma_n \Rightarrow \Delta_n) \ldots)) \tag{3.3}$$

Definition 8 (erotetic implication in $\mathscr{QL}$). Suppose that Q and Q^* are questions of $\mathscr{QL}$ and that $\mathfrak{X}$ is a set of sequents. We say that *question Q implies question Q^* on the basis of set $\mathfrak{X}$ of sequents* iff, for each admissible partition P of language $\mathscr{QL}$, the following holds:

1. for each $\mathfrak{t} \in dQ$: if $\mathfrak{X} \cup \{\mathfrak{t}\} \subseteq \mathsf{T}_\mathsf{P}$, then $dQ^* \cap \mathsf{T}_\mathsf{P} \neq \varnothing$; and
2. for each $\mathfrak{u} \in dQ^*$: there is a non-empty proper subset $\mathfrak{X}^*$ of dQ such that if $\mathfrak{X} \cup \{\mathfrak{u}\} \subseteq \mathsf{T}_\mathsf{P}$, then $\mathfrak{X}^* \cap \mathsf{T}_\mathsf{P} \neq \varnothing$.

The above construction leads to the following:

Corollary 3. Suppose that a sequence of questions $\langle Q_1, \ldots, Q_n \rangle$ is a Socratic proof of a certain sequent in $\mathcal{E}_{\mathscr{C}}$. Then question Q_1 implies question Q_n on the basis of the empty set of sequents.

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SEMI-HEYTING ALGEBRAS AND IDENTITIES OF ASSOCIATIVE TYPE

Abstract

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice, and it satisfies the identities : $x \wedge (x \to y) \approx x \wedge y$, $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$, and $x \to x \approx 1$. $\mathcal{SH}$ denotes the variety of semi-Heyting algebras. Semi-Heyting algebras were introduced by the second author as an abstraction from Heyting algebras. They share several important properties with Heyting algebras. An identity of *associative type of length* 3 is a groupoid identity, both sides of which contain the same three (distinct) variables that occur in any order and that are grouped in one of the two (obvious) ways. A subvariety of $\mathcal{SH}$ is *of associative type of length* 3 if it is defined by a single identity of associative type of length 3.

In this paper we describe all the distinct subvarieties of the variety $\mathcal{SH}$ of associative type of length 3. Our main result shows that there are 3 such subvarities of $\mathcal{SH}$.

Keywords: semi-Heyting algebra, Heyting algebra, identity of associative type, subvariety of associative type.

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1. Introduction

Semi-Heyting algebras were introduced by the second author in 1983–84, as a result of his research that went into [33] (which was still a preprint at the time). Some of the early results were announced in [35].

A closer look at the proofs of results proved in [33] led him to the following rather interesting observation:

The arguments in [33], for the most part, used only the following well known properties of Heyting algebras:

(1) Their lattice-reducts are pseudocomplemented,
(2) Their lattice-reducts are distributive, and
(3) Congruences on them are determined by filters.

This observation led him to the following conjecture.

Conjecture A: There exists a variety **V** of algebras such that

- it has the same language as that of Heyting algebras,
- it contains Heyting algebras, and
- it possesses the following well known properties of Heyting algebras:
    (1) The lattice reducts of the algebras in **V** are pseudocomplemented
    (2) The lattice-reducts are distributive,
    (3) Congruences on the algebras in **V** are determined by filters,

Around the same time (1983-85), he had also completed the research for [36] (which was still in the preprint form). Led by the striking similarities in the results and in the proofs of (the preprints of) the papers [33] and [36], he formulated the following conjecture which appeared in print much later in 1987:

Conjecture 1: There exists a variety **V** of algebras of type $\langle \vee, \wedge, \rightarrow, ', 0, 1\rangle$ which would provide a unifying framework to state and prove common generalizations of strikingly similar results, proved in the above-mentioned two papers.

The search for such a variety led him naturally to consider the following conjecture and strengthened his belief in the validity of Conjecture A.

Conjecture 2: There exists a common generalization of (dually) pseudocomplemented lattices and De Morgan algebras.

Conjecture 2 was easy to settle with the variety of semi-De Morgan algebras, since they were already known to the author in 1979. The results on these algebras, however, appeared in print in the paper [37].

Conjecture A was settled in 1983–84 with the discovery of semi-Heyting algebras. However, the first results on semi-Heyting algebras appeared in print only in 2008 in the Proceedings of 9th A. Monteiro Conference in Bahia Blanca, Argentina (see [38]), held in 2007. (It was predicted in [37] that semi-De Morgan algebras might be useful in resolving Conjecture 1. Indeed, it turned out to be the case. Conjecture 1 was settled later in [39], with the help of both semi-Heyting algebras and (a subvariety of) semi-De Morgan algebras.)

Definition 1.1. *An algebra $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:*

(SH1) $\langle A, \vee, \wedge, 0, 1\rangle$ *is a lattice with* 0 *and* 1,

(SH2) $x \wedge (x \to y) \approx x \wedge y$,

(SH3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(SH4) $x \to x \approx 1$.

A semi-Heyting algebra is a Heyting algebra if it satisfies the identity

(H) $(x \wedge y) \to x \approx 1$.

We will denote the variety of semi-Heyting algebras by $\mathcal{SH}$ and that of Heyting algebras by $\mathcal{H}$. It is clear that $\mathcal{H} \subset \mathcal{SH}$.

It turns out (see [38]) that semi-Heyting algebras share with Heyting algebras some rather strong properties, besides the three mentioned earlier. For example, semi-Heyting algebras share the following properties with Heyting algebras:
(1) every interval in a semi-Heyting algebra is also pseudocomplemented,
(2) the variety $\mathcal{SH}$ is arithmetical, and
(3) The variety $\mathcal{SH}$ has EDPC (equationally definable principal congruences).
Moreover, there is a rich supply of algebras in $\mathcal{SH}$. It is known that there are in $\mathcal{SH}$, up to isomorphism, two 2-element algebras, ten 3-element algebras, only one of which, of course, is a Heyting algebra and 160 algebras on a 4-element chain (see [38] and [4]).

It is well-known that Heyting algebras form an equivalent algebraic semantics for intuitionistic logic; and there is a vast literature on the lattice of subvarieties of $\mathcal{H}$ (equivalently, on the lattice of intermediate logics), both from algebraic and logical points of view. Recently, the first author, in

[11], has introduced "semi-intuitionistic logic", whose equivalent algebraic semantics is the the variety of semi-Heyting algebras and which has the intuitionistic logic and classical logic as extensions, thus implying that the lattice of intermediate logics (extensions of the intuitionistic logic) is an interval in the lattice of extensions of the semi-intuitionistic logic (see [13] for a more stream-lined version). These observations led us naturally to the following problem.

PROBLEM: Investigate the structure of the lattice of subvarieties of the variety of semi-Heyting algebras, algebraically and logically.

It should perhaps be mentioned here that already Problem 14.2 of [38] had called for an investigation into the structure of the lattice of subvarieties of the variety of semi-Heyting algebras (algebraically).

There exists already some literature related to this problem. The papers that deal with this problem algebraically include [38], [2], [3], [4], [5], [15] and [17]. The paper [4] investigates the properties of semi-Heyting chains and the structure of the variety $\mathcal{CSH}$ generated by all semi-Heyting chains. In [2], it is proved, among other things, that the variety of Boolean semi-Heyting algebras (algebras with an underlying structure of Boolean lattice) constitutes a reflective subcategory of $\mathcal{SH}$, extending the corresponding result for Heyting algebras (see [6, Corollary IX.5.4], and that the free algebras in a subvariety $\mathcal{V}$ of $\mathcal{SH}$ are directly indecomposable if and only if $\mathcal{V}$ satisfies the Stone identity, extending a known result for Heyting algebras. Article [3] presents two other subvarieties of semi-Heyting algebras that are term-equivalent to the variety of Goedel algebras (linear Heyting algebras), and that they are the only other subvarieties in L with this property. The variety of semi-Nelson algebras is introduced in [17] so that the well-known and well-exploited relationship between Heyting and Nelson algebras extends to semi-Heyting and semi-Nelson algebras. It is also proved that the variety of semi-Nelson algebras is arithmetical, has equationally definable principal congruences, has the congruence extension property, along with a description of the semisimple subvarieties. In [15] an equivalence is exhibited between the category of semi-Heyting algebras and the category of centered semi-Nelson algebras, extending Cignoli's result that the categories of Heyting algebras and centered Nelson algebras are equivalent.

The papers that deal with the above problem logically include [10], [11], [13] and [16]. In [10] the authors introduce a Gentzen style sequent calculus $\mathcal{LSJ}$ for the semi-intuitionistic logic. The advantage of this presentation of the logic is that they prove a cut-elimination theorem for $\mathcal{LSJ}$ that allows them to check the decidability of the logic. As a direct consequence, they also obtain the decidability of the equational theory of semi-Heyting algebras. In [16], a propositional calculus called "semi-intuitionistic logic with strong negation" is introduced and proved to be complete with respect to the variety of semi-Nelson algebras. It has intuitionstic logic with strong negation as an axiomatic extension.

The present paper is an addition to the above-mentioned papers. In the quest for finding new varieties of semi-Heyting algebras, we systematically investigate, in this paper, the identities of associative type.

1.1. Identities of Associative Type

A look at the associative law would reveal at least the following characteristics:

(1) Length of the left side term = length of the right side term = 3,
(2) The number of distinct variables on the left = the number of distinct variables on right = the number of occurrences of variables on either side,
(3) The order of the variables on the left side is the same as the order of the variables on the right side,
(4) The bracketings used in the left side term and in the right side term are different from each other.

One way to generalize the associative law is to relax (3) and second half of (1), while keeping (2), (4) and the first half of (1). So, we are led to the following definition.

Definition 1.2. *An identity of associative type of length n is an identity of the form $p \approx q$ of length n such that*

(a) *each of p and q contains the same n (an integer ≥ 3) distinct variables,*

(b) *p and q are terms obtained by distinct bracketings of a permutation of the n variables.*

The above definition is taken from [14]. We do not know whether the notion of "identities of associative type of length n" in such a generality as given above has occurred in the literature earlier. However, we do know that specific instances of the identities of associative type have already appeared in the literature. We mention a few examples below, using $\cdot$ for the binary operation instead of $\to$. (The interested reader may refer to [14] for more such examples.)

- The identity $x\cdot(y\cdot z) \approx (z\cdot x)\cdot y$ was considered in [42] by Suschkewitsch (see also [40, Theorem 11.5]).
- Abbott [1] uses the identity $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$ as one of the defining identities in his definition of implication algebras.
- The identities $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$, $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$, and $x \cdot (y \cdot z) \approx (z \cdot x) \cdot y$ were investigated for quasigroups by Hossuzú in [23].
- The identity $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$ is investigated by Pushkashu in [30].
- The identities $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$ and $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$ have appeared in [26] of Kazim and Naseeruddin.

The following problem was first mentioned in [14].

PROBLEM: Let $\mathcal{V}$ be a given variety of algebras (whose language includes a binary operation symbol, say, '$\to$'). Investigate the mutual relationships among the subvarieties of $\mathcal{V}$, each of which is defined by a single identity of associative type of length n.

We will now consider the above problem for the variety $\mathcal{SH}$. We begin a systematic analysis of the relationships among the identities of associative type of length 3 relative to the variety $\mathcal{SH}$. For reader's convenience we repeat the special case of Definition 1.2, when $n = 3$.

DEFINITION 1.3. *An identity $p \approx q$, in the groupoid language $\langle \to \rangle$, is called an identity of associative type of length* 3 *if p and q have exactly* 3 *(distinct) variables, say x,y,z, and these variables are grouped according to one of the following two ways of grouping:*

(a) $o \to (o \to o)$ (b) $(o \to o) \to o$.

A subvariety $\mathcal{V}$ of $\mathcal{SH}$ is called a subvariety of associative type of length 3 if it is defined by a single identity of associative type of lenth 3.

In the rest of the paper, we refer to an "identity of associative type of length 3" and a variety of associative type of lenth 3 as simply an identity of associative type and a variety of associative type, respectively.

We wish to determine distinct subvarieties of associative type and their mutual relationships, as well as their relationships with other known subvarieties of $\mathcal{SH}$.

Our main theorem says that there are 3 such subvarieties of $\mathcal{SH}$ that are distinct from each other and describes explicitly, by a Hasse diagram, the poset formed by them.

2. Preliminaries

We refer to [9] for concepts and results in universal algebra and to [6] for distributive lattices.

In this section, we recall some known subvarieties of $\mathcal{SH}$ and also recall some results that will be useful in later sections.

Lemma 2.1. [38] *Let* $\mathbf{A} \in \mathcal{SH}$ *and* $a, b \in A$.

(a) *If* $a \to b = 1$ *then* $a \leq b$.
(b) *If* $a \leq b$ *then* $a \leq a \to b$.
(c) $1 \to a = a$.

Theorem 2.2. [5, Theorem 1.8] *Let* $\mathbf{A} \in \mathcal{SH}$. *The following conditions are equivalent:*

(1) $\mathbf{A} \models x \to y \approx y \to x$,
(2) $\mathbf{A} \models x \to 1 \approx x$,
(3) $\mathbf{A} \models y \wedge (x \to y) \approx x \wedge y$.

The varieties of associative semi-Heyting algebras and commutive semi-Heyting algebras, denoted, respectively, by $\mathcal{A}$ and $\mathcal{C}$, are defined (see [38]), relative to $\mathcal{SH}$, by

(A) $x \to (y \to z) \approx (x \to y) \to z$,
(C) $x \to y \approx y \to x$.

In [5] it is proved that the identity $x \to (y \to z) \approx (x \to y) \to z$ characterizes the variety $\mathcal{V}(\mathbf{L}_2)$, where $\mathcal{V}(\mathbf{L}_2)$ is the variety generated by $\mathbf{L}_2$ (see bellow). The variety $\mathcal{C}$ has the interesting property that $0 \to 1 = 0$, quite opposite to the behavior of Heyting algebras. The variety $\mathcal{C}$ is, we think, also of interest from the philosophical point of view.

Theorem 2.3. [5, Theorem 1.12] $\mathcal{A} = \mathcal{V}(\mathbf{L}_2)$.

Lemma 2.4. [5, Lemma 1.10] *If* $\mathbf{A} \in \mathcal{A}$, *then* $\mathbf{A}$ *satisfies* $x \to 1 \approx x$.

The following examples of semi-Heyting algebras will be useful in the rest of the paper.

- Algebras defined on a 2-element chain $\{0, 1\}$ with $0 < 1$:

$\mathbf{L}_1$

$\to$:	0	1
0	1	1
1	0	1

$\mathbf{L}_2$

$\to$:	0	1
0	1	0
1	0	1

- Algebras defined on a 3-element chain $\{0, a, 1\}$, with $0 < a < 1$:

$\mathbf{L}_3$

$\to$:	0	1	a
0	1	a	1
1	0	1	a
a	0	1	1

$\mathbf{L}_4$

$\to$:	0	1	a
0	1	0	0
1	0	1	a
a	0	a	1

$\mathbf{L}_5$

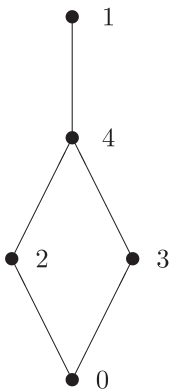
$\to$:	0	1	a
0	1	1	1
1	0	1	a
a	0	a	1

- Algebras defined on a 4-element chain $\{0, a, b, 1\}$, with $0 < a < b < 1$:

$\mathbf{L}_6$

$\to$:	0	1	b	a
0	1	1	1	b
1	0	1	b	a
b	0	1	1	a
a	0	1	1	1

- A 5-element algebra with the following lattice reduct and the $\to$ operation:



$\mathbf{L}_7$

$\to$:	0	1	2	3	4
0	1	0	3	2	0
1	0	1	2	3	4
2	3	2	1	0	2
3	2	3	0	1	3
4	0	1	2	3	1

3. Identities of Associative Type

We now turn our attention to identities of associative type of length 3. Recall that such an identity will contain three distinct variables that occur in any order and that are grouped in one of the two (obvious) ways. The following identities play a crucial role in the sequel.

Let Σ denote the set consisting of the following 14 identities of associative type in the binary language $\langle\to\rangle$):

($A1$) $x \to (y \to z) \approx (x \to y) \to z$, (Associative law,)

($A2$) $x \to (y \to z) \approx x \to (z \to y)$,

($A3$) $x \to (y \to z) \approx (x \to z) \to y$,

($A4$) $x \to (y \to z) \approx y \to (x \to z)$,

($A5$) $x \to (y \to z) \approx (y \to x) \to z$,

($A6$) $x \to (y \to z) \approx y \to (z \to x)$,

($A7$) $x \to (y \to z) \approx (y \to z) \to x$,

($A8$) $x \to (y \to z) \approx (z \to x) \to y$,

($A9$) $x \to (y \to z) \approx z \to (y \to x)$,

($A10$) $x \to (y \to z) \approx (z \to y) \to x$,

($A11$) $(x \to y) \to z \approx (x \to z) \to y$,

($A12$) $(x \to y) \to z \approx (y \to x) \to z$,

($A13$) $(x \to y) \to z \approx (y \to z) \to x$,

($A14$) $(x \to y) \to z \approx (z \to y) \to x$.

We will denote by $\mathcal{A}_i$ the subvariety of $\mathcal{SH}$ defined by the identity (Ai), for $1 \leq i \leq 14$. Such varieties will be referred to as subvarieties of $\mathcal{SH}$ of associative type. Sometimes we will use (A) for (A_1) and $\mathcal{A}$ for $\mathcal{A}_1$.

The following proposition, whose proof is routine, is crucial for the rest of the paper.

Proposition 3.1. [14] *Let $\mathcal{G}$ be the variety of all groupoids of type $\{\to\}$ and Let $\mathcal{V}$ denote the subvariety of $\mathcal{G}$ defined by a single identity of associative type. Then $\mathcal{V} = \mathcal{A}_i$, for some $i \in \{1, 2, \cdots, 14\}$.*

Our goal, in this paper, is to determine the distinct subvarieties of $\mathcal{SH}$ associative type and to describe the poset of subvarieties of $\mathcal{SH}$. It suffices to concentrate on the varieties defined by identities (A1)-(A14), in view of the above proposition.

3.1. Properties of subvarieties of $\mathcal{SH}$ of Associative type

In this section we present properties of several subvarieties of $\mathcal{SH}$ which will play a crucial role in our analysis of the identities of associative type relative to $\mathcal{SH}$.

The proof of the following lemma is straightforward.

Lemma 3.2. *If $\mathbf{A} \in \mathcal{SH}$ satisfies the identities $x \to y \approx y \to x$ and $x \to (y \to z) \approx (x \to y) \to z$ then $\mathbf{A} \in \mathcal{A}_j$ for all $j \in \{1, 2, \ldots, 14\}$.*

Lemma 3.3. *If $\mathbf{A} \in \mathcal{A}_j$ with $j \in \{1, 5, 8, 10, 12, 13, 14\}$, then $\mathbf{A} \models x \to 1 \approx x$.*

Proof: Let $a \in A$.

- $j = 1$: This case follows from Lemma 2.4.
- $j = 5$:

$$\begin{array}{rcll} a \to 1 & = & a \to (a \to a) & \text{by (SH4)} \\ & = & (a \to a) \to a & \text{by (A5)} \\ & = & 1 \to a & \text{by (SH4)} \\ & = & a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 8$:

$$\begin{array}{rcll} a & = & (1 \to 1) \to a & \text{by Lemma 2.1 (c)} \\ & = & 1 \to (a \to 1) & \text{by (A8)} \\ & = & a \to 1 & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 10$:

$$\begin{array}{rcll} a \to 1 & = & a \to (1 \to 1) & \text{by Lemma 2.1 (c)} \\ & = & (1 \to 1) \to a & \text{by (A10)} \\ & = & 1 \to a & \text{by Lemma 2.1 (c)} \\ & = & a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 12$: By Lemma 2.1 (b), $a \leq a \to 1$. Also note that

$$\begin{array}{rcll} (a \to 1) \to a & = & (1 \to a) \to a & \text{by (A12)} \\ & = & a \to a & \text{by Lemma 2.1 (c)} \\ & = & 1 & \text{by (SH4).} \end{array}$$

  Hence, using Lemma 2.1 (a), $a \to 1 \leq a$.
- $j = 13$:

$$\begin{array}{rcll} a & = & 1 \to a & \text{by Lemma 2.1 (c)} \\ & = & (1 \to 1) \to a & \text{by Lemma 2.1 (c)} \\ & = & (1 \to a) \to 1 & \text{by (A13)} \\ & = & a \to 1 & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 14$: By Lemma 2.1 (b), $a \leq a \to 1$. Also note that

$$\begin{array}{rcll} a \to 1 & = & (a \to 1) \wedge 1 & \\ & = & (a \to 1) \wedge ((a \to 1) \to 1) & \text{by (SH2)} \\ & = & (a \to 1) \wedge ((1 \to 1) \to a) & \text{by (A14)} \\ & = & (a \to 1) \wedge a & \text{by Lemma 2.1 (c).} \end{array}$$

  Therefore, $a \to 1 \leq a$,

proving the lemma. □

Lemma 3.4. *If* $\mathbf{A} \in \mathcal{A}_j$ *for* $1 \leq j \leq 14$ *and* $j \neq 4$ *then* $\mathbf{A} \in \mathcal{C}$.

Proof: Let $a, b \in A$.

- If $j \in \{1, 5, 8, 10, 12, 13, 14\}$ the result follows from Theorem 2.2 and Lemma 3.3.
- $j = 2$:

$$\begin{array}{rcll} a \to b & = & 1 \to (a \to b) & \text{by Lemma 2.1 (c)} \\ & = & 1 \to (b \to a) & \text{by (A2)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 3$:

$$\begin{array}{rcll} a \to b & = & 1 \to (a \to b) & \text{by Lemma 2.1 (c)} \\ & = & (1 \to b) \to a & \text{by (A3)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 6$:

$$\begin{array}{rcll} a \to b & = & a \to (1 \to b) & \text{by Lemma 2.1 (c)} \\ & = & 1 \to (b \to a) & \text{by (A6)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 7$:

$$\begin{array}{rcll} a \to b & = & a \to (1 \to b) & \text{by Lemma 2.1 (c)} \\ & = & (1 \to b) \to a & \text{by (A7)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 9$:

$$\begin{array}{rcll} a \to b & = & a \to (1 \to b) & \text{by Lemma 2.1 (c)} \\ & = & b \to (1 \to a) & \text{by (A9)} \\ & = & b \to a & \text{by Lemma 2.1 (c).} \end{array}$$

- $j = 11$:

$$\begin{array}{rcll} a \to b & = & (1 \to a) \to b & \text{by Lemma 2.1 (c)} \\ & = & (1 \to b) \to a & \text{by (A11)} \\ & = & b \to a & \text{by Lemma 2.1 (c),} \end{array}$$

proving the lemma. □

Lemma 3.5. *If* $\mathbf{A} \in \mathcal{A}_j$ *with* $j \in \{3, 5, 6, 8, 9, 11, 13, 14\}$ *then* $\mathbf{A} \in \mathcal{A}$.

Proof: If $\mathbf{A} \in \mathcal{A}_3$, then

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & (a \to b) \to c & \text{by (A3).} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_5$, then

$$\begin{array}{rcll} a \to (b \to c) & = & (b \to a) \to c & \text{by (A5)} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_6$, then

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & c \to (b \to a) & \text{by (A6)} \\ & = & c \to (a \to b) & \text{by Lemma 3.4} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_8$, then

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & (b \to a) \to c & \text{by (A8)} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_9$, then

$$\begin{array}{rcll} a \to (b \to c) & = & c \to (b \to a) & \text{by (A9)} \\ & = & c \to (a \to b) & \text{by Lemma 3.4} \\ & = & (a \to b) \to c & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_{11}$, then

$$\begin{array}{rcll} (a \to b) \to c & = & (b \to a) \to c & \text{by Lemma 3.4} \\ & = & (b \to c) \to a & \text{by (A11)} \\ & = & a \to (b \to c) & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_{13}$, then

$$\begin{array}{rcll} (a \to b) \to c & = & (b \to c) \to a & \text{by (A13)} \\ & = & a \to (b \to c) & \text{by Lemma 3.4.} \end{array}$$

If $\mathbf{A} \in \mathcal{A}_{14}$,

$$\begin{array}{rcll} (a \to b) \to c & = & (c \to b) \to a & \text{by (A14)} \\ & = & a \to (c \to b) & \text{by Lemma 3.4} \\ & = & a \to (b \to c) & \text{by Lemma 3.4.} \end{array}$$

The lemma is now proved. □

THEOREM 3.6. $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_3 = \mathcal{A}_5 = \mathcal{A}_6 = \mathcal{A}_8 = \mathcal{A}_9 = \mathcal{A}_{11} = \mathcal{A}_{13} = \mathcal{A}_{14}$.

PROOF: The proof follows directly from Lemma 3.2, Lemma 3.4 and Lemma 3.5. □

The proof of the following lemma is straightforward.

LEMMA 3.7. *If* $\mathbf{A} \in \mathcal{C}$ *then* $\mathbf{A} \in \mathcal{A}_j$ *with* $j \in \{2, 7, 10, 12\}$.

THEOREM 3.8. $\mathcal{C} = \mathcal{A}_2 = \mathcal{A}_7 = \mathcal{A}_{10} = \mathcal{A}_{12}$.

PROOF: This result is easy to check by using Lemma 3.4 and Lemma 3.7. □

4. Main Theorem

We are now ready to present the main theorem of this paper.

Theorem 4.1. *We have*

(a) *The following are the* 3 *subvarieties of* $\mathcal{SH}$ *of associative type that are distinct from each other:*

$$\mathcal{A}, \mathcal{C} \text{ and } \mathcal{A}_4.$$

(b) *They satisfy the following relationships:*

1. $\mathcal{T} \subset \mathcal{A} \subset \mathcal{C} \subset \mathcal{SH}$, *where* $\mathcal{T}$ *denotes the trivial variety,*
2. $\mathcal{A} \subset \mathcal{A}_4 \subset \mathcal{SH}$,
3. $\mathcal{C} \,||\, \mathcal{A}_4$,
4. $\mathcal{H} \subset \mathcal{A}_4$.

Proof: Observe that, in view of Theorem 3.6 and Theorem 3.8 we can conclude that each of the 14 subvarieties of associative type of $\mathcal{SH}$ is equal to one of the following varieties:

$$\mathcal{A}, \mathcal{C} \text{ and } \mathcal{A}_4.$$

We first wish to prove (b). By Lemma 3.2, $\mathcal{A} \subseteq \mathcal{A}_4$. The algebra $\mathbf{L}_1$ shows that the inclusion is proper using $x = 0, y = 0, z = 0$ in the identity (A). It is clear that $\mathcal{A}_4 \subseteq \mathcal{SH}$. The algebra $\mathbf{L}_3$ shows that the inclusion is proper using $x = 0, y = a, z = 1$ in the identity (A4) proving b2.

Let us check item (b3). The algebra $\mathbf{L}_4$ shows that $\mathcal{C} \nsubseteq \mathcal{A}_4$ using $x = 0, y = a, z = 0$ in the identity (A4). The algebra $\mathbf{L}_1$ shows that $\mathcal{A}_4 \nsubseteq \mathcal{C}$ using $x = 0, y = 1$ in the identity (C).

The condition $\mathcal{H} \subseteq \mathcal{A}_4$ is clear since if $\mathbf{A} \in \mathcal{H}$ and $a, b \in A$ then $a \to (b \to c) = (a \wedge b) \to c = (b \wedge a) \to c = b \to (a \to c)$, the latter being well-known. Let us consider the algebra $\mathbf{L}_2$. It shows that $\mathcal{A}_4 \nsubseteq \mathcal{H}$ using $x = 0, y = 1$ in the identity (H). Then the proof of item (b4) is done.

The inclusion $\mathcal{A} \subseteq \mathcal{C}$ follows from Lemma 2.4 and Theorem 2.2. The algebra $\mathbf{L}_4$ shows that the inclusion is proper since (A) fails in it at $x = 0$, $y = 0$, $z = a$. The proof of the theorem is now complete since (a) is an immediate consequence of (b). $\square$

Further relationship between $\mathcal{C}$, $\mathcal{A}_4$ and $\mathcal{A}$ is given in the following theorem.

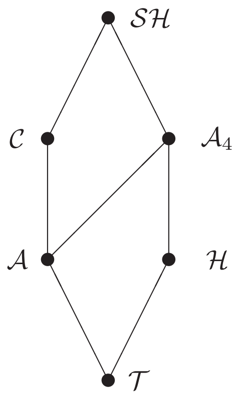
THEOREM 4.2. $\mathcal{C} \cap \mathcal{A}_4 = \mathcal{A}$.

PROOF: Let $\mathbf{A} \in \mathcal{A}_4$ and $a, b, c \in A$. Notice that

$$\begin{array}{rcll} a \to (b \to c) & = & a \to (c \to b) & \text{by (C)} \\ & = & c \to (a \to b) & \text{by (A4)} \\ & = & (a \to b) \to c & \text{by (C)} \end{array}$$

Hence $\mathcal{C} \cap \mathcal{A}_4 \subseteq \mathcal{A}$. In view of Theorem 4.1, $\mathcal{C} \cap \mathcal{A}_4 = \mathcal{A}$. □

The Hasse diagram of the poset (in fact, $\wedge$-semilattice) of subvarieties of $\mathcal{SH}$ of associative type, together with $\mathcal{SH}$, $\mathcal{T}$ and $\mathcal{H}$, is given below.



Next, we will study some relationships of this interesting new subvariety $\mathcal{A}_4$ with some of the other earlier known subvarieties of $\mathcal{SH}$.

In [38, Definition 8.1], Sankappanavar introduced the following subvarieties of $\mathcal{SH}$ by providing defining identities relative to $\mathcal{SH}$ for each of them (where * is the operation of pseudocomplementation):

Subvariety	Defining identity within $\mathcal{SH}$
$\mathcal{FTT}$ (**F**alse implies **T**rue is **T**rue)	$0 \to 1 \approx 1$
$\mathcal{FTD}$ (**F**alse implies **T**rue is **D**ense)	$(0 \to 1)^* \approx 0$
$\mathcal{QH}$ (Quasi-Heyting algebras)	$y \leq x \to y$
$\mathcal{SH}^S$ (Stone semi-Heyting algebras)	$x^* \vee x^{**} \approx 1$
$\mathcal{FTF}$ (**F**alse implies **T**rue is **F**alse)	$0 \to 1 \approx 0$

Theorem 4.3. *The variety $\mathcal{A}_4$ is incomparable to each of the subvarieties*

$$\mathcal{FTT}, \mathcal{FTD}, \mathcal{QH}, \mathcal{SH}^S \textit{ and } \mathcal{FTF}.$$

Proof: The algebra $\mathbf{L}_2$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{FTT}$, $\mathcal{A}_4 \not\subseteq \mathcal{FTD}$ and $\mathcal{A}_4 \not\subseteq \mathcal{QH}$.
The algebra $\mathbf{L}_3$ shows that $\mathcal{FTD} \not\subseteq \mathcal{A}_4$ and $\mathcal{SH}^S \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = a$, $z = 1$.
The algebra $\mathbf{L}_5$ shows that $\mathcal{FTT} \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = a$, $z = 0$.
The algebra $\mathbf{L}_6$ shows that $\mathcal{QH} \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = b$, $z = a$.
The algebra $\mathbf{L}_7$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{SH}^S$ with $x = 2$.
The algebra $\mathbf{L}_1$ shows that $\mathcal{A}_4 \not\subseteq \mathcal{FTF}$.
The algebra $\mathbf{L}_4$ shows that $\mathcal{FTF} \not\subseteq \mathcal{A}_4$ using $x = 0$, $y = a$, $z = 0$. □

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FULL CUT ELIMINATION AND INTERPOLATION FOR INTUITIONISTIC LOGIC WITH EXISTENCE PREDICATE[1]

Abstract

In previous work by Baaz and Iemhoff, a Gentzen calculus for intuitionistic logic with existence predicate is presented that satisfies partial cut elimination and Craig's interpolation property; it is also conjectured that interpolation fails for the implication-free fragment. In this paper an equivalent calculus is introduced that satisfies full cut elimination and allows a direct proof of interpolation via Maehara's lemma. In this way, it is possible to obtain much simpler interpolants and to better understand and (partly) overcome the failure of interpolation for the implication-free fragment.

Keywords: intuitionistic logic, existence predicate, sequent calculi, cut elimination, interpolation, Maehara's lemma.

1. Introduction

In [9] Scott introduced intuitionistic logic with existence predicate (ILE) to make explicit the existential assumptions in an intuitionistic theory, that is to indicate which objects exist. In ILE the language of first-order intuitionistic logic is extended with an existence predicate $\mathcal{E}$ and $\mathcal{E}t$ is interpreted as saying that (the object denoted by) t exists. In [9] ILE is presented as an Hilbert system extending the standard axiomatization of intuitionistic propositional logic with new rules for quantifiers and axioms for $\mathcal{E}$. In [1, 2] Baaz and Iemhoff introduced Gentzen systems equivalent to

[1] Thanks to an anonymous referee for many helpful comments.

Scott's axiomatization and showed that they satisfy partial cut elimination as well as Craig's interpolation property. The proof of the latter, however, is indirect in the sense that interpolation is proved not for the original calculus, but in an equivalent one where structural rules are not admissible. Moreover, it is conjectured that interpolation fails for the implication-free fragment.

The aim of this paper is to improve on [1, 2]. We introduce an alternative Gentzen calculus for ILE that satisfies full cut elimination and in which Craig's interpolation property can be proved via Maehara's lemma using exclusively the rules of the calculus. The advantage is that our proof is direct and delivers much simpler interpolants. This helps to improve on the conjecture of Baaz and Iemhoff [1, §5.1] that their calculi do not interpolate for the fragment of the language without implication (nor $\perp$). Specifically, we prove (Proposition 12) that although the interpolants for the implication-free fragment may contain implications, the antecedent of such implications is always an existence atom—and not an arbitrary formula as in [1]. Moreover, we are able to calculate a precise upper bound to the number of such implications. Finally, we prove (Proposition 13) that under an arguably plausible assumption our calculi interpolate for the fragment of the language without implication (nor $\perp$).

The paper also improves on other works in the area of interpolation for first-order theories, especially [4] where it is shown how to extend interpolation to a class of first-order theories, called singular geometric, where individual constants do not occur. Since in ILE, constants do occur in existential axioms $\mathcal{E}t$, it is clear that the proof of interpolation presented here indicates a way to generalize the results of [4].

2. The calculi LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$

To make the paper self-contained we recall basic definitions and results from [1, 2]. Let $\mathcal{L}'$ be a first-order language without identity and let LJE be a Gentzen calculus consisting of the initial sequents and rules given in Table 1.

In the rules $R\forall$ and $L\exists$, the variable y is *eigenvariable*, i.e. it does not occur free in the conclusion of the rule.[2] Moreover, let $\Sigma_{\mathcal{L}}$ be the set of all

[2] In [1, 2] the substitution occurs in the conclusion of the rules $R\forall$ and $L\exists$ instead of in their premises, but this difference is immaterial.

Table 1. The calculus LJE

$$P, \Gamma \Rightarrow P \qquad \frac{}{\bot, \Gamma \Rightarrow C}\ L\bot$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R\vee_1 \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R\vee_2$$

$$\frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\ L\!\to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ R\!\to$$

$$\frac{\forall x A, \Gamma \Rightarrow \mathcal{E}t \quad A[\tfrac{t}{x}], \forall x A, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ L\forall^* \qquad \frac{\mathcal{E}y, \Gamma \Rightarrow A[\tfrac{y}{x}]}{\Gamma \Rightarrow \forall x A}\ R\forall$$

$$\frac{\mathcal{E}y, A[\tfrac{y}{x}], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ L\exists \qquad \frac{\Gamma \Rightarrow \mathcal{E}t \quad \Gamma \Rightarrow A[\tfrac{t}{x}]}{\Gamma \Rightarrow \exists x A}\ R\exists^*$$

sequents $\Gamma \Rightarrow \mathcal{E}t$, where t is a term of a language $\mathcal{L} \subseteq \mathcal{L}'$ which contains no variable and at least one constant (hence t is a constant). Since $\mathcal{L}$ contains at least one constant, $\Sigma_{\mathcal{L}}$ is not empty; and since $\mathcal{L}$ contains no variable, all sequents in $\Sigma_{\mathcal{L}}$ are closed. Consider now the calculus LJE($\Sigma_{\mathcal{L}}$) obtained from LJE by adding all the sequents in the language $\mathcal{L}'$ that are LJE-derivable from $\Sigma_{\mathcal{L}}$. In other words, LJE($\Sigma_{\mathcal{L}}$) is obtained from LJE by adding "axiomatic sequents" $\Gamma \Rightarrow \mathcal{E}t$.

In [2] it is shown that LJE and LJE($\Sigma_{\mathcal{L}}$) are equivalent to the standard axiomatizations IQCE and IQCE$^+$ of ILE due to Scott [9] and Beeson [3], respectively. In [2] it is also shown that in LJE and LJE($\Sigma_{\mathcal{L}}$) weakening and contraction

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ Wkn \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ Ctr$$

are height-preserving admissible (Lemma 4.3 and 4.4). However, the presence of axiomatic sequents in LJE($\Sigma_{\mathcal{L}}$) impairs cut elimination: although LJE is fully cut-free, LJE($\Sigma_{\mathcal{L}}$) only allows a partial cut elimination. Specifically, the cut rule with $\mathcal{E}$-formulas principal (with t term in $\mathcal{L}$)

$$\frac{\Gamma \Rightarrow \mathcal{E}t \quad \mathcal{E}t, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ {\scriptstyle Cut}$$

is not eliminable (Theorem 4.6).

Interpolation in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ is investigated in [1] where it is shown LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ have Craig's interpolation property (Corollary 2). The proof is indirect in the sense that interpolation is not proved for $\mathsf{LJE}(\Sigma_{\mathcal{L}})$, but for an equivalent system where weakening and contraction are not admissible and cuts on axiomatic sequents are replaced by instances of weakening (Theorem 4). Towards the end of [1] it is conjectured that interpolation fails for the implication-free fragment of LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ because the proof of interpolation in the case of the rule $L\forall$ only deliver implicative interpolants, i.e. formulas of the form $A \to B$.

3. The cut-free calculi $\mathsf{G3ie}$ and $\mathsf{G3ie}^{\mathcal{T}}$

To overcome the limitation of partial cut elimination, we consider a calculus equivalent to $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ where the rules for quantifiers are aptly modified and each existential sequent $\Gamma \Rightarrow \mathcal{E}t$ in $\Sigma_{\mathcal{L}}$ is replaced by an inference rule. The modification of the quantifier rules has been largely inspired by the modal rules of labelled sequent calculi of [7] and consists into replacing $L\forall^*$ and $R\exists^*$ of $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ by the following rules:

$$\frac{A[{}^{t}_{x}], \forall x A, \mathcal{E}t, \Gamma \Rightarrow C}{\forall x A, \mathcal{E}t, \Gamma \Rightarrow C}\ {\scriptstyle L\forall} \qquad \frac{\mathcal{E}t, \Gamma \Rightarrow A[{}^{t}_{x}]}{\mathcal{E}t, \Gamma \Rightarrow \exists x A}\ {\scriptstyle R\exists}$$

Let $\mathsf{G3ie}$ be the result of replacing $L\forall^*$ and $R\exists^*$ of $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ with $L\forall$ and $R\exists$, respectively. The key feature of $L\forall$ and $R\exists$ in $\mathsf{G3ie}$ is that existential atoms may be active (principal) only in the left-hand side of the sequent arrow $\Rightarrow$.

Next, instead of axiomatic sequents $\Gamma \Rightarrow \mathcal{E}t$, we consider (extensions of $\mathsf{G3ie}$ with) existential rules $Ex(t)$ of the form

$$\frac{\mathcal{E}t, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ {\scriptstyle Ex(t)}$$

where t is a term of $\mathcal{L}$. We agree that if $\mathcal{T} = \{t : \Gamma \Rightarrow \mathcal{E}t \in \Sigma_{\mathcal{L}}\}$, then $\mathsf{G3ie}^{\mathcal{T}}$ is the extension of $\mathsf{G3ie}$ with a rule $Ex(t)$ for each $t \in \mathcal{T}$.

Now we show that $\mathsf{G3ie}^{\mathcal{T}}$ (as well as $\mathsf{G3ie}$) satisfies full cut elimination. As usual, we shall assume that derivations satisfy the pure-variable convention: in a derivation no variable occurs both free and bound and the *eigenvariables* are pairwise disjoint. Next, we begin with some preparatory lemmas. First, height-preserving admissibility of substitution in $\mathsf{G3ie}^{\mathcal{T}}$ (we shall omit to specify $\mathsf{G3ie}^{\mathcal{T}}$, unless it is necessary).

LEMMA 1 (substitution). *If* $\vdash^h \Gamma \Rightarrow C$ *and* t *is free for* x *in* Γ, C *then* $\vdash^h \Gamma[\frac{t}{x}] \Rightarrow C[\frac{t}{x}]$.

PROOF: By induction on height h of the derivation of $\Gamma \Rightarrow C$. If $h = 0$ or $h = n + 1$ and the last rule instance R is by a propositional rule, see [8, Theorem 4.1.2]. If ($h = n + 1$ and) R is $L\forall$ and $x \equiv y$ then the claim holds since the substitution $[\frac{t}{x}]$ is vacuous. Otherwise, if $x \not\equiv y$ then we apply IH on the premise of $L\forall$ and then $L\forall$ again. The case of $R\exists$ is similar. If R is $R\forall$ with conclusion $\Gamma \Rightarrow \forall y A$, then we take the premise $\mathcal{E}z, \Gamma \Rightarrow A[\frac{z}{y}]$ (with z *eigenvariable*) and we apply IH so as to replace z with a new variable u and obtain $\vdash^n \mathcal{E}u, \Gamma \Rightarrow A[\frac{u}{y}]$. By IH again and $R\forall$ we conclude $\Gamma[\frac{t}{x}] \Rightarrow (\forall y A)[\frac{t}{x}]$. The case of $L\exists$ is similar. □

Next is height-preserving admissibility of weakening.

LEMMA 2 (Weakening). *If* $\vdash^h \Gamma \Rightarrow C$ *then* $\vdash^h A, \Gamma \Rightarrow C$.

PROOF: By induction on h. If $h = 0$ or $h = n+1$ and the last rule instance R is by a propositional rule, see [8, Theorem 4.2.2]. If ($h = n + 1$ and) R is a quantifier rule without variable condition, then the claim holds by IH and R. If R is a quantifier rule with *eigenvariable* then use Lemma 1, IH and R. Finally, if R is $Ex(t)$, then its premise is $\mathcal{E}t, \Gamma \Rightarrow C$ and the conclusion $A, \Gamma \Rightarrow C$ is obtained by applying IH on the premise of $Ex(t)$ and then $Ex(t)$ (since t is a constant, we know $\mathcal{E}t$ is not affected by the substitution). □

To prove that contraction is height-preserving admissible, we need height-preserving invertibility of some rules.

LEMMA 3 (Inversion). *All rules, except* $R\lor$, $L\rightarrow$ *and* $R\exists$, *are height-preserving invertible. However,* $L\rightarrow$ *is height-preserving invertible with respect to its right premise.*

PROOF: For height-preserving invertibility of the propositional rules, see [8, Theorem 2.3.5]. The height-preserving invertibility of $L\forall$ and $Ex(t)$ follows by height-preserving admissibility of weakening (Lemma 2), whereas the case of $L\exists$ is as in [2, Lemma 4.4] and the case of $R\forall$ is similar and hence left to the reader. $\square$

Now we can prove height-preserving admissibility of contraction.

Lemma 4 (Contraction). *If* $\vdash^h A, A, \Gamma \Rightarrow C$ *then* $\vdash^h A, \Gamma \Rightarrow C$.

Proof: By induction on h. If $h = 0$ or $h = n + 1$ and the last rule instance R is by a propositional rule, see [8, Theorem 2.4.1]. If R is a height-preserving invertible quantifier rule, then the claim holds by IH and Lemma 3. If R is $R\exists$, then A cannot be principal in it and $C \equiv \exists x B$. We need to consider two cases: either $A \equiv \mathcal{E}t$ or $\Gamma \equiv \mathcal{E}t, \Gamma'$. In the first case, the premise of $R\exists$ is $\mathcal{E}t, \mathcal{E}t, \Gamma \Rightarrow B[{}^t_x]$ and the sequent $\mathcal{E}t, \Gamma \Rightarrow \exists x B$ is obtained by applying IH on $\mathcal{E}t$ and then $R\exists$ again. In the second case, the premise of $R\exists$ is $A, A, \mathcal{E}t, \Gamma' \Rightarrow B[{}^t_x]$ and the sequent $A, \mathcal{E}t, \Gamma' \Rightarrow \exists x B$ is obtained similarly. Finally, if R is $Ex(t)$, then A cannot be principal in it and the premise of $Ex(t)$ is $A, A, \mathcal{E}t, \Gamma \Rightarrow C$. Thus, $A, \mathcal{E}t, \Gamma \Rightarrow C$ is obtained by applying IH and then $Ex(t)$. $\square$

We are now ready to prove (full) cut elimination.

Theorem 5 (Cut). *If* $\vdash \Gamma \Rightarrow A$ *and* $\vdash A, \Delta \Rightarrow C$ *then* $\vdash \Gamma, \Delta \Rightarrow C$.

Proof: The proof is by induction on the weight of the cut formula A with a sub-induction on the sum of heights of derivation of the two premises (cut-height, for short). If at least one the two premises of cut is initial or concluded by $L\bot$, then the proof is the same as in [8, Theorem 2.4.3]. Otherwise, if none of the premises of cut is initial or concluded by $L\bot$, we consider three cases: (i) A is not principal in $\Gamma \Rightarrow A$; (ii) A is principal in $\Gamma \Rightarrow A$ only; (iii) A is principal in $\Gamma \Rightarrow A$ and $A, \Delta \Rightarrow C$.

In case (i), we proceed by cases according to the rule R concluding $\Gamma \Rightarrow A$. Since A is not principal by hypothesis, $\Gamma \Rightarrow A$ can only be concluded by a left rule or $Ex(t)$. We consider only the case of (ia) $L\forall$, (ib) $L\exists$ and (ic) $Ex(t)$, the rest being the same as in [8, Theorem 2.4.3]. If (ia) $\Gamma \Rightarrow A$ is concluded by $L\forall$, then $\Gamma \equiv \mathcal{E}t, \forall x B, \Gamma'$ and we have:

$$\dfrac{\dfrac{B[{}^t_x], \mathcal{E}t, \forall x B, \Gamma' \Rightarrow A}{\mathcal{E}t, \forall x B, \Gamma' \Rightarrow A}\,L\forall \qquad A, \Delta \Rightarrow C}{\mathcal{E}t, \forall x B, \Gamma', \Delta \Rightarrow C}\,Cut$$

We apply IH on the premise of $L\forall$ and then $L\forall$ as follows.

$$\cfrac{\cfrac{B[\genfrac{}{}{0pt}{}{t}{x}], Et, \forall x B, \Gamma' \Rightarrow A \quad A, \Delta \Rightarrow C}{B[\genfrac{}{}{0pt}{}{t}{x}], \mathcal{E}t, \forall x B, \Gamma', \Delta \Rightarrow C}\ \text{IH}}{\mathcal{E}t, \forall x B, \Gamma', \Delta \Rightarrow C}\ L\forall$$

If (ib) then $\Gamma \Rightarrow A$ is concluded by $L\exists$ then Γ is $\exists x B, \Gamma'$. In this case the procedure is similar to case (a), except that we need first to apply Lemma 1 on the premise $\mathcal{E}y, B[\genfrac{}{}{0pt}{}{y}{x}], \Gamma' \Rightarrow A$ of $L\exists$ so as to replace y with a new variable z.
Finally, if (ic) $\Gamma \Rightarrow A$ is concluded by $Ex(t)$ cut is "permuted upwards" as above.

In case (ii), we proceed by cases according to the rule R concluding $A, \Delta \Rightarrow C$. We consider here only the cases of the quantifier rules and $Ex(t)$. If R is $L\forall$ or $R\exists$ then we reason as in case (ia), whereas if R is $R\forall$ or $L\exists$ the reasoning is similar to (ib). Finally, If R is $Ex(t)$ we proceed as in (ic).

We now consider the case (iii). If the cut formula A is propositional, then see [8, Theorem 2.4.3]. If $A \equiv \forall x B$, then $\Delta \equiv \mathcal{E}t, \Delta'$ and we have

$$\cfrac{\cfrac{\mathcal{E}y, \Gamma \Rightarrow B[\genfrac{}{}{0pt}{}{y}{x}]}{\Gamma \Rightarrow \forall x B}\ R\forall \qquad \cfrac{B[\genfrac{}{}{0pt}{}{t}{x}], \forall x B, \mathcal{E}t, \Delta' \Rightarrow C}{\forall x B, \mathcal{E}t, \Delta' \Rightarrow C}\ L\forall}{\Gamma, \mathcal{E}t, \Delta' \Rightarrow C}\ \mathit{Cut}$$

where y is *eigenvariable* in $R\forall$. First, we apply height-preserving admissibility of substitution (Lemma 1) on the premise of $R\forall$ in order to replace y with t; thus, we obtain $\vdash^{n-1} \mathcal{E}t, \Gamma \Rightarrow B[\genfrac{}{}{0pt}{}{t}{x}]$, where n is the derivation height of the conclusion of $R\forall$. Then we apply IH twice and height-preserving admissibility of contraction (Lemma 4) as follows.

$$\cfrac{\cfrac{\cfrac{\mathcal{E}y, \Gamma \Rightarrow B[\genfrac{}{}{0pt}{}{y}{x}]}{\mathcal{E}t, \Gamma \Rightarrow B[\genfrac{}{}{0pt}{}{t}{x}]}\ [\genfrac{}{}{0pt}{}{t}{y}] \qquad \cfrac{\Gamma \Rightarrow \forall x B \quad B[\genfrac{}{}{0pt}{}{t}{x}], \forall x B, \mathcal{E}t, \Delta' \Rightarrow C}{\Gamma, B[\genfrac{}{}{0pt}{}{t}{x}], \mathcal{E}t, \Delta' \Rightarrow C}\ \text{IH}}{\mathcal{E}t, \Gamma, \Gamma, \mathcal{E}t, \Delta' \Rightarrow C}\ \text{IH}}{\Gamma, \mathcal{E}t, \Delta' \Rightarrow C}\ \mathit{Ctr}$$

The case in which the cut formula A is $\exists x B$ is similar. Notice the if $A \equiv \mathcal{E}t$, then such a formula cannot be both principal of a right rule and a left rule.
□

Thus, the calculus $\mathsf{G3ie}^{\mathcal{T}}$ satisfies full cut elimination. We now need to prove that it is equivalent to $\mathsf{LJE}(\Sigma_{\mathcal{L}})$.

THEOREM 6. $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ *and* $\mathsf{G3ie}^{\mathcal{T}}$ *are equivalent.*

PROOF: Any existential sequent $\Gamma \Rightarrow \mathcal{E}t$ in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ is clearly derivable in $\mathsf{G3ie}^{\mathcal{T}}$ by the corresponding existential rule $Ex(t)$. On the other hand, if the existential rule $Ex(t)$ is in $\mathsf{G3ie}^{\mathcal{T}}$, then it is admissible in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ since it can be simulated by a (non-eliminable) cut with the axiomatic sequent $\Rightarrow \mathcal{E}t$. Thus we have only to prove that rule $L\forall$ ($R\exists$) is equivalent to the rule $L\forall^*$ ($R\exists^*$). The following derivation shows that $L\forall$ is admissible in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$:

$$\dfrac{\mathcal{E}t, \forall x A, \Gamma \Rightarrow \mathcal{E}t \quad A[\tfrac{t}{x}], \forall x A, \mathcal{E}t, \Gamma \Rightarrow C}{\forall x A, \mathcal{E}t, \Gamma \Rightarrow C}\ {\scriptstyle L\forall^*}$$

and, by Lemmas 2 and 4 and Theorem 5, the following one shows that $L\forall^*$ is admissible in $\mathsf{G3ie}^{\mathcal{T}}$:

$$\dfrac{\dfrac{\forall x A, \Gamma \Rightarrow \mathcal{E}t \qquad \dfrac{\dfrac{A[\tfrac{t}{x}], \forall x A, \Gamma \Rightarrow C}{\mathcal{E}t, A[\tfrac{t}{x}], \forall x A, \Gamma \Rightarrow C}\ {\scriptstyle Wkn}}{\mathcal{E}t, \forall x A, \Gamma \Rightarrow C}\ {\scriptstyle L\forall}}{\forall x A, \forall x A, \Gamma, \Gamma \Rightarrow C}\ {\scriptstyle Cut}}{\forall x A, \Gamma \Rightarrow C}\ {\scriptstyle Ctr}$$

The cases of $R\exists$ in $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ and $R\exists^*$ in $\mathsf{G3ie}^{\mathcal{T}}$ are left to the reader. $\square$

4. Interpolation

We now turn to interpolation. The standard proof of interpolation for Gentzen's calculi LK and LJ rests on cut elimination and a result due to Maehara [5]. We recall from [10] some basic definitions.

DEFINITION 7 (Formula-language). Given a formula A the set $\mathsf{Ter}(A)$ is the set of free variables and individual constants occurring in A; the set $\mathsf{Rel}(A)$ is the set of non-logical relational symbols—i.e., all relational symbols except $\mathcal{E}$—occurring in A; and $\mathsf{Lan}(A)$ is the union of $\mathsf{Ter}(A)$ and $\mathsf{Rel}(A)$. These notions are extended to multisets and to sequents as expected.

DEFINITION 8 (partition, split-interpolant). A partition of a sequent $\Gamma \Rightarrow C$ is an expression $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow C$, where $\Gamma = \Gamma_1, \Gamma_2$ (where $=$ is the multiset-identity). A split-interpolant of a partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow C$ is a formula I such that:

I	$\vdash \Gamma_1 \Rightarrow I$
II	$\vdash I, \Gamma_2 \Rightarrow C$
III	$\mathsf{Lan}(I) \subseteq \mathsf{Lan}(\Gamma_1) \cap \mathsf{Lan}(\Gamma_2, C)$

We use $\Gamma_1 \; ; \; \Gamma_2 \stackrel{I}{\Rightarrow} C$ to indicate that I is a split-interpolant for $\Gamma_1 \; ; \Gamma_2 \Rightarrow C$.

Moreover, we say that a formula I satisfying conditions (I) and (II) satisfies the *derivability conditions* for being a split-interpolant for the partition $\Gamma_1 \; ; \; \Gamma_2 \Rightarrow I$, whereas if I satisfies (III) we say that it satisfies the *language condition* for being a split-interpolant for the same partition. Given a split sequent $\Gamma_1 \; ; \; \Gamma_2 \Rightarrow C$, we call Γ_1 (Γ_2) its *first* (*second*) *component*. Finally, having assumed that $\mathcal{E} \notin \mathsf{Rel}(A)$ for each formula A, we say that $\mathcal{E}$ is a *logical predicate*.

To prove Maehara's lemma we need first to prove a generalized version of Lemma 1 that allows arbitrary terms (either free variables or individual constants) to be replaced. Thus, we consider a general substitution $[\,{}^{t}_{u}\,]$ of terms for terms and we show its height-preserving admissibility.

LEMMA 9 (General substitution). *If $\vdash^n \Gamma \Rightarrow C$ and t is free for u in Γ, C and no instance of the rule Ex(u) has been applied in the derivation of $\Gamma \Rightarrow C$, then $\vdash^n \Gamma[\,{}^{t}_{u}\,] \Rightarrow C[\,{}^{t}_{u}\,]$.*

PROOF: If u is a variable, the claim holds by Lemma 1. Otherwise, let u be an individual constant. We can think of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow C$ as

$$\frac{\Gamma' \Rightarrow C'}{\Gamma'[\,{}^{u}_{z}\,] \Rightarrow C[\,{}^{u}_{z}\,]}\;[\,{}^{u}_{z}\,]$$

where $\Gamma' \Rightarrow C'$ is like $\Gamma \Rightarrow C$ save that it has a fresh variable z in place of u. Note that this is always feasible for purely logical derivations, and it is feasible for derivations involving no instance of rule *Ex(u)*. We transform $\mathcal{D}$ into

$$\frac{\Gamma' \Rightarrow C'}{\Gamma'[\,{}^{t}_{z}\,] \Rightarrow C'[\,{}^{t}_{z}\,]}\;[\,{}^{t}_{z}\,]$$

where t is free for z since we assumed it is free for u in $\Gamma \Rightarrow C$. We have thus found a derivation ($\mathcal{D}[\,{}^{t}_{u}\,]$) of $\Gamma[\,{}^{t}_{u}\,] \Rightarrow C[\,{}^{t}_{u}\,]$ that has the same height as the derivation $\mathcal{D}$ of $\Gamma \Rightarrow C$. □

LEMMA 10 (Maehara's lemma for $\mathsf{G3ie}^{\mathcal{T}}$). *Every partition $\Gamma_1 \; ; \; \Gamma_2 \Rightarrow C$ of a derivable sequent $\Gamma \Rightarrow C$ has a split-interpolant.*

PROOF: The proof is by induction on the height h of the derivation $\mathcal{D}$. The cases when $h = 0$ or $h = n + 1$ and the last step in $\mathcal{D}$ is by a propositional rule are identical to the ones for $\mathsf{G3i}$ and the reader is referred to [10, §4.4.2] for a sketch of the proof. Hence we have to consider only the rules for the quantifiers and the non-logical rule *Ex(t)*. We consider first the rules for the existential quantifier.

Suppose that the last step in $\mathcal{D}$ is by $L\exists$, i.e.

$$\frac{\mathcal{E}y, A[\frac{y}{x}], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ {\scriptstyle L\exists}$$

where y is *eigenvariable*. We have to consider the following two partitions of the conclusion:

1. $\exists x A, \Gamma_1 \ ; \ \Gamma_2 \Rightarrow C$
2. $\Gamma_1 \ ; \ \exists x A, \Gamma_2 \Rightarrow C$

The split-interpolants for these partitions are, respectively,

$$\boxed{\frac{A[\frac{y}{x}], \mathcal{E}y, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} C}{\exists x A, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} C} \qquad \text{and} \qquad \frac{\Gamma_1 \ ; \ A[\frac{y}{x}], \mathcal{E}y, \Gamma_2 \overset{I}{\Rightarrow} C}{\Gamma_1 \ ; \ \exists x A, \Gamma_2 \overset{I}{\Rightarrow} C}}$$

We give the details of the proof only for the first partition since the proof for the other one is almost identical. By induction hypothesis (IH), there is a formula I such that:

(i) $\vdash A[\frac{y}{x}], \mathcal{E}y, \Gamma_1 \Rightarrow I$

(ii) $\vdash I, \Gamma_2 \Rightarrow C$

(iii) $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(A[\frac{y}{x}], \mathcal{E}y, \Gamma_1) \cap \mathsf{Lan}(\Gamma_2, C)$

The following derivations show that I satisfies the derivability conditions for being a split-interpolant of the partition under consideration.

$$\frac{\dfrac{}{A[\frac{y}{x}], \mathcal{E}y, \Gamma_1 \Rightarrow I}\ {\scriptstyle (i)}}{\exists x A, \Gamma_1 \Rightarrow I}\ {\scriptstyle L\exists} \qquad \text{and} \qquad \frac{}{I, \Gamma_2 \Rightarrow C}\ {\scriptstyle (ii)}$$

Moreover, since $\mathcal{E} \notin \mathsf{Lan}(I)$ by Definition 7, to see that (iii) implies that I satisfies the language condition it is enough to notice that y cannot be in $\mathsf{Ter}(I)$ because y is the *eigenvariable* of this rule instance and, hence, it is not in $\mathsf{Ter}(\Gamma_2, C)$.[3] Now we consider the rule $R\exists$.

Suppose the last step in $\mathcal{D}$ is by $R\exists$:

$$\dfrac{\mathcal{E}t, \Gamma \Rightarrow A[\,{}^{t}_{x}\,]}{\mathcal{E}t, \Gamma \Rightarrow \exists x A}\ {\scriptstyle R\exists}$$

Once again, we have to consider two partitions:

1. $\Gamma_1 \ ; \ \mathcal{E}t, \Gamma_2 \Rightarrow \exists x A$
2. $\mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \Rightarrow \exists x A$

For the first partition we have the following split-interpolant:

$$\boxed{\dfrac{\Gamma_1 \ ; \ \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} A[\,{}^{t}_{x}\,]}{\Gamma_1 \ ; \ \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} \exists x A}}$$

To see this, notice that by IH we know that there is a formula I such that:

(i) $\vdash \Gamma_1 \Rightarrow I$
(ii) $\vdash I, \mathcal{E}t, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]$
(iii) $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(\Gamma_1) \cap \mathsf{Lan}(\mathcal{E}t, \Gamma_2, A[\,{}^{t}_{x}\,])$

From (i) and (ii) it immediately follows that I satisfies the derivability condition—we only need to apply $R\exists$ to the sequent in (ii). Moreover, (iii) implies that I satisfies the language condition too, since $\mathsf{Lan}(\mathcal{E}t, \Gamma_2, \exists x A) = \mathsf{Lan}(\mathcal{E}t, \Gamma_2, A[\,{}^{t}_{x}\,])$ (for t already occurs in both). For the second partition the proof is more complicated. By IH we can assume there is a formula I such that:

(i) $\vdash \mathcal{E}t, \Gamma_1 \Rightarrow I$
(ii) $\vdash I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]$
(iii) $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(\mathcal{E}t, \Gamma_1) \cap \mathsf{Lan}(\Gamma_2, A[\,{}^{t}_{x}\,])$

[3] In [1, p. 11] the split-interpolant of the given partition is identified with $\forall z I[\,{}^{z}_{y}\,]$ where the $\forall z$ can be dropped when y does not occur free in I. Our reasoning shows that we are always in this latter case.

Now we need to consider three mutually incompatible and exhaustive cases:

(a) $t \notin \mathsf{Ter}(\Gamma_2, \exists x A)$
(b) $t \in \mathsf{Ter}(\Gamma_2, \exists x A)$ and $t \in \mathcal{T}$
(c) $t \in \mathsf{Ter}(\Gamma_2, \exists x A)$ and $t \notin \mathcal{T}$

If case (a) holds, the partition has the following split-interpolant:

$$\boxed{\dfrac{\mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} A[\,{}^{t}_{x}\,]}{\mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{\exists y I[\,{}^{y}_{t}\,]} \exists x A}}$$

Indeed, the following derivations show that $\exists y I[\,{}^{y}_{t}\,]$ satisfies the derivability conditions:

$$\dfrac{\dfrac{\dfrac{\overline{\mathcal{E}t, \Gamma_1 \Rightarrow I}\ (i)}{\mathcal{E}y, \Gamma_1[\,{}^{y}_{t}\,] \Rightarrow (I[\,{}^{y}_{t}\,])[\,{}^{y}_{y}\,]}\ [\,{}^{y}_{t}\,][\,{}^{y}_{y}\,]}{\mathcal{E}y, \Gamma_1[\,{}^{y}_{t}\,] \Rightarrow \exists y I[\,{}^{y}_{t}\,]}\ R\exists}{\mathcal{E}t, \Gamma_1 \Rightarrow \exists y I[\,{}^{y}_{t}\,]}\ [\,{}^{t}_{y}\,] \quad \text{and} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]}\ (ii)}{\mathcal{E}t, I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]}\ Wkn}{I, \mathcal{E}t, \Gamma_2 \Rightarrow \exists x A}\ R\exists}{\exists y I[\,{}^{y}_{t}\,], \mathcal{E}y, \Gamma_2 \Rightarrow \exists x A}\ [\,{}^{y}_{t}\,][\,{}^{y}_{y}\,]}{\exists y I[\,{}^{y}_{t}\,], \Gamma_2 \Rightarrow \exists x A}\ L\exists$$

where y is a new variable and in both derivations the inference steps where we have applied substitutions are height-preserving admissible by Lemma 9. In particular, Lemma 9 allows us to apply the substitution $[\,{}^{y}_{t}\,]$ because, thanks to the admissibility of contraction (Lemma 4), we can eliminate any instance of rule *Ex(t)* from the derivation of a sequent where $\mathcal{E}t$ occurs. Moreover, the assumption for case (a), i.e. $t \notin \mathsf{Ter}(\Gamma_2, \exists x A)$, ensures that the substitution $[\,{}^{y}_{t}\,]$ has no effect on $\Gamma_2, \exists x A$ in the right derivation. It is also immediate to see that (iii) entails that $\exists y I[\,{}^{y}_{t}\,]$ satisfies the language condition since $t \notin \mathsf{Ter}(\exists y I[\,{}^{y}_{t}\,])$. Thus, the split-interpolant is $\exists y I[\,{}^{y}_{t}\,]$ and whenever $t \notin \mathsf{Ter}(I)$, we can drop the vacuous quantification.

In case (b), the partition has the following split-interpolant:

$$\boxed{\dfrac{\mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} A[\,{}^{t}_{x}\,]}{\mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} \exists x A}}$$

Too see this notice that (iii) entails that I satisfies the language condition and the following derivations, where $Ex(t)$ is applicable, since $t \in \mathcal{T}$ by assumption for case (b), show it satisfies the derivability conditions as well.

$$\dfrac{}{\mathcal{E}t, \Gamma_1 \Rightarrow I}\ (i) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]}\ (ii)}{\mathcal{E}t, I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]}\ Wkn}{\mathcal{E}t, I, \Gamma_2 \Rightarrow \exists x A}\ R\exists}{I, \Gamma_2 \Rightarrow \exists x A}\ Ex(t)$$

Finally, in case (c) the split-interpolant is:

$$\boxed{\dfrac{\mathcal{E}t, \Gamma_1\ ;\ \Gamma_2 \stackrel{I}{\Rightarrow} A[\,{}^{t}_{x}\,]}{\mathcal{E}t, \Gamma_1\ ;\ \Gamma_2 \xRightarrow{I \wedge \mathcal{E}t} \exists x A}}$$

On the one hand, the following derivations show that $I \wedge \mathcal{E}t$ satisfies the derivability conditions:

$$\dfrac{\dfrac{}{\mathcal{E}t, \Gamma_1 \Rightarrow I}\ (i) \qquad \mathcal{E}t, \Gamma_1 \Rightarrow \mathcal{E}t}{\mathcal{E}t, \Gamma_1 \Rightarrow I \wedge \mathcal{E}t}\ R\wedge \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]}\ (ii)}{\mathcal{E}t, I, \Gamma_2 \Rightarrow A[\,{}^{t}_{x}\,]}\ Wkn}{I, \mathcal{E}t, \Gamma_2 \Rightarrow \exists x A}\ R\exists}{I \wedge \mathcal{E}t, \Gamma_2 \Rightarrow \exists x A}\ L\wedge$$

On the other hand, the formula $I \wedge \mathcal{E}t$ satisfies the language condition since both I and $\mathcal{E}t$ satisfy it. Indeed, that I satisfies it follows immediately from (iii). To see that the same holds for $\mathcal{E}t$, notice that both t and $\mathcal{E}$ satisfy the language condition in virtue of the assumption for case (c) and the fact that $\mathcal{E}$ is a logical predicate, respectively.[4] This completes the proof for the rules of the existential quantifier. Next we move to the rules for the universal quantifier.

[4] If, instead, $\mathcal{E}$ weren't a logical predicate, interpolation would fail for ILE altogether: for $Pt \wedge \mathcal{E}t \rightarrow \exists x Px$ is a theorem whose interpolant is $Pt \wedge \mathcal{E}t$ but $\mathcal{E} \notin \mathsf{Lan}(\exists x Px)$. This is analogous to the case of Maehara's lemma for first-order logic with identity in [4].

Suppose that the last step in $\mathcal{D}$ is by $L\forall$:

$$\dfrac{A[\,{}^{t}_{x}\,], \forall x A, \mathcal{E}t, \Gamma \Rightarrow C}{\forall x A, \mathcal{E}t, \Gamma \Rightarrow C}\ L\forall$$

We have to consider four partitions of the conclusion:

1. $\forall x A, \mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \Rightarrow C$
2. $\Gamma_1 \ ; \ \forall x A, \mathcal{E}t, \Gamma_2 \Rightarrow C$
3. $\mathcal{E}t, \Gamma_1 \ ; \ \forall x A, \Gamma_2 \Rightarrow C$
4. $\forall x A, \Gamma_1 \ ; \ \mathcal{E}t, \Gamma_2 \Rightarrow C$

The reader can easily see that the split-interpolants for the first two partitions are, respectively:

$$\boxed{\dfrac{A[\,{}^{t}_{x}\,], \forall x A, \mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} C}{\forall x A, \mathcal{E}t, \Gamma_1 \ ; \ \Gamma_2 \overset{I}{\Rightarrow} C} \qquad \text{and} \qquad \dfrac{\Gamma_1 \ ; \ A[\,{}^{t}_{x}\,], \forall x A, \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}{\Gamma_1 \ ; \ \forall x A, \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}}$$

The third partition can be dealt with as the second partition for rule $R\exists$. In particular, by IH we can assume that:

(i) $\vdash \mathcal{E}t, \Gamma_1 \Rightarrow I$

(ii) $\vdash I, A[\,{}^{t}_{x}\,], \forall x A, \Gamma_2 \Rightarrow C$

(iii) $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(\mathcal{E}t, \Gamma_1) \cap \mathsf{Lan}(A[\,{}^{t}_{x}\,], \forall x A, \Gamma_2, C)$

and we have to consider three mutually incompatible and exhaustive cases:

(a) $t \notin \mathsf{Ter}(\forall x A, \Gamma_2, C)$

(b) $t \in \mathsf{Ter}(\forall x A, \Gamma_2, C)$ and $t \in \mathcal{T}$

(c) $t \in \mathsf{Ter}(\forall x A, \Gamma_2, C)$ and $t \notin \mathcal{T}$

For each case the split-interpolant for the partition is, respectively:

$$\frac{\mathcal{E}t, \Gamma_1 \; ; \; A[\,^t_x\,], \forall x A, \Gamma_2 \overset{I}{\Rightarrow} C}{\mathcal{E}t, \Gamma_1 \; ; \; \forall x A, \Gamma_2 \xrightarrow{\exists y I[\,^y_t\,]} C} \qquad \frac{\mathcal{E}t, \Gamma_1 \; ; \; A[\,^t_x\,], \forall x A, \Gamma_2 \overset{I}{\Rightarrow} C}{\mathcal{E}t, \Gamma_1 \; ; \; \forall x A, \Gamma_2 \overset{I}{\Rightarrow} C}$$

$$\frac{\mathcal{E}t, \Gamma_1 \; ; \; A[\,^t_x\,], \forall x A, \Gamma_2 \overset{I}{\Rightarrow} C}{\mathcal{E}t, \Gamma_1 \; ; \; \forall x A, \Gamma_2 \xrightarrow{I \wedge \mathcal{E}t} C}$$

We now deal with the the fourth partition. We assume by IH that:

(i) $\vdash A[\,^t_x\,], \forall x A, \Gamma_1 \Rightarrow I$
(ii) $\vdash I, \mathcal{E}t, \Gamma_2 \Rightarrow C$
(iii) $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(A[\,^t_x\,], \forall x A, \Gamma_1) \cap \mathsf{Lan}(\mathcal{E}t, \Gamma_2, C)$

We have, once again, to consider three mutually incompatible and exhaustive cases:

(a) $t \notin \mathsf{Ter}(\forall x A, \Gamma_1)$
(b) $t \in \mathsf{Ter}(\forall x A, \Gamma_1)$ and $t \in \mathcal{T}$
(c) $t \in \mathsf{Ter}(\forall x A, \Gamma_1)$ and $t \notin \mathcal{T}$

Mutatis mutandis, in cases (a) and (b) we reason as in the corresponding cases for the second partition of rule $R\exists$ and we find that the split-interpolants for the partition are, respectively:

$$\frac{A[\,^t_x\,], \forall x A, \Gamma_1 \; ; \; \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}{\forall x A, \Gamma_1 \; ; \; \mathcal{E}t, \Gamma_2 \xrightarrow{\forall y I[\,^y_t\,]} C} \qquad \frac{A[\,^t_x\,], \forall x A, \Gamma_1 \; ; \; \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}{\forall x A, \Gamma_1 \; ; \; \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}$$

sIn case (c), instead, the split-interpolant is:

$$\frac{A[\,^t_x\,], \forall x A, \Gamma_1 \; ; \; \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}{\forall x A, \Gamma_1 \; ; \; \mathcal{E}t, \Gamma_2 \xrightarrow{\mathcal{E}t \to I} C}$$

Indeed (iii) entails that the formula $\mathcal{E}t \to I$ satisfies the language condition since, in virtue of the assumption for case (c), we know that $t \in$

$\mathsf{Ter}(\forall xA, \Gamma_1)$ and since $\mathcal{E}$ is a logical predicate (cf. footnote 4). Next the following derivations show that $\mathcal{E}t \to I$ satisfies the derivability conditions, too:

$$\dfrac{\dfrac{\dfrac{\overline{A[\frac{t}{x}], \forall xA, \Gamma_1 \Rightarrow I}\ (i)}{\mathcal{E}t, A[\frac{t}{x}], \forall xA, \Gamma_1 \Rightarrow I}\ Wkn}{\mathcal{E}t, \forall xA, \Gamma_1 \Rightarrow I}\ L\forall}{\forall xA, \Gamma_1 \Rightarrow \mathcal{E}t \to I}\ R\to \qquad \dfrac{\mathcal{E}t, \Gamma_2 \Rightarrow \mathcal{E}t \quad \overline{I, \mathcal{E}t, \Gamma_2 \Rightarrow C}\ (ii)}{\mathcal{E}t \to I, \mathcal{E}t, \Gamma_2 \Rightarrow C}\ L\to$$

This completes the proof for $L\forall$ and we can now consider rule $R\forall$.

Suppose that the last step in $\mathcal{D}$ is by $R\forall$:

$$\dfrac{\mathcal{E}y, \Gamma \Rightarrow A[\frac{y}{x}]}{\Gamma \Rightarrow \forall xA}\ R\forall$$

where y is *eigenvariable*. As in [1] (omitting the vacuous quantifier, cf. footnote 3), we have to consider only one partition of the conclusion, whose split-interpolant is:

$$\boxed{\dfrac{\Gamma_1\ ;\ \mathcal{E}y, \Gamma_2 \overset{I}{\Rightarrow} A[\frac{y}{x}]}{\Gamma_1\ ;\ \Gamma_2 \overset{I}{\Rightarrow} \forall xA}}$$

The proof that I satisfies the language and derivability conditions is as in [1]. Finally, we have to consider rule *Ex(t)*.

Suppose the final step in $\mathcal{D}$ is by $Ex(t)$:

$$\dfrac{\mathcal{E}t, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ Ex(t)$$

We have only one partition to consider, namely $\Gamma_1\ ;\ \Gamma_2 \Rightarrow C$. We consider two cases according to whether $t \in \mathsf{Ter}(\Gamma_1)$ or not and we have, respectively:

$$\boxed{\dfrac{\mathcal{E}t, \Gamma_1\ ;\ \Gamma_2 \overset{I}{\Rightarrow} C}{\Gamma_1\ ;\ \Gamma_2 \overset{I}{\Rightarrow} C} \qquad \dfrac{\Gamma_1\ ;\ \mathcal{E}t, \Gamma_2 \overset{I}{\Rightarrow} C}{\Gamma_1\ ;\ \Gamma_2 \overset{I}{\Rightarrow} C}}$$

The fact that I satisfies the derivability conditions is obvious in both cases: we just have to apply IH and then an instance of rule *Ex(t)*. It is also easy to see that I satisfies the language condition too: in the first case I satisfies the language condition because ($\mathcal{E} \notin \mathsf{Lan}(I)$ and) if $t \in \mathsf{Ter}(I)$ then, by IH, it must be in $\mathsf{Ter}(\Gamma_2, C)$ and we are assuming that it is in $\mathsf{Ter}(\Gamma_1)$; but so does in the second case, since here we are assuming that $t \notin \mathsf{Ter}(\Gamma_1)$ and hence $t \notin \mathsf{Ter}(I)$. □

From Maehara's lemma for $\mathsf{G3ie}^{\mathcal{T}}$, it is immediate to prove Craig's interpolation theorem.

Theorem 11 (Craig's interpolation for $\mathsf{G3ie}^{\mathcal{T}}$). *If $A \Rightarrow B$ is derivable in $\mathsf{G3ie}^{\mathcal{T}}$ then there exists a formula I such that $\vdash A \Rightarrow I$ and $\vdash I \Rightarrow B$ and $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(A) \cap \mathsf{Lan}(B)$.*

Proof: Let $A \Rightarrow B$ be derivable in $\mathsf{G3ie}^{\mathcal{T}}$ and let us consider the partition $A \,;\, \varnothing \Rightarrow B$ of $A \Rightarrow B$. By Lemma 10, this partition has a split-interpolant, namely there exists a I such that $A \,;\, \varnothing \overset{I}{\Rightarrow} B$. Hence $\vdash A \Rightarrow I$ and $\vdash I \Rightarrow B$ and $\mathsf{Lan}(I) \subseteq \mathsf{Lan}(A) \cap \mathsf{Lan}(B)$ by Definition 8. □

Comparing our proof of interpolation with that of [1], it appears that ours is direct in the sense that it relies exclusively on the rules of $\mathsf{G3ie}^{\mathcal{T}}$, with no need to go through an equivalent system.

5. Interpolation for the implication-free fragment

In this section we consider interpolation for sequents in the fragment of the language without implication (nor $\bot$). First, we show that for the procedure given in Lemma 10 (henceforth `Proc`) there exists a bound on the number of implications introduced in the interpolants. Next, we consider the class of implication-free derivable partitions $\Pi_1 \,;\, \Pi_2 \Rightarrow F$ such that Π_2 contains no existential atom. We show that for sequents in this class, `Proc` can be modified in such a way that it outputs implication-free interpolants. Notice that the assumption that no existence atom occurs in the second component of the end-sequent is not restrictive as long as one aims to prove interpolation for sequents representing theorems of ILE, since their second component is empty (cf. Theorem 11).

We say that a sequent calculus G *interpolates for a fragment $\mathcal{F}$* of the language $\mathcal{L}'$ if whenever $\vdash \Gamma_1 \,;\, \Gamma_2 \overset{I}{\Rightarrow} C$ in G with $\Gamma_1, \Gamma_2, C \in \mathcal{F}$, we have that $I \in \mathcal{F}$. It is well-known that LJ interpolates for the $\{\wedge, \vee, \exists, \forall\}$-fragment. Nevertheless, in [1, §5.1] it is conjectured LJE and $\mathsf{LJE}(\Sigma_{\mathcal{L}})$ do not interpolate for the $\{\wedge, \vee, \exists, \forall\}$-fragment because rule $L\forall$ does not interpolate for this fragment. The problem, roughly, is that the procedure given in [1, Theorem 4] might introduce an implication in the interpolant of the conclusion of an instance of $L\forall$.

We are now going to show that the calculi $\mathsf{G3ie}$ and $\mathsf{G3ie}^{\mathcal{T}}$ are better behaved with respect to interpolation for the $\{\wedge, \vee, \exists, \forall\}$-fragment in that:

Proposition 12. *If* $\Gamma_1 \,;\, \Gamma_2 \Rightarrow C$ *is a* $\mathsf{G3ie}^{(\mathcal{T})}$*-derivable sequent in the* $\{\wedge, \vee, \exists, \forall\}$*-fragment and we apply* `Proc` *to its derivation, we obtain an interpolant* I *such that:*

(α) *If an implication occurrs in* I*, its antecedent is of the form* $\mathcal{E}t$*;*[5]

(β) *If* $\#_{\circ}(\Gamma)$ *stands for the number of occurrences of the symbol* $\circ$ *in* Γ*, then*

$$\#_{\rightarrow}(I) \quad \leq \quad [\#_{\forall}(\Gamma_1)] \times [\#_{\mathcal{E}}(\Gamma_2) + \#_{\forall}(C) + \#_{\exists}(\Gamma_2)\,]$$

Proof: First of all, by inspecting `Proc` we immediately see that an implication may occur in the interpolant of an implication-free sequent only when in its derivation there is an instance of $L\forall$ whose conclusion is an instance of subcase (c) of the fourth partition considered in Lemma 10, i.e., it is of the form

$$(\dagger)\ \frac{A[\,^{t}_{x}\,], \forall x A, \Delta_1 \,;\, \mathcal{E}t, \Delta_2 \Rightarrow D}{\forall x A, \Delta_1 \,;\, \mathcal{E}t, \Delta_2 \Rightarrow D}\, L\forall \qquad (\ddagger) \text{ with } t \notin \mathcal{T} \text{ and } t \in \mathsf{Ter}(\forall x A, \Delta_1)$$

Let us call *quasi-implicative* an arbitrary instance of the fourth partition of $L\forall$ and *fully-implicative* one that falls under case (c). Thus, an implication may occur in the interpolant of an implication-free conclusion of an instance of a rule of $\mathsf{G3ie}^{\mathcal{T}}$ if and only if it is a fully-implicative instance of $L\forall$.

The claim (α) holds since the interpolant of a conclusion of a fully-implicative instance of $L\forall$ is $\mathcal{E}t \rightarrow J$ (where J is the interpolant of the premiss).

To prove claim (β) we analyze the derivation of $\Gamma_1 \,;\, \Gamma_2 \Rightarrow C$ bottom-up, as is normally done in proof-search procedures. The first thing to notice is that no case of `Proc` for the $\{\wedge, \vee, \exists, \forall\}$-fragment switches the position of

[5]And not a formula of arbitrary complexity as in [1, Thm. 4]

a (sub)formula occurring in a rule instance: in moving from the conclusion to the premiss(es) nothing goes from one component of the antecedent (or from one side of the sequent) to the other. As a consequence we have that $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow C$ has been concluded by a quasi-implicative instance of $L\forall$ only if:

1. its principal formula $\forall x A$ is a subformula of (some formula in) Γ_1, and
2. its principal formula $\mathcal{E}t$
   (a) is a subformula of Γ_2, or
   (b) it has been introduced (bottom-up) by an instance of $Ex(t)$, or
   (c) it has been introduced (bottom-up) by an instance either of $R\forall$ whose principal formula is a subformula of C or of $L\exists$ whose principal formula is a subformula of Γ_2.

We immediately have that the number of (quasi- and) fully-implicative instances of $L\forall$ is bounded by a function of the number of universal quantifiers occurring in Γ_1, namely $\#_{\rightarrow}(I) \ \leq \ [\#_{\forall}(\Gamma_1)] \times m$, for some m.

Now we show that $m = \#_{\mathcal{E}}(\Gamma_2) + \#_{\forall}(C) + \#_{\exists}(\Gamma_2)$. This will done by identifying when the quasi-implicative instance of $L\forall$ in (†) is fully-implicative, i.e. it satisfies (‡). If its principal formula $\mathcal{E}t$ is a subformula of Γ_2, it is fully-implicative if and only if $t \notin \mathcal{T}$ and $t \in \forall x A, \Delta_1$. Hence $\#_{\mathcal{E}}(\Gamma_2)$ goes into m. If, instead, its principal formula $\mathcal{E}t$ has been introduced by a lower instance of $Ex(t)$, it is never fully-implicative since $t \in \mathcal{T}$. Hence nothing goes into m. Lastly, if its principal formula $\mathcal{E}t$ has been introduced by a lower instance of one of $R\forall$ and $L\exists$, say

$$\frac{\Sigma_1 \ ; \ \mathcal{E}t, \Sigma_2 \Rightarrow B[\,{}^{t}_{z}\,]}{\Sigma_1 \ ; \ \Sigma_2 \Rightarrow \forall z B} \ R\forall$$

by the variable condition on $R\forall$ we immediately get that t is a variable (hence we know $t \notin \mathcal{T}$) that does not occur in Σ_1. Therefore, t can be in $\mathsf{Ter}(\forall x A, \Delta_1)$ only if it has been introduced (bottom-up) by some rule instance occurring between $R\forall$ and (†). The only rule that can introduce new (free) occurrences of t in the first component of its premiss is $L\forall$. Hence the first quasi-implicative instance of $L\forall$ (with $\mathcal{E}t$ principal) cannot be fully-implicative (the only occurrence of t in its antecedent is the one in $\mathcal{E}t$) and each other one is fully-implicative (provided the quantification in the principal formula of the first one wasn't vacuous). Hence $\#_{\forall}(C)$ and $\#_{\exists}(\Gamma_2)$ goes into m. □

Now we give an alternative procedure showing that, under a plausible assumption, $\mathsf{G3ie}^{\mathcal{T}}$ interpolates for the $\{\wedge, \vee, \exists, \forall\}$-fragment.

Proposition 13. *The calculus* $\mathsf{G3ie}^{(\mathcal{T})}$ *interpolates for the* $\{\wedge, \vee, \exists, \forall\}$*-fragment provided that we exclude end-sequents with existence atoms occurring in their second component.*

Proof: Suppose we are applying `Proc` to the derivation of a sequent $\Pi_1 \; ; \; \Pi_2 \Rightarrow F$ satisfying the hypothesis of the Proposition 13. By Proposition 12, we know that an instance of $L\forall$ is fully-implicative only if its principal formulas $\mathcal{E}y$ has been introduced in the second component by a lower instance of either $L\exists$ or $R\forall$, and y has been introduced in its first-component by another (in-between) instance of $L\forall$. In particular, the first instance of $L\forall$, i.e. the one introducing free occurrences of y in the first component, is a (non-fully-implicative) instance of case (a) of the fourth partition for $L\forall$ in Lemma 10.

Let us consider a procedure `Proc`* that is like `Proc` save that in case (a) of the fourth partition for $L\forall$ it moves the existence atom $\mathcal{E}t$ from the second component of its conclusion to the first component of its premiss:

$$\boxed{\dfrac{\mathcal{E}t, A[\tfrac{t}{x}], \forall x A, \Delta_1 \; ; \; \Delta_2 \overset{I}{\Rightarrow} D}{\forall x A, \Delta_1 \; ; \; \mathcal{E}t, \Delta_2 \xrightarrow{\forall y I[\frac{y}{t}]} D} \; L\forall, \quad t \notin \mathsf{Ter}(\forall x A, \Delta_1)}$$

Maehara's lemma holds for `Proc`*: we have just moved the existence atom $\mathcal{E}t$ from one component to the other. This difference has no impact for the language condition since $\mathcal{E}$ is a logical predicate and $t \notin \mathsf{Ter}(\forall y I[\frac{y}{t}])$. As for the derivability conditions, the only difference is that now we have to introduce $\mathcal{E}t$ via an instance of weakening in the derivation of $\forall y I[\frac{y}{t}], \mathcal{E}t, \Delta_2 \Rightarrow C$ instead of introducing it via weakening in the one of $\forall x A, \Delta_1 \Rightarrow \forall y I[\frac{y}{t}]$ (as in lemma 10).

Let us consider an arbitrary quasi-implicative instance of $L\forall$ occurring in the derivation of $\Pi_1 \; ; \; \Pi_2 \Rightarrow F$ with principal formula $\mathcal{E}y$ (introduced by a lower instance of one of $R\forall$ and $L\exists$). We can easily show that it is not a fully-implicative instance since it must fall under the modified case (a) above. To witness, we have already shown that the first quasi-implicative instance of $L\forall$ occurring above the introduction of $\mathcal{E}y$ falls under case (a) and, given that `Proc`* moves $\mathcal{E}y$ to the first component of the premiss of this rule instance and no rule instance can move it back to the second component nor introduce another instance of $\mathcal{E}y$ (since we are considering derivations satisfying the pure-variable convention), no other quasi-implicative instance of $L\forall$ with principal formula $\mathcal{E}y$ can occur above the first one. Thus we never apply a fully-implicative instance of $L\forall$ under $\texttt{Proc}^*$, and this is enough to prove the proposition.

From the perspective of the numeric bound given for $\texttt{Proc}$ in Proposition 12, we now have that $\#_{\mathcal{E}}(\Gamma_2) = 0$ by hypothesis of the proposition and that $\texttt{Proc}^*$ is defined so that $\#_{\forall}(C) + \#_{\exists}(\Gamma_2)$ is replaced by 0. Hence $\#_{\rightarrow}(I) = 0$. □

6. Conclusion

In this paper we presented an improvement on the previous work by Baaz and Iemhoff on cut elimination and interpolation for ILE. In particular, we have shown that ILE admits a fully cut-free systematization in sequent calculus, which allows a direct constructive proof of interpolation, and we have shown that if an implication occurs in the interpolant of an implication-free sequent, then its antecedent must be an atom of existence. Moreover, we have also shown that (under a plausible assumption) our cut-free calculi interpolate for the $\{\wedge, \vee, \exists, \forall\}$-fragment.

This paper is also an improvement on the previous work on interpolation in first-order theories, especially [4]. In [4] it is shown how to extend interpolation from classical and intuitionistic logic to singular geometric theories, a subclass of geometric theories investigated in [6]. Interestingly, singular geometric theories are subjected to the condition that individual constants do not occur in any axiom. ILE is clearly an example of a theory outside the singular geometric class, since individual constants occur necessarily in existential axioms. Therefore, $\mathsf{G3ie}^{\mathcal{I}}$ is a calculus not falling within the singular geometric class for which interpolation holds. This motivates further interest in generalizing the approach of [4] and we leave the task to future work.

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