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Sadegh Khosravi Shoar 

Rajab Ali Borzooei 

SOME NEW KINDS OF BCK-FILTERS IN BOUNDED BCK-ALGEBRAS

Abstract

This paper investigates and characterizes some new types of BCK-filters in bounded BCK-algebras. We introduce the notions of EQI-filters, implicative BCK-filters, and commutative BCK-filters. We clarify the relationships among these filters and prove that every implicative BCK-filter is a commutative BCK-filter and every commutative BCK-filter is an EQI-filter, although the converses do not hold in general. We also study these filters in several important subclasses of BCK-algebras. For instance, in involutory BCK-algebras, the notions of EQI-filters, implicative BCK-filters, and commutative BCK-filters coincide, whereas they are distinct in general.

Keywords: BCK-filter, implicative BCK-filter, commutative BCK-filter, EQI-filter, EQI-algebras.

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1. Introduction

A BCK-algebra is an algebraic structure introduced by Y. Imai and K. Iséki [5] in 1966. It originated from set-theoretic considerations and is closely related to classical and non-classical logics. Since logical systems play a crucial role in computer science and artificial intelligence, research in this area is highly significant.

The class of bounded BCK-algebras includes important subclasses such as bounded commutative BCK-algebras, bounded implicative BCK-algebras, and involutory BCK-algebras. These structures contain various types of BCK-lattices. For example, S. Khosravi Shoar et al. [11] introduced the concept of PC-lattices in bounded BCK-algebras and proved that in PC-lattices satisfying condition (S), commutative BCK-algebras and involutory BCK-algebras coincide.

Filter theory, together with ideal theory, plays a vital role in studying the structure of BCK/BCI-algebras. J. Meng [8] introduced the concept of BCK-filters. W. A. Dudek and Y. B. Jun [2] studied poor and crazy filters in bounded BCK-algebras. In this paper, we introduce EQI-filters, implicative BCK-filters, and commutative BCK-filters in bounded BCK-algebras and investigate the relationships among them. We also examine these filters in several important subclasses of BCK-algebras.

2. Preliminaries

DEFINITION 2.1 ([6, 3]). Let X be a set with a binary operation $*$ and a constant 0 . Then $(X; *, 0)$ is called a BCK-algebra if it satisfies the following axioms for all $x, y, z \in X$:

$$(BCK-1) \quad ((x * y) * (x * z)) * (z * y) = 0.$$

$$(BCK-2) \quad (x * (x * y)) * y = 0.$$

$$(BCK-3) \quad x * x = 0.$$

$$(BCK-4) \quad x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y.$$

$$(BCK-5) \quad 0 * x = 0.$$

A partial ordering $\leq$ on X is defined by $x \leq y$ if and only if $x * y = 0$.
In any BCK-algebra X , for all $x, y, z \in X$ the following statements hold:

- (I) $(x * y) * z = (x * z) * y$.
- (II) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
- (III) $x * (x * (x * y)) = (x * y)$.
- (IV) $(z * x) * (y * x) \leq z * y$ and $(x * y) * (x * z) \leq z * y$.

DEFINITION 2.2 ([1, 5, 12, 9]). Let X be a BCK-algebra. Then for all $x, y \in X$:

- (i) X is called an *implicative* BCK-algebra if $x * (y * x) = x$.
- (ii) X is called a *commutative* BCK-algebra if $x * (x * y) = y * (y * x)$.
- (iii) X is called *bounded*, if there exists the greatest element of X , which is denoted by 1 (In this case for any $x \in X$, $1 * x$ is denoted by Nx).
- (iv) If X is bounded, then $x \in X$ is called *involution* of X , if $NNx = x$ where $NNx = N(Nx)$.
- (v) X is called an *involutory* BCK-algebra, if for all $x \in X$, $NNx = x$.

DEFINITION 2.3 ([6, 7]). Let I be a nonempty subset of BCK-algebra X . Then for all $x, y, z \in X$:

- (i) I is called an *ideal* of X if $0 \in I$, $y \in I$ and $x * y \in I$ imply $x \in I$.
- (ii) I is called an *implicative ideal* of X if $0 \in I$, $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$.

DEFINITION 2.4 ([8, 2]). Let F be a nonempty subset of bounded BCK-algebra X . Then for all $x, y, z \in X$:

- (i) F is called a *BCK-filter* of X if $1 \in F$ and $N(Nx * Ny) \in F$ and $y \in F$ imply $x \in F$.
- (ii) F is called a *crazy filter* of X if $1 \in F$, $N(y * x) \in F$ and $y \in F$ imply $x \in F$.

THEOREM 2.5 ([8, 2]). *Let F be a nonempty subset of bounded BCK-algebra X . Then for all $x, y \in X$:*

- (i) *If F is a BCK-filter, $x \leq y$ and $x \in F$, then $y \in F$.*
- (ii) *If F is a crazy filter, $x \leq y$ and $x \in F$, then $y \in F$.*

PROPOSITION 2.6 ([6, 9]). *Let X be a bounded BCK-algebra with the greatest element 1. Then for all $x, y \in X$:*

- (i) $N1 = 0$ and $N0 = 1$.
- (ii) $NNx \leq x$.
- (iii) $Nx * Ny \leq y * x$.
- (iv) $y \leq x$ implies $Nx \leq Ny$.
- (v) $Nx * y = Ny * x$.
- (vi) $NNNx = Nx$.

LEMMA 2.7 ([4]). *Let X be a bounded BCK-algebra. Then for $x, y \in X$:*

- (i) *If the greatest lower bound $x \wedge y$ of x and y exists, then the least upper bound $Nx \vee Ny$ of Nx and Ny exists, and $Nx \vee Ny = N(x \wedge y)$.*
- (ii) *If X is involutory and the least upper bound $x \vee y$ exists, then the greatest lower bound $Nx \wedge Ny$ exists, and $Nx \wedge Ny = N(x \vee y)$.*

DEFINITION 2.8 ([3]). *Let X be a BCK-algebra. Then $(X; *, 0)$ is called a BCK-lattice, if $(X, \leq)$ is a lattice, where $\leq$ is the partial BCK-order on X .*

PROPOSITION 2.9 ([4]). *Let X be a BCK-algebra. Then for all $x, y, z \in X$:*

- (i) *if X is a BCK-lattice, then $x * (y \wedge z) = (x * y) \vee (x * z)$.*
- (ii) *if X is an involutory BCK-lattice, then $(x \vee y) * z = (x * z) \vee (y * z)$.*

THEOREM 2.10 ([2]). *Every BCK-filter is a crazy filter (but the converse does not hold in general).*

DEFINITION 2.11 ([11]). Let X be a BCK-lattice. Then X is called a PC-lattice if for all $x, y, z \in X$:

$$(z * x) * (y * x) = z * (x \vee y).$$

DEFINITION 2.12 ([10]). Let X be a bounded BCK-algebra. Then X is called an EQI-algebra if for all $x \in X$:

$$N(x * NNx) = 1.$$

THEOREM 2.13 ([10, 11, 9]). Let X be a bounded BCK-algebra. Then

- (i) Every commutative BCK-algebra is an involutory BCK-algebra.
- (ii) A bounded implicative BCK-algebra is a PC-lattice, but the converse does not hold in general.
- (iii) Every involutory BCK-algebra and PC-lattice is an EQI-algebra, but the converse does not hold in general.

DEFINITION 2.14 ([10]). Let I be an ideal of X . Then I is called an EQI-ideal of X if $NN(x * NNx) \in I$ for all $x \in X$.

3. EQI-filters in bounded BCK-algebras

LEMMA 3.1. Let X be a bounded BCK-algebra. Then for all $x, y, v \in X$:

- (i) $NN(NNx * y) = NNx * y$.
- (ii) $NN(Nx * y) = Nx * y$.
- (iii) $NN(Nx * v) * N(Nx * v) = ((Nx * x) * v) * v$.

PROOF: Let $x, y, v \in X$.

- (i) By Proposition 2.6 (ii), $(NN(NNx * y)) * (NNx * y) = 0$.

Moreover, we have

$$\begin{aligned}
 (NNx * y) * NN(NNx * y) & \stackrel{(I)}{=} (NNx * NN(NNx * y)) * y \\
 & \stackrel{(v)}{=} (NNN(NNx * y) * Nx) * y \\
 & \stackrel{\text{Prop. 2.6(vi)}}{=} (N(NNx * y) * Nx) * y \\
 & \stackrel{\text{Prop. 2.6(v)}}{=} (NNx * (NNx * y)) * y \\
 & \stackrel{(v)}{=} (NNx * y) * (NNx * y) \\
 & \stackrel{(\text{BCK-3})}{=} 0.
 \end{aligned}$$

Hence, by BCK-4, $NN(NNx * y) = NNx * y$.

(ii) By (i) and Proposition 2.6 (vi) we have

$$NN(Nx * y) = NN(NN(Nx) * y) = NN(Nx * y) = Nx * y. \quad (3.1)$$

(iii)

$$\begin{aligned}
 NN(Nx * v) * N(Nx * v) & \stackrel{(ii)}{=} (Nx * v) * N(Nx * v) \\
 & \stackrel{(I)}{=} (Nx * N(Nx * v)) * v \\
 & \stackrel{(I)}{=} (NN(Nx * v) * x) * v \\
 & \stackrel{(ii)}{=} ((Nx * v) * x) * v \\
 & \stackrel{(I)}{=} ((Nx * x) * v) * v. \quad \square
 \end{aligned}$$

DEFINITION 3.2. A BCK-filter F of X is called an *EQI-filter* if for all $x \in X$, $N(x * NNx) \in F$.

The following example shows that EQI-filters exist and that not every BCK-filter is an EQI-filter in general.

Example 3.3. Let $X = \{0, a, b, c, 1\}$ be a set, and the operation $*$ and the partial ordering $\leq$ on X be defined as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	a	a	0	0
1	1	b	a	b	0

(3.2)

Then $(X; *, 0)$ is a bounded BCK-algebra. It is easy to check that $\{1\}$ and $\{1, b\}$ are BCK-filters, $\{1, b\}$ is an EQI-filter, but $\{1\}$ is not an EQI-filter, because $N(c * NNc) = 1 * (c * (1 * (1 * c))) = 1 * (c * (1 * b)) = 1 * (c * a) = 1 * a = b \notin \{1\}$.

LEMMA 3.4. *Let X be a bounded BCK-algebra, F and E be BCK-filters with $F \subseteq E$. If F is an EQI-filter, then so is E .*

PROOF: The proof follows directly from the definition of an EQI-filter. $\square$

THEOREM 3.5. *Let X be a bounded BCK-algebra. Then the following statements are equivalent:*

- (i) $\{1\}$ is an EQI-filter.
- (ii) Every BCK-filter is an EQI-filter.
- (iii) X is an EQI-algebra.

PROOF: The theorem holds by Lemma 3.4, and Definition 2.12. $\square$

If we define $NF = \{Nx \mid x \in F\}$, then we have the following theorem.

THEOREM 3.6. *Let F be a nonempty subset of X . Then F is an EQI-filter if and only if NF is an EQI-ideal.*

PROOF: Let F be a nonempty subset of X . Then by Definition 2.14 and 3.2, for all $x \in X$ we have

$$\begin{aligned}
 F \text{ is an EQI-filter} &\Leftrightarrow N(x * NNx) \in F \\
 &\Leftrightarrow NN(x * NNx) \in NF \\
 &\Leftrightarrow NF \text{ is an EQI-ideal.} \quad \square
 \end{aligned}$$

THEOREM 3.7. *In any EQI-algebra, the concepts of BCK-filters, EQI-filters and crazy filters coincide.*

PROOF: Suppose that X is an EQI-algebra. Then by Theorem 3.5, the concepts of BCK-filters and EQI-filters coincide. By Theorem 2.10, it is sufficient to prove that any crazy filter is a BCK-filter. Hence, we must prove that if $y \in F$ and $N(Nx * Ny) \in F$, then $x \in F$. By Lemma 3.1, we have

$$\begin{aligned}
 N(Nx * Ny) * N(y * x) &\stackrel{\text{Lem. 3.1(ii)}}{=} NN[N(Nx * Ny) * N(y * x)] \\
 &\stackrel{(I)}{=} NN[NN(y * x) * (Nx * Ny)] \\
 &\stackrel{(II) \text{ and Prop. 2.6(ii)}}{\leq} NN[(y * x) * (Nx * Ny)] \\
 &\stackrel{(I)}{=} NN[(y * x) * (NNy * x)] \\
 &\stackrel{(IV)}{\leq} NN(y * NNy) \\
 &\stackrel{(\text{Def. 2.12})}{=} 0.
 \end{aligned}$$

Therefore by Theorem 2.5(ii), we have $N(y * x) \in F$, which implies that $x \in F$. $\square$

COROLLARY 3.8. Let X be a bounded BCK-algebra. Then in PC-lattices, involutory BCK-algebras, commutative BCK-algebras and implicative BCK-algebras, the concepts of BCK-filters, EQI-filters and crazy filters coincide.

PROOF: The corollary holds by Theorem 2.13, and Theorem 3.5. $\square$

THEOREM 3.9. *Let X be a bounded BCK-algebra and F a nonempty subset of X . Then in PC-lattices, involutory BCK-algebras, commutative BCK-algebras and implicative BCK-algebras, F is a BCK-filter if and only if NF is an ideal of X .*

PROOF: The theorem holds by Corollary 3.8, Theorem 3.6, Theorem 2.13 and Theorem 3.5. $\square$

4. Implicative BCK-filters in bounded BCK-algebras

DEFINITION 4.1. A nonempty subset F of X is said to be an *implicative BCK-filter* if $1 \in F$ and for all $x, y, z \in X$: $N((Nx * (Ny * Nx)) * Nz) \in F$ and $z \in F$ imply $x \in F$.

THEOREM 4.2. Any implicative BCK-filter of X is a BCK-filter, but the converse does not hold in general.

PROOF: Let F be an implicative BCK-filter of X , and suppose $N(Nx * Ny) \in F$ and $y \in F$. Then

$$\begin{aligned} N((Nx * (N1 * Nx)) * Ny) &= N((Nx * (0 * Nx)) * Ny) \\ &= N((Nx * 0) * Ny) = N(Nx * Ny) \in F. \end{aligned}$$

Hence by Definition 4.1, we have $x \in F$. □

The following example shows that the converse of Theorem 4.2 does not hold in general.

Example 4.3. Let $X = \{0, a, b, 1\}$ be a set, and define the operation $*$ and the partial ordering $\leq$ on X as follows:

$*$	0	a	b	1	1 b a 0	
0	0	0	0	0		
a	a	0	0	0		
b	b	b	0	0		
1	1	b	a	0		

(4.1)

Then $(X; *, 0)$ is a bounded BCK-algebra. It is easy to check that $\{1\}$ and $\{1, b\}$ are BCK-filters. $\{1, b\}$ is an implicative BCK-filter, but $\{1\}$ is not an implicative BCK-filter because $N((Nb * (N0 * Nb)) * N1) = N((a * (1 * a)) * 0) = N((a * b)) = N0 = 1 \in \{1\}$ and $1 \in \{1\}$, yet $b \notin \{1\}$.

LEMMA 4.4. Let F be a BCK-filter of X . Then the following statements are equivalent for all $x, y \in X$:

- (i) F is an implicative BCK-filter.

(ii) $N(Nx * (Ny * Nx)) \in F$ implies $x \in F$.

(iii) $N(Nx * x) \in F$ implies $x \in F$.

PROOF: (i) $\Rightarrow$ (ii): Let F be an implicative BCK-filter and $N(Nx * (Ny * Nx)) \in F$. We have $1 \in F$ and

$$\begin{aligned} N((Nx * (Ny * Nx)) * N1) &= N((Nx * (Ny * Nx)) * 0) \\ &= N(Nx * (Ny * Nx)) \in F. \end{aligned}$$

So by Definition 4.1, we get $x \in F$.

(ii) $\Rightarrow$ (iii): Assume $N(Nx * x) \in F$. Then by Proposition 2.6(vi) and (I),

$$\begin{aligned} N(Nx * (N0 * Nx)) &= N(Nx * (1 * Nx)) \\ &= N(Nx * NNx) \\ &= N(NNNx * x) \\ &= N(Nx * x) \in F. \end{aligned}$$

Therefore by (ii) we have $x \in F$.

(iii) $\Rightarrow$ (ii): Suppose $N(Nx * (Ny * Nx)) \in F$. Given that $Ny * Nx \leq 1 * Nx = NNx$, by (II) we have $Nx * NNx \leq Nx * (Ny * Nx)$, so $N(Nx * (Ny * Nx)) \leq N(Nx * NNx)$. Consequently $N(Nx * NNx) \in F$. Moreover, by (I) and Proposition 2.6(vi), $N(Nx * x) = N(NNNx * x) = N(Nx * NNx) \in F$, so by (iii) we obtain $x \in F$.

(ii) $\Rightarrow$ (i): This follows directly from the definition of a BCK-filter. $\square$

THEOREM 4.5. *Let F be a BCK-filter of X . Then the following are equivalent for all $x, y, z \in X$:*

(i) F is an implicative BCK-filter.

(ii) $N((Nx * Ny) * Ny) \in F$ implies $N(Nx * Ny) \in F$.

(iii) $N((Nx * Ny) * Nz) \in F$ implies $N((Nx * Nz) * (Ny * Nz)) \in F$.

PROOF: (i) $\Rightarrow$ (ii): Suppose F is an implicative BCK-filter and $N((Nx * Ny) * Ny) \in F$. We have

$$\begin{aligned}
(Nx * Ny) * Ny &\stackrel{\text{Prop. 2.6(ii) and (II)}}{\geq} NN(Nx * Ny) * Ny \\
&\stackrel{\text{(IV)}}{\geq} (Nx * Ny) * (Nx * NN(Nx * Ny)) \\
&\stackrel{\text{Prop. 2.6(ii) and (II)}}{\geq} NN(Nx * Ny) * (Nx * NN(Nx * Ny)).
\end{aligned}$$

Hence $N((Nx * Ny) * Ny) \leq N(NN(Nx * Ny) * (Nx * NN(Nx * Ny)))$. Given that F is a BCK-filter, $N(NN(Nx * Ny) * (Nx * NN(Nx * Ny))) \in F$. Consequently by Lemma 4.4 (ii), we have $N(Nx * Ny) \in F$.

(ii) $\Rightarrow$ (i): Let $N(Nx * x) \in F$. Then by (I),

$$N((NNx * (Nx * x)) * Nx) = N((NNx * Nx) * (Nx * x)).$$

Hence

$$\begin{aligned}
N((N(Nx * NNx) * NNx) * NNx) &\stackrel{\text{(I)}}{=} N((NNNx * (Nx * NNx)) * NNx) \\
&\stackrel{\text{Prop. 2.6(iv)}}{=} N((Nx * (Nx * NNx)) * NNx) \\
&\stackrel{\text{(I)}}{=} N((Nx * NNx) * (Nx * NNx)) \\
&= N(0) \\
&= 1 \in F.
\end{aligned}$$

Thus by (ii) we have $N((N(Nx * NNx) * NNx)) \in F$. On the other hand,

$$\begin{aligned}
N(Nx * NN(Nx * x)) &\stackrel{\text{Prop. 2.6(iv)}}{=} N(NNNx * NN(Nx * x)) \\
&\stackrel{\text{(I)}}{=} N(NNN(Nx * x) * NNx) \\
&\stackrel{\text{Prop. 2.6(iv)}}{=} N(N(Nx * x) * NNx) \\
&\stackrel{\text{Prop. 2.6(iv)}}{=} N(N(NNNx * x) * NNx) \\
&\stackrel{\text{(I)}}{=} N(N(Nx * NNx) * NNx).
\end{aligned}$$

Hence $N(Nx * NN(Nx * x)) \in F$. Given that F is a BCK-filter, we get $x \in F$. Therefore by Lemma 4.4, F is an implicative BCK-filter.

(ii) $\Rightarrow$ (iii): Suppose $N((Nx * Ny) * Nz) \in F$. Then

$$\begin{aligned}
 & (((Nx * (Ny * Nz)) * Nz) * Nz) * ((Nx * Ny) * Nz) \\
 & \stackrel{(IV)}{\leq} ((Nx * (Ny * Nz)) * Nz) * (Nx * Ny) \\
 & \stackrel{(I)}{\leq} ((Nx * (Ny * Nz)) * (Nx * Ny)) * Nz \\
 & \stackrel{(IV),(II)}{\leq} (Ny * (Ny * Nz)) * Nz \\
 & \stackrel{(I)}{=} (Ny * Nz) * (Ny * Nz) \\
 & = 0.
 \end{aligned}$$

It follows that $N((Nx * Ny) * Nz) \leq N(((Nx * (Ny * Nz)) * Nz) * Nz)$. Hence by Theorem 2.5, $N(((Nx * (Ny * Nz)) * Nz) * Nz) \in F$. Consequently by (ii),

$$N((Nx * Nz) * (Ny * Nz)) = N((Nx * (Ny * Nz)) * Nz) \in F.$$

(iii) $\Rightarrow$ (ii): Put $z = y$; the statement clearly holds. $\square$

THEOREM 4.6. *Let F and E be BCK-filters of X with $F \subseteq E$. If F is an implicative BCK-filter, then so is E .*

PROOF: Suppose F is an implicative BCK-filter, E is a BCK-filter, $F \subseteq E$, and $N((Nx * Ny) * Ny) \in E$. Set $U = (Nx * Ny) * Ny$. Then

$$\begin{aligned}
 (NN(Nx * U) * Ny) * Ny & \stackrel{\text{Lem. 3.1(ii)}}{=} ((Nx * U) * Ny) * Ny \\
 & \stackrel{(I)}{=} ((Nx * Ny) * U) * Ny \\
 & \stackrel{(I)}{=} ((Nx * Ny) * Ny) * U \\
 & = 0.
 \end{aligned}$$

Therefore $N((NN(Nx * U) * Ny) * Ny) = N0 = 1 \in F$. Now by Theorem 4.5 (ii), we have $N(NN(Nx * U) * Ny) \in F$. By Lemma 3.1, and Theorem 2.6 (vi), $N((Nx * Ny) * U) \in F$. On the other hand, by Proposition 2.6 (iii),

$$NN(Nx * Ny) * NNU \leq (Nx * Ny) * U.$$

Thus $N((Nx * Ny) * U) \leq N(NN(Nx * Ny) * NNU)$. Given that F is a BCK-filter, $N(NN(Nx * Ny) * NNU) \in F \subseteq E$. Because $NU \in E$ and E is a BCK-filter, we obtain $N(Nx * Ny) \in E$. By Theorem 4.5, E is an implicative BCK-filter. $\square$

THEOREM 4.7. *Let F be a BCK-filter of X . Then F is an implicative BCK-filter of X if and only if for every $a \in X$, the set*

$$B^F(a) = \{x \in X : N(Nx * Na) \in F\}$$

is the smallest BCK-filter containing F and a .

PROOF: Suppose that for every $a \in X$, $B^F(a)$ is a BCK-filter of X and $N((Nx * Ny) * Ny) \in F$. By Theorem 4.5, it suffices to show $N(Nx * Ny) \in F$. By Lemma 3.1, $N(NN(Nx * Ny) * Ny) = N((Nx * Ny) * Ny) \in F$, so $N(Nx * Ny) \in B^F(y)$. Given that $y \in B^F(y)$ and $B^F(y)$ is a BCK-filter, we have $x \in B^F(y)$. Therefore $N(Nx * Ny) \in F$.

Conversely, assume F is an implicative BCK-filter. If $N(Nx * Ny) \in B^F(y)$ and $x \in B^F(y)$, then by definition of $B^F(y)$ and Lemma 3.1, we obtain $N((Nx * Ny) * Ny) = N(NN(Nx * Ny) * Ny) \in F$. Hence by Theorem 4.5, $N(Nx * Ny) \in F$, which shows $x \in B^F(y)$. Thus $B^F(a)$ is a BCK-filter. Clearly $a \in B^F(a)$. Moreover, for any $x \in F$, using Lemma 3.1(ii), we have $N(NN(Nx * Na) * Nx) = N((Nx * Na) * Nx) = N((Nx * Nx) * Na) = N0 = 1 \in F$. Given that F is a BCK-filter, $N(Nx * Na) \in F$, so $x \in B^F(a)$. Hence $F \subseteq B^F(a)$. If E is any BCK-filter containing F and a , then for $x \in B^F(a)$ we have $N(Nx * Na) \in F \subseteq E$, and because $a \in E$ and E is a BCK-filter, we get $x \in E$. Therefore $B^F(a) \subseteq E$. $\square$

THEOREM 4.8. *Let X be an involutory BCK-algebra. Then*

- (i) $\{1\}$ is an implicative BCK-filter.
- (ii) Every BCK-filter is an implicative BCK-filter.
- (iii) For every $a \in X$, $B(a) = B^{\{1\}}(a)$ is an implicative BCK-filter.
- (iv) X is an implicative BCK-algebra.

PROOF: (i) $\Leftrightarrow$ (ii) follows directly from Theorem 4.6.

(ii) $\Rightarrow$ (iii): $\{1\}$ is a BCK-filter, so by (ii) it is an implicative BCK-filter. By Theorem 4.7, $B(a) = B^{\{1\}}(a)$ is a BCK-filter, and then by (ii) it is an implicative BCK-filter.

(iii) $\Rightarrow$ (iv): Given that X is involutory, $NNx = x$ for all $x \in X$. For $x, y \in X$, set $s = Nx$, $t = Ny$, so $x = Ns$, $y = Nt$. Put $u = N(x * (y * x))$. Then $N(Ns * (Nt * Ns)) = N(x * (y * x)) = u \in B(u)$, and $B(u)$ is an implicative BCK-filter. By Lemma 4.4(ii), $s \in B(u)$, hence $N(Ns * Nu) \in \{1\}$. Thus $N(x * (x * (y * x))) = N(x * NN(x * (y * x))) = N(Ns * Nu) = 1$, so $NN(x * (x * (y * x))) = x * (x * (y * x)) = 0$. Therefore $x \leq x * (y * x)$. Clearly $x * (y * x) \leq x$, so $x * (y * x) = x$. By Definition 2.2, X is an implicative BCK-algebra.

(iv) $\Rightarrow$ (i): Let $N(Nx * x) \in \{1\}$. By (iv), $x * Nx = x * (1 * x) = x$. Hence $N(Nx * x) = N(Nx * (x * Nx)) = N(Nx) = NNx = x \in \{1\}$, which implies $x = 1 \in \{1\}$. By Lemma 4.4, $\{1\}$ is an implicative BCK-filter. $\square$

THEOREM 4.9. *In EQI-algebras, involutory BCK-algebras and PC-lattices, F is an implicative BCK-filter if and only if NF is an implicative ideal.*

PROOF: Given that $\{1\}$ is a BCK-filter, by Theorem 3.9, we have $N\{1\} = \{0\}$, which is an ideal. The result then follows from Theorem 4.8 and Definition 2.3. $\square$

5. Commutative BCK-filters in bounded BCK-algebras

DEFINITION 5.1. A BCK-filter F of X is called a *commutative BCK-filter* if for all $x, y \in X$, $N(Nx * Ny) \in F$ implies $N(Nx * (Ny * (Ny * Nx))) \in F$.

THEOREM 5.2. *Let F be an implicative BCK-filter of X . Then F is a commutative BCK-filter, but the converse does not hold in general.*

PROOF: Suppose F is an implicative BCK-filter, $N(Nx * Ny) \in F$, and let $v = Ny * (Ny * Nx)$. Then

$$\begin{aligned}
& N(Nx * Ny) * N(NN(Nx * v) * N(Nx * v)) \\
\text{Prop. 2.6(iii)} \quad & \leq (NN(Nx * v) * N(Nx * v)) * (Nx * Ny) \\
\text{Lem. 3.1(iii)} \quad & = (((Nx * x) * v) * v) * (Nx * Ny) \\
& \stackrel{(I)}{=} (((Nx * x) * (Nx * Ny)) * v) * v \\
& \stackrel{(IV)}{\leq} ((Ny * x) * v) * v \\
& = ((Ny * x) * (Ny * (Ny * Nx))) * v \\
& \stackrel{(IV)}{\leq} ((Ny * Nx) * x) * v \\
& \stackrel{(I)}{=} ((NNx * y) * x) * v \\
& \stackrel{(I)}{=} ((NNx * x) * y) * v \\
& = 0 * v \\
& = 0.
\end{aligned}$$

Therefore $N(Nx * Ny) \leq N(NN(Nx * v) * N(Nx * v))$. Given that $N(Nx * Ny) \in F$, we have $N(NN(Nx * v) * N(Nx * v)) \in F$, and by Lemma 4.4, we obtain $N(Nx * v) = N(Nx * (Ny * (Ny * Nx))) \in F$. Hence F is a commutative BCK-filter. $\square$

The following example shows that the converse does not hold in general.

Example 5.3. Let $X = \{0, a, b, c, 1\}$ be a set, and define the operation $*$ and the partial ordering $\leq$ on X as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	b	0	0
1	1	c	b	a	0

$$\begin{array}{c} 1 \\ | \\ c \\ | \\ b \\ | \\ a \\ | \\ 0 \end{array} \quad (5.1)$$

Then $(X; *, 0)$ is a bounded BCK-algebra. It is easy to check that $\{1\}$ and $\{1, c\}$ are BCK-filters. $\{1, c\}$ is a commutative BCK-filter, but it is not an implicative BCK-filter because $N(Nb * b) = N(b * b) = N(0) = 1 \in \{1, c\}$ while $b \notin \{1, c\}$.

THEOREM 5.4. *Let X be an EQI-algebra. Then every commutative BCK-filter is an EQI-filter.*

PROOF: Let F be a commutative BCK-filter of the EQI-algebra X . Given that every commutative BCK-filter is a BCK-filter, Theorem 3.5 implies that F is an EQI-filter. $\square$

The following example shows that the converse of Theorem 5.4 does not hold in general.

In Example 5.3, it is easy to check that $\{1\}$ is an EQI-filter, but it is not a commutative BCK-filter because $N(Nc * Na) = N(a * c) = N0 = 1 \in \{1\}$, yet $N(Nc * (Na * (Na * Nc))) = N(a * (c * (c * a))) = N(a * (c * c)) = N(a * 0) = N(a) = c \notin \{1\}$.

THEOREM 5.5. *Let X be an involutory BCK-algebra and a PC-lattice. Then the concepts of crazy filters, BCK-filters, EQI-filters and commutative BCK-filters coincide.*

PROOF: Assume X is an involutory BCK-algebra and a PC-lattice. By Corollary 3.8, the concepts of crazy filters, BCK-filters and EQI-filters coincide. Thus it suffices to prove that every BCK-filter is a commutative BCK-filter. Let F be a BCK-filter and $N(Nx * Ny) \in F$. Then

$$\begin{aligned}
N(Nx * (Ny * (Ny * Nx))) & \stackrel{(I)}{=} N(Nx * (Ny * (NNx * y))) \\
& \stackrel{\text{Def. 2.2(iii)}}{=} N(Nx * ((1 * y) * (NNx * y))) \\
& \stackrel{\text{Def. 2.11}}{=} N(Nx * (1 * (NNx \vee y))) \\
& \stackrel{\text{Def. 2.2(iii)}}{=} N(Nx * (N(NNx \vee y))) \\
& \stackrel{\text{Lem. 2.7(ii)}}{=} N(Nx * (NNNx \wedge Ny)) \\
& \stackrel{\text{Prop. 2.6(vi)}}{=} N(Nx * (Nx \wedge Ny)) \\
& \stackrel{\text{Prop. 2.9(i)}}{=} N((Nx * Nx) \vee (Nx * Ny)) \\
& = N(Nx * Ny) \in F.
\end{aligned}$$

Therefore, by Definition 5.1, F is a commutative BCK-filter. $\square$

LEMMA 5.6. *Let F and E be BCK-filters of X with $F \subseteq E$. If F is a commutative BCK-filter, then so is E .*

PROOF: Assume that F is a commutative BCK-filter, E is a BCK-filter, $F \subseteq E$, and $u = N(Nx * Ny) \in E$. By (I) and (BCK-3) we have

$$N((Nx * (Nx * Ny)) * Ny) = N((Nx * Ny) * (Nx * Ny)) = N(0) = 1 \in F.$$

Hence by the definition of a commutative BCK-filter,

$$N[(Nx * (Nx * Ny)) * (Ny * (Ny * (Nx * (Nx * Ny))))] = v \in F \subseteq E.$$

Therefore

$$\begin{aligned}
& N[Nx * (Ny * (Ny * (Nx * (Nx * Ny))))] * NN(Nx * Ny) \\
& \stackrel{\text{Lem. 3.1(ii)}}{=} N[Nx * (Ny * (Ny * (Nx * (Nx * Ny))))] * (Nx * Ny) \\
& \stackrel{(I)}{=} N(Nx * (Nx * Ny)) * (Ny * (Ny * (Nx * (Nx * Ny)))) \\
& = v \in E.
\end{aligned}$$

Thus by Definition 2.4 (i) we obtain

$$N(Nx * (Ny * (Ny * (Nx * (Nx * Ny)))) \in E.$$

On the other hand, by Proposition 2.6 (iii), (IV) and (BCK-3),

$$N(Nx * (Ny * (Ny * (Nx * (Nx * Ny)))) * N(Nx * (Ny * (Ny * Nx)))$$

Prop. 2.6(iii)

$$\leq (Nx * (Ny * (Ny * Nx))) * (Nx * (Ny * (Ny * (Nx * (Nx * Ny))))$$

$$\stackrel{(IV)}{\leq} (Ny * (Ny * (Nx * (Nx * Ny)))) * (Ny * (Ny * Nx))$$

$$\stackrel{(IV)}{\leq} (Ny * Nx) * (Ny * (Nx * (Nx * Ny)))$$

$$\stackrel{(IV)}{\leq} (Nx * (Nx * Ny)) * Nx$$

$$\stackrel{(I)}{\leq} (Nx * Nx) * (Nx * Ny)$$

$$\stackrel{(BCK-3)}{\leq} 0.$$

Hence by Theorem 2.5 (i), $N(Nx * (Ny * (Ny * Nx))) \in E$, and by Definition 5.1 the theorem holds. $\square$

THEOREM 5.7. *Let X be a bounded BCK-algebra. Then the following statements are equivalent for all $x, y \in X$:*

- (i) $\{1\}$ is a commutative BCK-filter.
- (ii) Every BCK-filter is a commutative BCK-filter.
- (iii) $Nx * (Nx * Ny) = Ny * (Ny * Nx)$.
- (iv) $Ny * (Nx * (Nx * Ny)) = Ny * Nx$.

PROOF: (i) $\Leftrightarrow$ (ii): This follows directly from Lemma 5.6.

(ii) $\Rightarrow$ (iii): Suppose that $\{1\}$ is a commutative BCK-filter and let $v = N(Nx * (Nx * Ny))$. Then

$$\begin{aligned}
N(Nv * Ny) &= N(NN(Nx * (Nx * Ny)) * Ny) \\
&\stackrel{\text{Lem. 3.1(ii)}}{=} N((Nx * (Nx * Ny)) * Ny) \\
&\stackrel{(I)}{=} N((Nx * Ny) * (Nx * Ny)) \\
&\stackrel{(\text{BCK-3})}{=} N(0) \\
&\stackrel{(\text{BCK-3})}{=} 1 \in \{1\}.
\end{aligned}$$

Therefore, by the definition of a commutative BCK-filter,

$$N(Nv * (Ny * (Ny * Nv))) \in \{1\}. \quad (\text{a})$$

By Lemma 3.1 (ii), $Nv \leq Nx$. Hence by (II) we have $Ny * Nx \leq Ny * Nv$. For the same reason, $Ny * (Ny * Nv) \leq Ny * (Ny * Nx)$ and $Nv * (Ny * (Ny * Nx)) \leq Nv * (Ny * (Ny * Nv))$. Consequently,

$$N(Nv * (Ny * (Ny * Nv))) \leq N(Nv * (Ny * (Ny * Nx))).$$

Thus by (a) and Theorem 2.5 (i), $N(Nv * (Ny * (Ny * Nx))) \in \{1\}$. Hence $NN(Nv * (Ny * (Ny * Nx))) = Nv * (Ny * (Ny * Nx)) = 0$, and therefore $Nx * (Nx * Ny) = NN(Nx * (Nx * Ny)) = Nv \leq Ny * (Ny * Nx)$. Similarly we obtain $Ny * (Ny * Nx) \leq Nx * (Nx * Ny)$, so $Ny * (Ny * Nx) = Nx * (Nx * Ny)$.

(iii) $\Rightarrow$ (iv): By (III) and the assumption,

$$Ny * (Nx * (Nx * Ny)) = Ny * (Ny * (Ny * Nx)) = Ny * Nx.$$

(iv) $\Rightarrow$ (ii): Suppose F is a BCK-filter and $N(Nx * Ny) \in F$. Then by (iv), $N(Nx * (Ny * (Ny * Nx))) = N(Nx * Ny) \in F$, which completes the proof. $\square$

THEOREM 5.8. *Let X be an involutory BCK-algebra. Then the following statements are equivalent for all $x, y \in X$:*

- (i) $\{1\}$ is a commutative BCK-filter.
- (ii) Every BCK-filter is a commutative BCK-filter.

$$(iii) \quad Ny * (Ny * Nx) = Nx * (Nx * Ny).$$

$$(iv) \quad Nx * (Ny * (Ny * Nx)) = Nx * Ny.$$

$$(v) \quad X \text{ is a commutative BCK-algebra.}$$

PROOF: The theorem follows from Theorem 5.7 and Definition 2.2 (ii). $\square$

6. Conclusion

To investigate and characterize the structure of logical systems, the concepts of ideals and filters are very important. In this research we introduced and studied EQI-filters, implicative and commutative BCK-filters in bounded BCK-algebras. We determined the relations between these BCK-filters and surveyed them in subclasses of bounded BCK-algebras such as EQI-algebras and involutory BCK-algebras.

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APPLICATIONS OF COMMUTATORS IN PSEUDO-BL-ALGEBRAS

Abstract

In this paper, the concepts of commutativity relation, commutant of a subset, center and solvable pseudo-BL-algebras are defined and their properties are investigated. The notion of n -Engel pseudo-BL-algebras as a natural generalization of BL-algebras is introduced, and we discuss Engel pseudo-BL-algebra, which is defined by left and right normed commutators.

Keywords: pseudo-BL-algebra, commutativity relation, commutator, solvable, Engel element.

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1. Introduction

BL-algebras were introduced by Hájek in 1998, an algebraic semantics of basic fuzzy logic which is generated by continuous t-norms on the interval $[0,1]$ and their residuals [4]. Georgescu et al introduced pseudo-BL-algebra as a non-commutative extension of BL-algebra [7, 8]. The idea of pseudo-BL-algebra originates not only in logic and algebra, but also in algebraic properties that come from the syntax of certain non-classical propositional logics and intuitionistic logic. Main properties of pseudo-BL-algebras were studied in [7, 8]. Pseudo-BL-algebras correspond to a pseudo-basic fuzzy logic (see [2]). It should be pointed out that the difference between BL-algebras and pseudo-BL-algebras is in the commutativity of the $\odot$ operation and therefore, in this paper, we intend to focus on this topic and investigate this difference. We present a definition for the notion of commutator in pseudo-BL-algebras. We define also the notion of Engel pseudo-BL-algebras based on commutators, give several characterizations of it and prove that the class of commutative pseudo-BL-algebras and 1-Engel pseudo-BL-algebras are equal.

2. Preliminaries

In this section, we recall the concept of a pseudo-BL-algebra and then present some definitions and properties which we will need in the next sections.

DEFINITION 2.1. A BL-algebra is an algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following [4]:

- (BL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
- (BL2) $(A, \odot, 1)$ is a commutative monoid;
- (BL3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $a \odot b \leq c$ iff $a \leq b \rightarrow c$;
- (BL4) $a \wedge b = a \odot (a \rightarrow b)$;
- (BL5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$; For all $a, b, c \in A$.

DEFINITION 2.2. A pseudo-BL-algebra is an algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following [7]:

- (PSBL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
- (PSBL2) $(A, \odot, 1)$ is a monoid;
- (PSBL3) $a \odot b \leq c$ iff $a \leq b \rightarrow c$ iff $b \leq a \rightsquigarrow c$, for all $a, b, c \in A$;
- (PSBL4) $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$;
- (PSBL5) $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$, for all $a, b \in A$.

It is proved in [7] that commutative pseudo-BL-algebras are BL-algebras. A pseudo-BL-algebra A is nontrivial if and only if $0 \neq 1$.

For any $a \in A$, we define $a^- = a \rightarrow 0$ and $a^\sim = a \rightsquigarrow 0$. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \mathbb{N}$.

The following properties hold in any pseudo-BL-algebra A and will be used in the sequel. See [7] for details.

- (psbl-c₁) $a \leq b$ iff $a \rightarrow b = 1$ iff $a \rightsquigarrow b = 1$;
- (psbl-c₂) $a \rightarrow a = a \rightsquigarrow a = 1$;
- (psbl-c₃) $a \rightarrow 1 = a \rightsquigarrow 1 = 1$;
- (psbl-c₄) $1 \rightarrow a = 1 \rightsquigarrow a = a$;
- (psbl-c₅) $a \odot b \leq a \wedge b \leq a, b$;
- (psbl-c₆) $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c$;
- (psbl-c₇) $a \rightsquigarrow (b \rightsquigarrow c) = (b \odot a) \rightsquigarrow c$;
- (psbl-c₈) $a \odot 0 = 0 \odot a = 0$;
- (psbl-c₉) $a \odot 1 = 1 \odot a = a$;
- (psbl-c₁₀) $a \odot a^\sim = a^- \odot a = 0$;
- (psbl-c₁₁) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;

(psbl-c₁₂) $a \leq b$ implies $b^- \leq a^-$ and $b^\sim \leq a^\sim$;

(psbl-c₁₃) $a \odot b = 0$ iff $a \leq b^\sim$ iff $b \leq a^-$;

(psbl-c₁₄) $a \leq a^{-\sim}$ and $a \leq a^{\sim-}$;

(psbl-c₁₅) $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$ and $c \odot (a \wedge b) = (c \odot a) \wedge (c \odot b)$;

(psbl-c₁₆) $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$ and $c \odot (a \vee b) = (c \odot a) \vee (c \odot b)$;

(psbl-c₁₇) $(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$ and $a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$;

(psbl-c₁₈) $(\bigwedge_{i \in I} b_i) \odot a = \bigwedge_{i \in I} (b_i \odot a)$ and $a \odot (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \odot b_i)$.

(psbl-c₁₉) if $a \vee b = 1$, then $a \odot b = a \wedge b$.

(psbl-c₂₀) if $a \vee b = 1$, then for each $n \geq 1$, $a^n \vee b^n = 1$.

(psbl-c₂₁) $(a \wedge b)^\sim = a^\sim \vee b^\sim$, $(a \wedge b)^- = a^- \vee b^-$.

(psbl-c₂₂) $(a \odot b)^- = a \rightarrow b^-$, $(a \odot b)^\sim = b \rightsquigarrow a^\sim$.

Let A be a pseudo-BL-algebra. A filter of A is a nonempty subset F of A such that for all $a, b \in A$ [8]

i) if $a, b \in F$, then $a \odot b \in F$; ii) if $a \in F$ and $a \leq b$, then $b \in F$.

A filter F of A is proper if $F \neq A$. Let $X \subseteq A$. The filter of A generated by X will be denoted by $\langle X \rangle$. We have that $\langle X \rangle = \{a \in A : x_1 \odot x_2 \odot \dots \odot x_n \leq a \text{ for some } n \in N \text{ and some } x_1, x_2, \dots, x_n \in X\}$. For any $a \in A$, $\langle a \rangle$ denotes the principal filter of A generated by $\{a\}$. Then, $\langle a \rangle = \{b \in A : b^n \leq a \text{ for some } n \in N\}$. If F is a filter and $a \notin F$, we denote by $F(a) = \langle F \cup \{a\} \rangle$. A filter H of A is called normal ([8]) if for every $a, b \in A$ we have the equivalence: $a \rightarrow b \in H$ iff $a \rightsquigarrow b \in H$. With any normal filter H of A we can associate a congruence relation $\equiv_H$ on A by defining $a \equiv_H b$ if and only if $(a \rightarrow b) \odot (b \rightarrow a) \in H$ if and only if $(a \rightsquigarrow b) \odot (b \rightsquigarrow a) \in H$. In [8] it is proved that the map $H \rightarrow \equiv_H$ is an isomorphism between the lattice of normal filters of A and the lattice of congruences of A . If we denote by A/H the quotient set $A / \equiv_H$, then A/H becomes a pseudo-BL-algebra A/H with the natural operations

induced from those of A . The function $f : A \longrightarrow B$ is a homomorphism of A to pseudo-BL-algebra B iff it satisfies the following conditions, for every $a, b \in A$:

$f(0) = 0$, $f(a \odot b) = f(a) \odot f(b)$, $f(a \rightarrow b) = f(a) \rightarrow f(b)$, $f(a \rightsquigarrow b) = f(a) \rightsquigarrow f(b)$. It follows that, $f(1) = 1$, $f(a \wedge b) = f(a) \wedge f(b)$, $f(a \vee b) = f(a) \vee f(b)$.

For any pseudo-BL-algebra A , let us denote

$$G(A) = \{a \in A : a \odot a = a\}$$

$$M(A) = \{a \in A : a = (a^-)^\sim = (a^\sim)^-\}$$

Recall that an element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. The set of the complemented elements of the bounded lattice reduct of A is called the Boolean center of A and is denoted by $B(A)$. Clearly $0, 1 \in B(A)$. The elements of $B(A)$ are called Boolean elements of A . It is known that $B(A)$ is a Boolean algebra, with the operation induced by those of A together with the complementation operation given by the negation in A . Also it is straightforward that $B(A)$ is a subalgebra of the pseudo-BL-algebra A . In [2] it proved that $B(A) = M(A) \cap G(A)$.

DEFINITION 2.3 ([7]). A pseudo-BL-algebra A is commutative iff $a \odot b = b \odot a$, for any $a, b \in A$.

THEOREM 2.4 ([7]). A pseudo-BL-algebra A is commutative iff $a \rightarrow b = a \rightsquigarrow b$, for any $a, b \in A$. Any commutative pseudo-BL-algebra A is a BL-algebra.

From now on, $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ or simply A is a pseudo-BL-algebra, unless otherwise specified.

3. Commutativity and the Center in Pseudo-BL-Algebras: Definitions and Properties

In this section, we introduce and study the commutativity relation in A . We define central elements and the center of A , and show that the center of A is a filter of A .

DEFINITION 3.1. For elements $a, b \in A$, if $a \odot b = b \odot a$, we write aCb and say that a commutes with b . The relation C is called the commutativity relation of A . It can be observe that the relation C is reflexive and symmetric, but not necessarily transitive. Furthermore, in every commutative pseudo-BL-algebra A (and in any BL-algebra A), aCb holds for all $a, b \in A$.

The following simple example of a non-commutative pseudo-BL-algebra will be used to characterize our concepts.

Example 3.2. Let $a, b, c, d \in R$, where R is the set of all real numbers. Define the order relation $\leq$ on R^2 by

$$(a, b) \leq (c, d) \Leftrightarrow a < c \text{ or } (a = c \text{ and } b \leq d).$$

For any $x, y \in R^2$, we define the operations $\vee$ and $\wedge$ as follows:
 $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Let

$$A = \{(\frac{1}{2}, b) \in R^2 : b \geq 0\} \cup \{(a, b) \in R^2 : \frac{1}{2} < a < 1\} \cup \{(1, b) \in R^2 : b \leq 0\}$$

For $(a, b), (c, d) \in A$, consider the operations

$$(a, b) \odot (c, d) = (\frac{1}{2}, 0) \vee (ac, bc + d)$$

$$(a, b) \rightarrow (c, d) = (\frac{1}{2}, 0) \vee [(\frac{c}{a}, \frac{d-b}{a}) \wedge (1, 0)]$$

$$(a, b) \rightsquigarrow (c, d) = (\frac{1}{2}, 0) \vee [(\frac{c}{a}, \frac{ad-bc}{a}) \wedge (1, 0)]$$

Then $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ is a pseudo-BL-algebra.

If, in this algebra, we choose $x = (1, -1)$ and $y = (1, -2)$, by routine calculations we obtain $(1, -1) \odot (1, -2) = (\frac{1}{2}, 0) \vee (1, -3) = (1, -3)$ and $(1, -2) \odot (1, -1) = (\frac{1}{2}, 0) \vee (1, -3) = (1, -3)$. Thus, xCy .

Also, if $x = (\frac{1}{2}, 2)$ and $y = (1, -2)$, then $(\frac{1}{2}, 2) \odot (1, -2) = (\frac{1}{2}, 0) \vee (\frac{1}{2}, 0) = (\frac{1}{2}, 0)$, and $(1, -2) \odot (\frac{1}{2}, 2) = (\frac{1}{2}, 0) \vee (\frac{1}{2}, 1) = (\frac{1}{2}, 1)$. Hence, x does not commute with y .

LEMMA 3.3. *Let a and b be elements of A . Then*

1. aCa , $aC0$, and $aC1$.
2. aCb if and only if bCa .
3. If $a \vee b = 1$, then a^nCb^n for each positive integer n .
4. If $a \wedge b = 0$, then $a^{\sim}Cb^{\sim}$ and $a^{-}Cb^{-}$.

PROOF: 1. These properties are obvious by psbl- c_8 and psbl- c_9 .

2. aCb if and only if $a \odot b = b \odot a$. This is equivalent to $b \odot a = a \odot b$, which by definition means bCa .

3. By psbl- c_{19} , if $a \vee b = 1$, then $a \odot b = a \wedge b = b \wedge a = b \odot a$, which implies aCb . Now, assume $a \vee b = 1$. By psbl- c_{20} , we have $a^n \vee b^n = 1$. Therefore, a^nCb^n for every $n \geq 1$.

4. If $a \wedge b = 0$, then $(a \wedge b)^{\sim} = 0^{\sim} = 1$. By psbl- c_{21} , $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$, so $a^{\sim} \vee b^{\sim} = 1$. Consequently, $a^{\sim}Cb^{\sim}$. Also, $1 = 0^{-} = (a \wedge b)^{-} = a^{-} \vee b^{-}$. Thus, $a^{-}Cb^{-}$. $\square$

THEOREM 3.4. *Let a, b, c be elements of A . If a commutes with b and c , then a also commutes with $b \odot c$, $b \wedge c$ and $b \vee c$.*

PROOF: Assume aCb and aCc . This means $a \odot b = b \odot a$ and $a \odot c = c \odot a$. We need to show that $a \odot (b \odot c) = (b \odot c) \odot a$, $a \odot (b \wedge c) = (b \wedge c) \odot a$ and $a \odot (b \vee c) = (b \vee c) \odot a$.

We know that $\odot$ is associative, then $a \odot (b \odot c) = (a \odot b) \odot c = (b \odot a) \odot c = b \odot (a \odot c) = b \odot (c \odot a) = (b \odot c) \odot a$, which means $aC(b \odot c)$.

Using psbl- c_{15} , we have $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$. Since aCb and aCc , we can substitute $b \odot a$ for $a \odot b$ and $c \odot a$ for $a \odot c$. So $(a \odot b) \wedge (a \odot c) = (b \odot a) \wedge (c \odot a)$. By the commutative property of the operation $\wedge$ with respect to $\odot$ (which is implied by the properties of pseudo-BL-algebras, specifically how $\wedge$ interacts with $\odot$), we get $(b \odot a) \wedge (c \odot a) = (b \wedge c) \odot a$. Thus, $a \odot (b \wedge c) = (b \wedge c) \odot a$, which means $aC(b \wedge c)$.

Similarly, using psbl- c_{16} we deduce $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$. Substituting as before $(a \odot b) \vee (a \odot c) = (b \odot a) \vee (c \odot a)$. Again, due to the

interaction between $\vee$ and $\odot$ we have $(b \odot a) \vee (c \odot a) = (b \vee c) \odot a$. Thus, $a \odot (b \vee c) = (b \vee c) \odot a$, which means $aC(b \vee c)$. $\square$

Remark 3.5. An immediate question arises: Given two elements a and b of a pseudo-BL-algebra, how can we determine if they commute? While direct computation using the definition is possible, it is not always the most efficient method. Instead, we can leverage properties such as, if $a \vee b = 1$, then aCb , and if $a \wedge b = 0$, then $a \sim Cb \sim$ and a^-Cb^- . Our aim is to discover and utilize such rules to streamline the commutativity check.

It is also beneficial to be able to compute the commutant of a subset of a pseudo-BL-algebra, as well as its individual elements.

DEFINITION 3.6. If M is a subset of A , the set

$$\{a \in A : aCm \ \forall m \in M\}$$

is called the commutant of M in A and is denoted by $C(M)$. The elements of $C(A)$ are referred to as central elements, and $C(A)$ is called the center of A .

By utilizing $\text{psbl-}c_8$ and $\text{psbl-}c_9$, we deduce that $0, 1 \in C(A)$. We use the notation $C(a)$ to denote the commutant of the singleton set $\{a\}$. Obvious, $0, 1, a \in C(a)$.

Example 3.7. Consider the pseudo-BL-algebra defined in Example 4.8. If $M = \{(1, -3), (1, -2)\}$, then through routine calculations, we find that $C(M) = \{(1, 0)\}$. For $x = (\frac{1}{2}, 2)$, we have $C(x) = \{(a, 4 - 4a) : 1 < a < 2\}$. This is because $(\frac{1}{2}, 2) \odot (a, 4 - 4a) = (\frac{1}{2}, 0) \vee (\frac{a}{2}, 4 - 2a)$ and $(a, 4 - 4a) \odot (\frac{1}{2}, 2) = (\frac{1}{2}, 0) \vee (\frac{a}{2}, 4 - 2a)$. Thus, $(2, 3)$ commutes with $(a, 4 - 4a)$. Furthermore, for any $(a, b) \in A$, $(a, b)C(\frac{1}{2}, 2)$ if and only if $b = 4 - 4a$.

PROPOSITION 3.8. Let $a, b \in C(A)$. Then

1. $a \odot b \in C(A)$.
2. $a \wedge b \in C(A)$.
3. $a \vee b \in C(A)$.

PROOF: 1. Let $a, b \in C(A)$. Then $a \odot x = x \odot a$ and $b \odot x = x \odot b$ for each $x \in A$. Hence, $(a \odot b) \odot x = a \odot (b \odot x) = a \odot (x \odot b) = (a \odot x) \odot b = (x \odot a) \odot b = x \odot (a \odot b)$. Since $(a \odot b) \odot x = x \odot (a \odot b)$ for all $x \in A$, we conclude that $a \odot b \in C(A)$.

2. By psbl-c₁₅, we have $(a \wedge b) \odot x = (a \odot x) \wedge (b \odot x) = (x \odot a) \wedge (x \odot b) = x \odot (a \wedge b)$. Thus, $(a \wedge b) \odot x = x \odot (a \wedge b)$ for all $x \in A$, and hence $a \wedge b \in C(A)$.

3. By using psbl-c₁₆, we get $(a \vee b) \odot x = (a \odot x) \vee (b \odot x) = (x \odot a) \vee (x \odot b) = x \odot (a \vee b)$. Thus, $(a \vee b) \odot x = x \odot (a \vee b)$ for all $x \in A$, and hence $a \vee b \in C(A)$.

□

According to Proposition 3.8, the center of A is a subalgebra of A , but it is not a filter of A , in general. For instance, in Example 4.8, we have $C(A) = \{(\frac{1}{2}, 0), (1, 0)\}$ and $a = (\frac{1}{2}, 0) \leq (\frac{1}{2}, 2) = b$, and $a \in C(A)$ but $b \notin C(A)$.

LEMMA 3.9. If $a \in C(A)$, then $a^- = a^\sim$.

PROOF: If $a \in C(A)$, then $a \odot b = b \odot a$ for all $b \in A$. Thus, $a \odot a^- = a^- \odot a$ and $a \odot a^\sim = a^\sim \odot a$. By psbl-c₁₀, we obtain $a \odot a^- = a^\sim \odot a = 0$, so $(a \odot a^-)^\sim = 0^\sim = 1$ and $(a^\sim \odot a)^- = 0^- = 1$. Using psbl-c₂₁, we obtain $a^\sim \rightarrow a^- = 1$ and $a^- \rightsquigarrow a^\sim = 1$, which means $a^\sim \leq a^-$ and $a^- \leq a^\sim$. Hence, $a^- = a^\sim$. □

If we choose $a = (1, -1)$ in the pseudo-BL-algebra defined in Example 4.8, then we have $a^- = a^\sim = (\frac{1}{2}, 1)$. By routine calculus we obtain $(1, -1) \odot (\frac{1}{2}, 2) = (\frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{2}, 2) \odot (1, -1) = (\frac{1}{2}, 1)$. Thus $a \notin C(A)$. Therefore the converse of Lemma 3.9 is not true, generally.

PROPOSITION 3.10. Let H be a normal filter of A and $a \in A \setminus H$. If $a \in C(A)$, then

- (i) $\langle a \rangle = \{x \in A : a \leq x\}$ is a normal filter of A .
- (ii) $H(a)$ is a normal filter of A , where $H(a) = \langle H \cup \{a\} \rangle$.

PROOF: (i) Suppose that $a \in C(A)$. If $x, y \in A$ such that $x \rightarrow y \in \langle a \rangle$, then

$$\begin{aligned}
a \leq x \rightarrow y &\iff a \odot x \leq y \\
&\iff x \odot a \leq y \\
&\iff a \leq x \rightsquigarrow y \\
&\iff x \rightsquigarrow y \in \langle a \rangle.
\end{aligned}$$

That is, $\langle a \rangle$ is a normal filter of A .

(ii) Since $a \in C(A)$, $a \odot b = b \odot a$ for all $b \in A$. Let $x, y \in A$ such that $x \rightarrow y \in H(a)$. Then there exists $h \in H$ such that

$$\begin{aligned}
h \odot a \leq x \rightarrow y &\iff (h \odot a) \odot x \leq y \\
&\iff h \odot (a \odot x) \leq y \\
&\iff h \odot (x \odot a) \leq y \\
&\iff (h \odot x) \odot a \leq y.
\end{aligned}$$

Since $h \in H$ and H is a normal filter, there exists $h' \in H$ such that $x \odot h' = h \odot x$. We obtain

$$(x \odot h') \odot a \leq y \iff x \odot (h' \odot a) \leq y \iff h' \odot a \leq x \rightsquigarrow y.$$

Therefore, $x \rightsquigarrow y \in H(a)$. Analogously, $x \rightsquigarrow y \in H(a)$ implies that $x \rightarrow y \in H(a)$. Hence, $H(a)$ is a normal filter of A . $\square$

PROPOSITION 3.11. $B(A) \subseteq G(A) \subseteq C(A)$.

PROOF: We know that $B(A) = M(A) \cap G(A)$, hence $B(A) \subseteq G(A)$. If $a \in G(A)$, then $a \odot b = a \wedge b = b \wedge a = b \odot a$ for all $b \in A$. Therefore, aCb , for all $b \in A$. Hence, $a \in C(A)$. $\square$

Remark 3.12. Since in any BL-algebra, the $\odot$ operation is commutative, $C(A) = A$ for any BL-algebra A . However, there exist BL-algebras A such that $G(A)$ is a proper subset of A . Hence, $G(A) \neq C(A)$ in general.

PROPOSITION 3.13. The algebraic structure $(C(A), \wedge, \vee, \odot, \rightarrow, 0, 1)$ forms a BL-algebra.

PROOF: $(C(A), \wedge, \vee, 0, 1)$ is a bounded lattice. From the properties of pseudo-BL-algebras and the definition of the center, it is known that $0, 1 \in$

$C(A)$. Furthermore, if $x, y \in C(A)$, then $x \wedge y \in C(A)$ and $x \vee y \in C(A)$. This implies that $C(A)$ inherits the lattice structure and is bounded by 0 and 1.

$(C(A), \odot, 1)$ is a commutative monoid. For any $x, y, z \in C(A)$, since $x, y, z \in A$, we have $(x \odot y) \odot z = x \odot (y \odot z)$ because $\odot$ is associative in A . $1 \in C(A)$ by definition of the center. For any $x \in C(A)$, $x \odot 1 = x$ because 1 is the identity element in A . For any $x, y \in C(A)$, by the definition of the center $C(A)$, x commutes with all elements of A . Therefore, x commutes with y , which means $x \odot y = y \odot x$. This holds for all $x, y \in C(A)$. Thus, $(C(A), \odot, 1)$ is a commutative monoid. The operations $\odot$ and $\rightarrow$ form an adjoint pair. Since this property holds for all elements in A , it certainly holds for all elements in $C(A)$. $x \wedge y = x \odot (x \rightarrow y)$ for all $x, y \in C(A)$. Since $x, y \in C(A)$, they are also elements of A . The property $x \wedge y = x \odot (x \rightarrow y)$ holds in A , then this property holds for elements in $C(A)$. $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for all $x, y \in C(A)$. Similar to the previous point, since $x, y \in C(A)$, they are elements of A . The property $(x \rightarrow y) \vee (y \rightarrow x) = 1$ holds in A . Given these points, especially the commutativity inherited from the definition of the center, $(C(A), \wedge, \vee, \odot, \rightarrow, 0, 1)$ fulfills all the requirements of a BL-algebra. $\square$

LEMMA 3.14. *Suppose that H is a normal filter of A and $a, b \in A$. Then:*

1. $C(0) = C(1) = A$.
2. $a \in C(b)$ if and only if $b \in C(a)$.
3. $C(a)/H \subseteq C(a/H)$.

PROOF:

1. Since $aC0$ and $aC1$ for all $a \in A$, $C(0) = C(1) = A$.
2. $a \in C(b)$ if and only if aCb if and only if bCa if and only if $b \in C(a)$.
3. Let $b/H \in C(a)/H$. Then $b \in C(a)$ and hence bCa . Therefore, $b \odot a = a \odot b$. Then $a/H \odot b/H = (a \odot b)/H = (b \odot a)/H = b/H \odot a/H$, so $a/HCb/H$. Hence, $b/H \in C(a/H)$.

$\square$

THEOREM 3.15. *Suppose that $a, b \in A$. Then*

1. $a \in C(A)$ if and only if $C(a) = A$.
2. $C(A) = \bigcap_{a \in A} C(a)$.
3. $C(a) \subseteq C(b)$ if and only if $b \in C(C(a))$.
4. $C(A) \subseteq C(a)$ for every $a \in A$.
5. A is commutative if and only if $C(a) = A$, for every $a \in A$.
6. $C(a) = C(b)$ if and only if $C(C(a)) = C(C(b))$.

PROOF:

1. Let $a \in C(A)$. Then for all $b \in A$, we obtain aCb . Let $x \in A$, then aCx . Therefore, $x \in C(a)$, that is, $A \subseteq C(a)$. We also have $C(a) \subseteq A$. Hence, $C(a) = A$.
Conversely, let $C(a) = A$. Since $C(a) = \{b \in A : aCb\} = A$, it follows that $a \in C(A)$.
2. Let $a \in C(A)$. Then aCb for every $b \in A$. So $b \in C(a)$ for every $a \in A$. That means $b \in \bigcap_{a \in A} C(a)$. Therefore, $C(A) \subseteq \bigcap_{a \in A} C(a)$.
Conversely, let $b \in \bigcap_{a \in A} C(a)$. Then $b \in C(a)$ for every $a \in A$. Hence, aCb for every $a \in A$. Then $b \in C(A)$. So $\bigcap_{a \in A} C(a) \subseteq C(A)$.
3. Let $C(a) \subseteq C(b)$. Then for any $x \in C(a)$, $x \in C(b)$. Hence, xCb for any $x \in C(a)$. Therefore, $b \in C(C(a))$.
Conversely, let $b \in C(C(a))$ and $x \in C(a)$. Therefore, xCb . Hence, $x \in C(b)$. Then $C(a) \subseteq C(b)$.
4. We know that $C(A) = \bigcap_{a \in A} C(a)$, it follows that $C(A) \subseteq C(a)$ for any $a \in A$.
5. A is commutative if and only if $C(A) = A$ if and only if $A = \bigcap_{a \in A} C(a)$ if and only if $A = C(a)$ for every $a \in A$.
6. Obviously, if $C(a) = C(b)$, then $C(C(a)) = C(C(b))$.

Conversely, let $C(C(a)) = C(C(b))$. Since $a \in C(C(a))$, $a \in C(C(b))$. Then $C(b) \subseteq C(a)$. Similarly, since $b \in C(C(b))$, $C(a) \subseteq C(b)$. Therefore, $C(a) = C(b)$. $\square$

THEOREM 3.16. *For a subset M of A , $C(M) = \bigcap_{m \in M} C(m)$.*

PROOF:

$$\begin{aligned} x \in C(M) &\Leftrightarrow \forall m \in M, xCm \\ &\Leftrightarrow \forall m \in M, x \in C(m) \\ &\Leftrightarrow x \in \bigcap_{m \in M} C(m). \end{aligned}$$

$\square$

In the following theorems, some of the properties of the operator C , such as symmetry and monotonicity, are examined.

THEOREM 3.17. *For subsets M and N of A , it holds that $M \subseteq C(C(M))$, and that $M \subseteq N$ implies $C(N) \subseteq C(M)$.*

PROOF: Suppose that $x \in M$. It is sufficient to show that for all $m \in C(M)$, xCm . If m is an arbitrary element of $C(M)$, then by Definition 3.6, mCn for all $n \in M$. Then xCm . Hence, $x \in C(C(M))$ and so $M \subseteq C(C(M))$. For the proof of the second part, let $M \subseteq N$ and $x \in C(N)$. Then xCn for all $n \in N$. Therefore, xCm for all $m \in M$. Hence, $x \in C(M)$. Thus, $C(N) \subseteq C(M)$. $\square$

THEOREM 3.18. *Suppose that M and N are two subsets of A . Then*

(i) $M \subseteq C(N)$ if and only if $N \subseteq C(M)$.

(ii) $C(C(C(M))) = C(M)$.

PROOF: (i) Let $M \subseteq C(N)$, and let $n \in N$. To show that $n \in C(M)$, we must show that for all $m \in M$, nCm . Suppose that m is an arbitrary element of M . Therefore, $m \in C(N)$. Then by definition, mCn for every $n \in N$. Hence, $n \in C(M)$. By symmetry, if $N \subseteq C(M)$, then we see $M \subseteq C(N)$.

(ii) Since $M \subseteq C(C(M))$, it follows that $C(C(C(M))) \subseteq C(M)$. We also obtain $C(M) \subseteq C(C(C(M)))$ by putting $C(M)$ instead of M in Theorem 3.17. Hence, $C(M) = C(C(C(M)))$. $\square$

4. Commutators relative to $\rightarrow$ and $\rightsquigarrow$

This section delves into the concept of commutators within pseudo-BL-algebras. Our main objective is to introduce and examine the properties of these commutators, which play a crucial role in understanding the structure and behavior of these algebraic structures.

DEFINITION 4.1. Suppose a and b are elements of A . We define the commutators of a and b relative to $\rightarrow$ and $\rightsquigarrow$ as follows:

- $com_{\rightarrow}(a, b) = (a \odot b) \rightarrow (b \odot a)$
- $com_{\rightsquigarrow}(a, b) = (b \odot a) \rightsquigarrow (a \odot b)$

The *commutator* of a and b , denoted by $[a, b]$, is defined as their meet:

$$[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b).$$

For elements $a_1, a_2, \dots, a_n \in A$ with $n \geq 2$, the *commutator* of weight n is defined recursively as $[a_1, a_2, \dots, a_n] = [[a_1, \dots, a_{n-1}], a_n]$, with the convention that $[a_1] = a_1$.

Example 4.2. In Example 4.8, let $a = (\frac{3}{4}, -2)$ and $b = (1, -2)$. Then,

- $com_{\rightarrow}(a, b) = (1, 0)$
- $com_{\rightsquigarrow}(a, b) = (1, -\frac{3}{8})$
- $[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b) = (1, 0) \wedge (1, -\frac{3}{8}) = (1, -\frac{3}{8})$.

Similarly, if $a = (1, -4)$ and $b = (\frac{3}{4}, 8)$, then

- $com_{\rightarrow}(a, b) = (1, -\frac{4}{3})$

- $com_{\rightsquigarrow}(a, b) = (1, 0)$
- $[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b) = (1, -\frac{4}{3}) \wedge (1, 0) = (1, -\frac{4}{3})$.

Remark 4.3. Referring to Example 4.8, for $a = (1, -4)$ and $b = (\frac{3}{4}, 8)$, we have:

- $com_{\rightarrow}(a, b) = (1, -\frac{4}{3})$
- $com_{\rightarrow}(b, a) = (1, 0)$
- $com_{\rightsquigarrow}(a, b) = (1, 0)$
- $com_{\rightsquigarrow}(b, a) = (1, -1)$

Therefore,

- $[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b) = (1, -\frac{4}{3}) \wedge (1, 0) = (1, -\frac{4}{3})$
- $[b, a] = com_{\rightarrow}(b, a) \wedge com_{\rightsquigarrow}(b, a) = (1, 0) \wedge (1, -1) = (1, -1)$.

Thus, in general, $[a, b] \neq [b, a]$ in pseudo-BL-algebras.

Next we establish some properties of the commutators.

THEOREM 4.4. *For any $a, b \in A$, we have*

- (i) $com_{\rightarrow}(a, a) = com_{\rightsquigarrow}(a, a) = 1$, hence $[a, a] = 1$;
- (ii) $com_{\rightarrow}(a, 1) = com_{\rightsquigarrow}(a, 1)$, hence $[a, 1] = 1$;
- (iii) $com_{\rightarrow}(1, a) = com_{\rightsquigarrow}(1, a) = 1$, hence $[1, a] = 1$;
- (iv) $com_{\rightarrow}(a, 0) = com_{\rightsquigarrow}(a, 0) = 1$, hence $[a, 0] = 1$;
- (v) $com_{\rightarrow}(0, a) = com_{\rightsquigarrow}(0, a)$, hence $[0, a] = 1$;
- (vi) $com_{\rightarrow}(\bar{a}, a) = com_{\rightsquigarrow}(a, \bar{a}) = 1$;
- (vii) $com_{\rightarrow}(a, \tilde{a}) = com_{\rightsquigarrow}(\tilde{a}, a) = 1$;
- (viii) $com_{\rightarrow}(a, b), com_{\rightsquigarrow}(a, b) \geq b$, hence $[a, b] \geq b$.

PROOF: The proof of (i), (ii), ..., (vi) are obvious.

(vii) $com_{\rightarrow}(\bar{a}, a) = (\bar{a} \odot a) \rightarrow (a \odot \bar{a}) = 0 \rightarrow (a \odot \bar{a}) = 1$ and $com_{\rightsquigarrow}(\tilde{a}, a) = (a \odot \tilde{a}) \rightsquigarrow (\tilde{a} \odot a) = 0 \rightsquigarrow (a \odot \tilde{a}) = 1$.

(viii) $b \rightarrow com_{\rightarrow}(a, b) = b \rightarrow ((a \odot b) \rightarrow (b \odot a)) = (b \odot (a \odot b)) \rightarrow (b \odot a) = 1$. Therefore $b \leq com_{\rightarrow}(a, b)$. Also $b \rightsquigarrow com_{\rightsquigarrow}(a, b) = b \rightsquigarrow ((b \odot a) \rightsquigarrow (a \odot b)) = ((b \odot a) \odot b) \rightsquigarrow (a \odot b) = 1$. Then $b \leq com_{\rightsquigarrow}(a, b)$. So $b \leq [a, b]$. $\square$

LEMMA 4.5. *Suppose H is a normal filter of A and $a, b \in A$. Then*

- $com_{\rightarrow}(a, b)/H = com_{\rightarrow}(a/H, b/H)$,
- $com_{\rightsquigarrow}(a, b)/H = com_{\rightsquigarrow}(a/H, b/H)$.

PROOF: We have

$$\begin{aligned} com_{\rightarrow}(a, b)/H &= ((a \odot b) \rightarrow (b \odot a))/H \\ &= (a \odot b)/H \rightarrow (b \odot a)/H \\ &= (a/H \odot b/H) \rightarrow (b/H \odot a/H) \\ &= com_{\rightarrow}(a/H, b/H). \end{aligned}$$

and similarly,

$$\begin{aligned} com_{\rightsquigarrow}(a, b)/H &= ((b \odot a) \rightsquigarrow (a \odot b))/H \\ &= (b \odot a)/H \rightsquigarrow (a \odot b)/H \\ &= (b/H \odot a/H) \rightsquigarrow (a/H \odot b/H) \\ &= com_{\rightsquigarrow}(a/H, b/H). \end{aligned} \quad \square$$

PROPOSITION 4.6. For elements $a, b \in A$, aCb holds if and only if $[a, b] = 1$.

PROOF: If aCb holds, then $a \odot b = b \odot a$. Therefore,

$$\begin{aligned} com_{\rightarrow}(a, b) &= (a \odot b) \rightarrow (b \odot a) = (a \odot b) \rightarrow (a \odot b) = 1 \\ com_{\rightsquigarrow}(a, b) &= (b \odot a) \rightsquigarrow (a \odot b) = (a \odot b) \rightsquigarrow (a \odot b) = 1 \end{aligned}$$

hence $[a, b] = com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b) = 1 \wedge 1 = 1$.

Conversely, if $[a, b] = 1$, then $com_{\rightarrow}(a, b) = com_{\rightsquigarrow}(a, b) = 1$. From $com_{\rightarrow}(a, b) = 1$, we deduce $a \odot b \leq b \odot a$ and $com_{\rightsquigarrow}(a, b) = 1$ implies $b \odot a \leq a \odot b$. Thus $a \odot b = b \odot a$, and hence aCb holds. $\square$

LEMMA 4.7. For elements $a, b \in A$

1. If $a \vee b = 1$, then $[a, b] = 1$,
2. If $b \leq \bar{a}$, then $com_{\rightsquigarrow}(a, b) = 1$,
3. If $b \leq \tilde{a}$, then $com_{\rightarrow}(a, b) = 1$,
4. $[a \rightarrow b, b \rightarrow a] = [a \rightsquigarrow b, b \rightsquigarrow a] = 1$.

PROOF:

1. Suppose $a \vee b = 1$. By psbl-c₁₉, $a \odot b = a \wedge b = b \wedge a = b \odot a$. Hence $com_{\rightarrow}(a, b) = com_{\rightsquigarrow}(a, b) = 1$, and so $[a, b] = 1$.
2. Let $b \leq \bar{a}$. Then $b \odot a = 0$ (psbl-c₁₃). So $com_{\rightsquigarrow}(a, b) = (b \odot a) \rightsquigarrow (a \odot b) = 0 \rightsquigarrow (a \odot b) = 1$.
3. Let $b \leq \tilde{a}$. Then $a \odot b = 0$ (psbl-c₁₃). So $com_{\rightarrow}(a, b) = (a \odot b) \rightarrow (b \odot a) = 0 \rightarrow (b \odot a) = 1$.
4. By ps-BL₅, $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$. So, by part (1), $[a \rightarrow b, b \rightarrow a] = [a \rightsquigarrow b, b \rightsquigarrow a] = 1$ for all $a, b \in A$. $\square$

PROPOSITION 4.8. Let $a, b_i \in A, (i \in I)$. If $[a, b_i] = 1$ for all $i \in I$, then $[a, \bigvee_{i \in I} b_i] = 1$ and $[a, \bigwedge_{i \in I} b_i] = 1$, whenever the arbitrary meets and joins exist.

PROOF: Let $[a, b_i] = 1$ for every $i \in I$. Then, by Proposition *efd1*, aCb_i for all $i \in I$. By psbl-c₁₇ and psbl-c₁₈, we have

$$\begin{aligned} a \odot \left(\bigwedge_{i \in I} b_i \right) &= \bigwedge_{i \in I} (a \odot b_i) = \bigwedge_{i \in I} (b_i \odot a) = \left(\bigwedge_{i \in I} b_i \right) \odot a \\ a \odot \left(\bigvee_{i \in I} b_i \right) &= \bigvee_{i \in I} (a \odot b_i) = \bigvee_{i \in I} (b_i \odot a) = \left(\bigvee_{i \in I} b_i \right) \odot a \end{aligned}$$

Therefore, $aC(\bigwedge_{i \in I} b_i)$ and $aC(\bigvee_{i \in I} b_i)$. Applying Proposition 4.6 again, we obtain $[a, (\bigwedge_{i \in I} b_i)] = 1$ and $[a, (\bigvee_{i \in I} b_i)] = 1$. $\square$

5. Engel elements and Engel sets in pseudo-BL-algebras

In this section, using the notion of commutator of two elements in a pseudo-BCI-algebra, we introduce and study Engel sets, which are defined by left and right normed commutators. Several properties of these sets are then discussed.

Suppose that a and b are elements of A . For a non-negative integer n , we define the n -Engel left normed commutator $[a, {}_n b]$ inductively as follows:

$$\begin{aligned} [a, {}_0 b] &= a \\ [a, {}_n b] &= [[a, {}_{n-1} b], b] \quad \text{for } n \geq 1 \end{aligned}$$

Similarly, the n -Engel right normed commutator $[{}_n a, b]$ of the pair (a, b) is defined by induction:

$$\begin{aligned} [{}_0 a, b] &= b \\ [{}_n a, b] &= [a, [{}_{n-1} a, b]] \quad \text{for } n \geq 1 \end{aligned}$$

Note that $[a, {}_1 b] = [{}_1 a, b] = [a, b]$.

For a positive integer k , an element $a \in A$ is called a *right k -Engel element* of A if $[a, {}_k b] = 1$, for all $b \in A$. An element $a \in A$ is called a *right Engel element* if it is right k -Engel for some non-negative integer k . We denote by $R(A)$ and $R_k(A)$ the set of right Engel elements and right k -Engel elements, respectively.

$$\begin{aligned} R_k(A) &= \{a \in A \mid [a, {}_k b] = 1, \forall b \in A\} \\ R(A) &= \bigcup_{k \in \mathbb{N}} R_k(A) \end{aligned}$$

Observe that the variable element b appears on the right of the bracket. If k can be chosen independently of b , then a is a right k -Engel element of A . Left Engel elements are defined analogously.

For a positive integer k , an element $a \in A$ is called a *left k -Engel element* of A , whenever $[{}_k b, a] = 1$, for all $b \in A$. Also, a is said to be a *left Engel*

element of A if it is left k -Engel for some non-negative integer k . We denote by $L(A)$ and $L_k(A)$ the set of left Engel elements and left k -Engel elements, respectively.

$$L_k(A) = \{a \in A \mid [b, {}_k a] = 1, \forall b \in A\}$$

$$L(A) = \bigcup_{k \in \mathbb{N}} L_k(A)$$

In $L_k(A)$ and $L(A)$, the variable element is on the left of the bracket. Furthermore, since $[a, 0] = [0, a] = [a, 1] = [1, a] = 1$ for every $a \in A$, we have $0, 1 \in R(A) \cap L(A)$.

An element $a \in A$ that is both a left and right Engel element is said to be an *Engel element*. The set of all Engel elements of A is denoted by $En(A)$. Obviously, 0 is an Engel element in any pseudo-BL-algebra.

THEOREM 5.1. *If a is in the center of A , then a is a right Engel element.*

PROOF: Let $a \in C(A)$. By definition of the center, aCb for all $b \in A$. This implies that $[a, b] = 1$ for all $b \in A$. Therefore, a is a right k -Engel element for $k = 1$, and thus $a \in R_1(A) \subseteq R(A)$. $\square$

LEMMA 5.2. *If A is commutative, then every $a \in A$ is an Engel element.*

PROOF: Let A be commutative and let $a \in A$. By the definition of a commutative algebra, $a \odot b = b \odot a$ for all $b \in A$. This means aCb for all $b \in A$. Consequently, $[a, b] = 1$ for all $b \in A$. Thus, a is a right 1-Engel element ($a \in R_1(A)$) and also a left 1-Engel element ($a \in L_1(A)$). Therefore, a is an Engel element of A . $\square$

COROLLARY 5.3. Any element a of a BL-algebra A is an Engel element.

PROOF: In a BL-algebra, it is a known property that $a \odot b = b \odot a$ for all $a, b \in A$. This means that every BL-algebra is commutative. By Lemma 5.2, if A is commutative, then any $a \in A$ is an Engel element. $\square$

LEMMA 5.4. *i) $L_1(A) \subseteq L_2(A) \subseteq \cdots \subseteq L_n(A) \subseteq \cdots \subseteq L(A)$.*

ii) $R_1(A) \subseteq R_2(A) \subseteq \cdots \subseteq R_n(A) \subseteq \cdots \subseteq R(A)$.

PROOF: i) Let $a \in L_1(A)$. Then for all $b \in A$, we have $[b, a] = 1$. This means $[b, {}_1 a] = 1$. Now, consider $[b, {}_2 a]$,

$$[b, {}_2a] = [[b, {}_1a], a] = [1, a] = 1.$$

Thus, $a \in L_2(A)$, which shows that $L_1(A) \subseteq L_2(A)$. By induction, assume $L_n(A) \subseteq L_{n+1}(A)$ for some $n \geq 1$. Let $a \in L_n(A)$, so $[b, {}_na] = 1$ for all $b \in A$. Then

$$[b, {}_{n+1}a] = [[b, {}_na], a] = [1, a] = 1.$$

This implies $a \in L_{n+1}(A)$, so $L_n(A) \subseteq L_{n+1}(A)$. Finally, by the definition of $L(A)$, it is clear that $L_n(A) \subseteq L(A) = \bigcup_{k \in \mathbb{N}} L_k(A)$ for all $n \in \mathbb{N}$.

ii) The proof for right Engel elements is analogous to the proof for left Engel elements. If $a \in R_1(A)$, then $[a, b] = 1$ for all $b \in A$. Then $[a, {}_2b] = [[a, {}_1b], b] = [1, b] = 1$. Thus, $a \in R_2(A)$, which shows $R_1(A) \subseteq R_2(A)$. The induction proceeds similarly. $\square$

PROPOSITION 5.5. Let f be a homomorphism from A to a pseudo-BL-algebra B , and let $a, b \in A$. Then $f[a, {}_nb] = [f(a), {}_nf(b)]$ for every $n \in \mathbb{N}$.

PROOF: Suppose $a, b \in A$ and $n \in \mathbb{N}$. We proceed by induction on n . If $n = 1$, then

$$\begin{aligned} f(\text{com}_{\rightarrow}(a, b)) &= f((a \odot b) \rightarrow (b \odot a)) \\ &= (f(a) \odot f(b)) \rightarrow (f(b) \odot f(a)) \\ &= \text{com}_{\rightarrow}(f(a), f(b)) \end{aligned}$$

and

$$\begin{aligned} f(\text{com}_{\rightsquigarrow}(a, b)) &= f((b \odot a) \rightsquigarrow (a \odot b)) \\ &= (f(b) \odot f(a)) \rightsquigarrow (f(a) \odot f(b)) \\ &= \text{com}_{\rightsquigarrow}(f(a), f(b)). \end{aligned}$$

Therefore,

$$\begin{aligned} f[a, b] &= f((\text{com}_{\rightarrow}(a, b)) \wedge (\text{com}_{\rightsquigarrow}(a, b))) \\ &= (f(\text{com}_{\rightarrow}(a, b)) \wedge f(\text{com}_{\rightsquigarrow}(a, b))) \\ &= \text{com}_{\rightarrow}(f(a), f(b)) \wedge \text{com}_{\rightsquigarrow}(f(a), f(b)) \\ &= [f(a), f(b)]. \end{aligned}$$

Thus, $f([a, b]) = [f(a), f(b)]$. Assume the hypothesis holds for n , i.e., $f([a, {}_n b]) = [f(a), {}_n f(b)]$. Now, by the inductive hypothesis,

$$\begin{aligned} f([a, {}_{n+1} b]) &= f([{}_n [a, b], b]) \\ &= [f([{}_n [a, b]), f(b)]) \\ &= [[f(a), {}_n f(b)], f(b)] \\ &= [f(a), {}_{n+1} f(b)]. \end{aligned}$$

Hence, the result holds for all positive integers n . $\square$

PROPOSITION 5.6. $L(A) = A$ if and only if $R(A) = A$ if and only if $En(A) = A$.

PROOF: Suppose that $A = L(A)$. Then $A = \{a \in A : \forall b \in A \exists n \in \mathbb{N} \text{ s.t. } [b, {}_n a] = 1\}$. That is, for all $a \in A$ and for every $b \in A$ there exists $n \in \mathbb{N}$ such that $[b, {}_n a] = 1$. Substituting a for b and b for a , we have that for any $b \in A$, there exists a positive integer n such that $[a, {}_n b] = 1$ for all $a \in A$. Hence $A = R(A)$. The proof that $A = R(A)$ implies $A = L(A)$ is similar.

Now, let $R(A) = A$ and $a \in A = R(A)$. Then, by definition of $R(A)$, for all $b \in A$, there exists $n \in \mathbb{N}$ such that $[a, {}_n b] = 1$. So, for each $b \in A$ there exists $n \in \mathbb{N}$ such that $[b, {}_n a] = 1$. Thus $a \in R(A) \cap L(A)$ and hence $A \subseteq En(A)$. Since $En(A) \subseteq A$ we have $A = En(A)$.

If $En(A) = A$, then $R(A) \cap L(A) = A$. So $R(A) = L(A) = A$. $\square$

Now, we introduce the concept of Engel pseudo-BL-algebras and study it in detail.

DEFINITION 5.7. A pseudo-BL-algebra A in which all elements are Engel is said to be an Engel pseudo-BL-algebra. We call the number $\sup\{n : [a, {}_n b] = 1, (a, b) \in A \times A\}$ the Engel degree of A .

By Proposition 5.6 a pseudo-BL-algebra A is an Engel pseudo-BL-algebra, if there exists $n \in \mathbb{N}$, such that $[a, {}_n b] = 1$, for any $a, b \in A$. Also a subset M of A is called an Engel set if, for all $a, b \in M$, there is a non-negative integer n such that $[a, {}_n b] = 1$.

Example 5.8. The pseudo-BL-algebra in Example 4.8 is not Engel, because

if we set $a = (1, -4)$ and $b = (\frac{3}{4}, 8)$, then $[a, b] = (1, -\frac{4}{3})$ and $[a, {}_2b] = (1, -\frac{4}{9})$. For any $n \in \mathbb{N}$, we obtain $[a, {}_nb] = (1, -\frac{4}{3^n}) \neq (1, 0)$.

COROLLARY 5.9. $A = L_n(A)$ if and only if $A = R_n(A)$.

PROOF: $A = L_n(A)$ if and only if $\{a \in A : \forall b \in A [b, {}_na] = 1\} = A$ if and only if for all $a, b \in A$, $[b, {}_na] = 1$ if and only if for any $a, b \in A$, $[a, {}_nb] = 1$ if and only if $A = R_n(A)$. $\square$

DEFINITION 5.10. A is called n -Engel if $A = L_n(A)$ or equivalently $A = R_n(A)$.

Since $L_1(A) \subseteq L_2(A) \subseteq \cdots \subseteq L_n(A) \subseteq \cdots \subseteq L(A)$, if A is an n -Engel pseudo-BL-algebra, then A is m -Engel for any $m \geq n$.

THEOREM 5.11. *1-Engel pseudo-BL-algebras are exactly commutative pseudo-BL-algebras.*

PROOF: A is commutative if and only if for any $a, b \in A$, aCb if and only if $[a, b] = 1$ for any $a, b \in A$ if and only if A is 1-Engel. $\square$

PROPOSITION 5.12. Let $f : A \rightarrow B$ be a homomorphism between two pseudo-BL-algebras. If A is Engel, then $f(A)$ is also Engel.

PROOF: Suppose that $y_1, y_2 \in f(A)$. Then, for some $a, b \in A$ we have $y_1 = f(a)$ and $y_2 = f(b)$. Since A is Engel, there exists $n \in \mathbb{N}$ such that $[a, {}_nb] = 1$. Hence $[y_1, {}_ny_2] = [f(a), {}_nf(b)] = f([a, {}_nb]) = f(1) = 1$. So $f(A)$ is Engel. $\square$

6. Solvable pseudo-BL-algebras

In mathematics, specifically in group theory, a solvable group is a group that can be constructed from abelian groups using extensions. At this point, it is possible to mimic the solvability conditions of group theory.

Let $A_1, A_2, \dots, A_n$ be nonempty subsets of A . Define the commutator of A_1 and A_2 to be

$$[A_1, A_2] = \langle \{[a_1, a_2] \mid a_1 \in A_1, a_2 \in A_2\} \rangle$$

More generally, for $n \geq 2$, we define $[A_1, \dots, A_n] = [[A_1, \dots, A_{n-1}], A_n]$.

When $A_1 = A_2 = A$, $[A, A]$ is the filter of A generated by all the commutators of A , denoted by A' . For any $n \in \mathbb{N}$, define $A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$.

THEOREM 6.1. *Let H be a normal filter of A . Then A/H is commutative if and only if $A' \subseteq H$.*

PROOF: Let H be a normal filter of A . Then A/H is commutative if and only if $a/H \odot b/H = b/H \odot a/H$ for all $a, b \in A$, which is equivalent to $(a \odot b)/H = (b \odot a)/H$. This holds if and only if $(a \odot b) \rightarrow (b \odot a) \in H$ and $(b \odot a) \rightsquigarrow (a \odot b) \in H$ for all $a, b \in A$, which is equivalent to $com_{\rightarrow}(a, b) \in H$ and $com_{\rightsquigarrow}(a, b) \in H$ for all $a, b \in A$. This is equivalent to $[a, b] \in H$ for all $a, b \in A$. Thus, A/H is commutative if and only if $A' \subseteq H$. $\square$

Hence, if A' is a normal filter of A , then A' is the unique smallest normal filter H of A such that A/H is commutative.

THEOREM 6.2. *The following conditions are equivalent:*

- (i) A is commutative.
- (ii) $C(A) = A$.
- (iii) $A' = \{1\}$.

PROOF: (i $\Leftrightarrow$ ii) A is commutative if and only if $a \odot b = b \odot a$ for all $a, b \in A$, which is equivalent to aCb for all $a, b \in A$. This holds if and only if $C(A) = A$.

(ii $\Leftrightarrow$ iii) We know that $C(A) = \{a \in A \mid aCb \text{ for all } b \in A\} = \{a \in A \mid [a, b] = 1 \text{ for all } b \in A\}$. Since $C(A) = A$, for every $a, b \in A$, we have $[a, b] = 1$. Therefore, $A' = [A, A] = \langle \{[a_1, a_2] \mid a_1 \in A, a_2 \in A\} \rangle = \langle \{1\} \rangle = \{1\}$.

Conversely, if $A' = \{1\}$, then $[a, b] = 1$ for every $a, b \in A$. Hence, aCb for every $a, b \in A$. Then $C(A) = \{a \in A \mid aCb \text{ for every } b \in A\} = A$. $\square$

COROLLARY 6.3. *The following conditions on A are equivalent:*

- (i) A is commutative.
- (ii) $A' = \{1\}$.
- (iii) A is 1-Engel.
- (iv) $A = C(A)$.

It is clear that $R_0(A) = L_0(A) = \{1\}$ and $R_1(A) = L_1(A) = C(A)$. So x is a left 1-Engel or right 1-Engel element if and only if x is an element of $C(A)$, and A is 1-Engel if and only if A is commutative.

PROPOSITION 6.4. If $f : A \longrightarrow B$ is a homomorphism from A to a pseudo-BL-algebra B , and A_1, A_2 are nonempty subsets of A , then

$$(i) \quad f([A_1, A_2]) = [f(A_1), f(A_2)].$$

$$(ii) \quad f(A^{(n)}) = (f(A))^{(n)}, \text{ for any positive integer } n \in \mathbb{N}.$$

PROOF: i) Let $y \in f([A_1, A_2])$. Then there exists $x \in [A_1, A_2]$ such that $y = f(x)$. So $x = [a_1, a_2]$ for some $a_1 \in A_1$ and $a_2 \in A_2$, and hence $y = f(x) = f([a_1, a_2]) = [f(a_1), f(a_2)] \in [f(A_1), f(A_2)]$. Therefore $f([A_1, A_2]) \subseteq [f(A_1), f(A_2)]$.

Conversely, let $y \in [f(A_1), f(A_2)]$. Then there exist $y_1 \in f(A_1)$ and $y_2 \in f(A_2)$ such that $y = [y_1, y_2]$. Thus, $y_1 = f(a_1)$ and $y_2 = f(a_2)$ for some $a_1 \in A_1$ and $a_2 \in A_2$. Hence $y = [y_1, y_2] = [f(a_1), f(a_2)] = f([a_1, a_2]) \in f([A_1, A_2])$. Whence $[f(A_1), f(A_2)] \subseteq f([A_1, A_2])$. Therefore $f([A_1, A_2]) = [f(A_1), f(A_2)]$.

ii) We prove by induction on n . If $n = 1$, then $f(A^{(1)}) = f(A') = f([A, A]) = [f(A), f(A)] = (f(A))'$. Now assume that $f(A^{(k)}) = (f(A))^{(k)}$ for some $k \in \mathbb{N}$. Then $f(A^{(k+1)}) = f([A^{(k)}, A^{(k)}]) = [f(A^{(k)}), f(A^{(k)})] = [(f(A))^{(k)}, (f(A))^{(k)}] = (f(A))^{(k+1)}$. $\square$

PROPOSITION 6.5. Let H be a normal filter of A . Then A/H is Engel pseudo-BL-algebra if and only if A is Engel pseudo-BL-algebra.

PROOF: Assume A is an Engel pseudo-BL-algebra. Then by definition, for any $a, b \in A$, there exists $n \in \mathbb{N}$ such that $[a, {}_nb] = 1$. We claim that $[a/H, {}_nb/H] = [a, {}_nb]/H$. We prove this by induction on n . For $n = 1$:

$$\begin{aligned} [a/H, b/H] &= (com_{\rightarrow}(a/H, b/H) \wedge com_{\rightsquigarrow}(a/H, b/H)) \\ &= (com_{\rightarrow}(a, b)/H) \wedge (com_{\rightsquigarrow}(a, b)/H) \\ &= (com_{\rightarrow}(a, b) \wedge com_{\rightsquigarrow}(a, b))/H \\ &= [a, b]/H \end{aligned}$$

Now, assume that $[a/H, {}_kb/H] = [a, {}_kb]/H$ for any positive integer k . Then

$$\begin{aligned} [a/H, {}_{k+1}b/H] &= [[a/H, {}_kb/H], b/H] \\ &= [[a, {}_kb]/H, b/H] \\ &= [[a, {}_kb], b]/H \\ &= [a, {}_{k+1}b]/H \end{aligned}$$

Thus, the claim holds for every positive integer n . Since A is Engel, there exists $n \in \mathbb{N}$ such that $[a, {}_nb] = 1$ for all $a, b \in A$. Then $[a/H, {}_nb/H] = [a, {}_nb]/H = 1/H$. So A/H is an Engel pseudo-BL-algebra.

Conversely, assume A/H is Engel. If a is an arbitrary element of A , then $a/H \in A/H$. Therefore, for every $b \in A$, there exists a positive integer n such that $[a/H, {}_nb/H] = 1/H$. As shown by the claim, $[a/H, {}_nb/H] = [a, {}_nb]/H$. So $[a, {}_nb]/H = 1/H$, which implies $[a, {}_nb] \in H$. Since H is Engel, there exists $m \in \mathbb{N}$ such that $[[a, {}_nb], {}_mb] = 1$. This means $[a, {}_{n+m}b] = 1$. Therefore, A is an Engel pseudo-BL-algebra. $\square$

DEFINITION 6.6. A is called solvable if there exists $n \in \mathbb{N}$ such that $A^{(n)} = \{1\}$.

THEOREM 6.7. Let f be an isomorphism from a pseudo-BL-algebra A to a pseudo-BL-algebra B , and let $n \in \mathbb{N}$. Then

- i) A is an n -Engel pseudo-BL-algebra if and only if B is an n -Engel pseudo-BL-algebra.
- ii) A is solvable if and only if B is solvable.

PROOF: i) Suppose A is n -Engel. Let h, k be arbitrary elements of B . Since f is an isomorphism, there exist unique $a, b \in A$ such that $f(a) = h$ and $f(b) = k$. Since A is n -Engel, we have $[a, {}_nb] = 1$. Applying f to this equation, we get $f([a, {}_nb]) = f(1)$. Since f is a homomorphism, $f([a, {}_nb]) = [f(a), {}_nf(b)]$. Also, $f(1) = 1_B$, the identity element of B . Thus, $[h, {}_nk] = 1_B$. This shows that B is an n -Engel pseudo-BL-algebra.

Conversely, suppose B is an n -Engel pseudo-BL-algebra. Let $a, b \in A$. Then $f(a), f(b) \in B$. Since B is n -Engel, we have $[f(a), {}_nf(b)] = 1_B$. Applying the inverse isomorphism f^{-1} to this equation, we get $f^{-1}([f(a), {}_nf(b)]) = f^{-1}(1_B)$. Since f is a homomorphism,

$f^{-1}([f(a), {}_n f(b)]) = [f^{-1}(f(a)), {}_n f^{-1}(f(b))]$. Also, $f^{-1}(1_B) = 1_A$, the identity element of A . Thus, $[a, {}_n b] = 1_A$. This shows that A is an n -Engel pseudo-BL-algebra.

ii) Since f is an isomorphism from A to B , we have $f(A) = B$. From Proposition 6.4 (ii), $f(A^{(n)}) = (f(A))^{(n)} = B^{(n)}$ for every positive integer n . If A is solvable, then there exists $m \in \mathbb{N}$ such that $A^{(m)} = \{1_A\}$. Applying f , we get $f(A^{(m)}) = f(\{1_A\})$. Since f is a homomorphism, $f(\{1_A\}) = \{f(1_A)\} = \{1_B\}$. Thus, $f(A^{(m)}) = B^{(m)} = \{1_B\}$. This shows that B is a solvable pseudo-BL-algebra.

Conversely, if B is solvable, then there exists $m \in \mathbb{N}$ such that $B^{(m)} = \{1_B\}$. Since $f(A^{(m)}) = B^{(m)}$, we have $f(A^{(m)}) = \{1_B\}$. Since f is an isomorphism, $f(A^{(m)}) = \{1_B\}$ implies $A^{(m)} = f^{-1}(\{1_B\}) = \{1_A\}$. This shows that A is solvable. $\square$

THEOREM 6.8. *Let H be a normal filter of a pseudo-BL-algebra A . If H and A/H are solvable pseudo-BL-algebras, then A is a solvable pseudo-BL-algebra.*

PROOF: Let $f : A \rightarrow A/H$ be the natural homomorphism. Since A/H is solvable, there exists a positive integer n such that $(A/H)^{(n)} = \{H\}$. By Proposition 6.4 (ii), $(A/H)^{(n)} = f(A^{(n)})$. Thus, $f(A^{(n)}) = \{H\}$. By the definition of the kernel of a homomorphism, $H = \ker(f)$. Therefore, $f(A^{(n)}) = \{H\}$ implies that $A^{(n)} \subseteq \ker(f) = H$. Since H is solvable, there exists a positive integer k such that $H^{(k)} = \{1_A\}$. Since $A^{(n)} \subseteq H$, we have $(A^{(n)})^{(k)} \subseteq H^{(k)} = \{1_A\}$. The $(n+k)$ -th derived chain of A is $A^{(n+k)} = (A^{(n)})^{(k)}$. Therefore, $A^{(n+k)} \subseteq \{1_A\}$. Since $A^{(n+k)}$ is a subset of A containing 1_A , and it must be closed under the commutator operation, the only possibility is $A^{(n+k)} = \{1_A\}$. This shows that A is solvable. $\square$

7. Conclusions

In this paper, we have introduced the concepts of commutativity relations, commutators, and Engel elements in pseudo-BL-algebras, and we have investigated several of their properties. To further develop the theory of pseudo-BL-algebras, one promising direction would be to investigate the

Engel degree of these algebras and to establish relationships between various subclasses. For example, 1-Engel BCI-algebras are strictly commutative pseudo-BL-algebras. We hope that this work will contribute to further studies in this area. We believe that the results presented here, along with forthcoming works, can pave the way for a bright future in the theory of pseudo-BL-algebras. A major goal of Engel theory in pseudo-BL-algebras could be stated as follows: to identify conditions on A that ensure $L(A)$ and $R(A)$ are filters, if such conditions exist.

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RADICAL AND JACOBSON RADICAL IN L -ALGEBRAS

Abstract

In this paper, inspired by the concepts of the radical of an ideal and the Jacobson radical in ring theory, we introduce these notions in the context of L -algebras. Our main goal is to explore the nature of the radical of an ideal and Jacobson radical in specific L -algebras. We successfully identify the elements of these radicals in special classes of L -algebras, including proper Glivenko algebras and finite Brouwerian semi-lattices using a closure operator. To achieve this goal, we introduce a new notion that is associated with any element $a \in L$ as well as with an ideal of an L -algebra L .

Keywords: L -algebra, CKL -algebra, ideal, radical of an ideal, Jacobson radical.

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1. Introduction

The quantum Yang-Baxter equation (QYBE), a fundamental concept in mathematical physics, was formulated independently by Zhenning Yang in 1967 and R.J. Baxter in 1972. The QYBE is connected to mathematical structures including quantum binomial algebras [7, 20], I-type semigroups [8], colorings of plane curves [6], dyeing of bijective cocycles, semimultipolar small triangular Hopf algebras, dynamic systems, and geometric crystals. Numerous initial solutions to the Quantum Yang-Baxter Equation have been found, and these solutions have spurred significant research. Furthermore, the inherent algebraic structures that are linked and related to these QYBE solutions have been the subject of extensive and widespread investigation.

In 2005, W. Rump demonstrated that any set X with a binary operation $\rightarrow$ satisfying

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z),$$

corresponds to a solution of the quantum Yang-Baxter equation if left multiplication is bijective [16]. Using this equation, he provided a precise definition for L -algebras and conducted an investigation into their properties. His work included showing that several established algebraic structures can be classified as L -algebras. More specifically, he proved that Hilbert algebras, locales, (left) hoops, (pseudo) MV -algebras, and l -group cones all satisfy the conditions necessary to be considered L -algebras according to his definition [17].

Rump introduced the concept of an ideal in an L -algebra [17]. We continued the work on ideals based on his definition and were able to define concepts such as the radical of an ideal, prime ideal, and primary ideal. We were also able to find the relationship between prime ideals and maximal ideals [14]. Significant results have been achieved in this area, as documented in [2, 4, 9, 10, 12, 15, 17, 22, 24, 25, 26]. For us, the question arose as to which elements in different L -algebras are members of the radical of an ideal. This was the motivation for writing this paper. Additionally, the Jacobson radical plays a crucial role in structural theorems of Ring Theory

and Module Theory, such as Nakayama's lemma, and provides insight into the non-semisimple part of a ring. Therefore, we were motivated to extend these foundational ideas to L -algebras, exploring how analogous radicals can be defined and what structural information they reveal in this broader algebraic context.

In the present paper, our primary aim is to delve into the nature of the radical of an ideal and the Jacobson radical within the context of specific L -algebras. Initially, we provide an introduction to the radical of an ideal, outlining its properties. Following this, we define the Jacobson radical. The main thrust of our investigation then becomes pinpointing the elements that belong to these radicals.

In Sections 3 and 4, we introduce special L -algebras such as proper Glivenko algebras and Brouwerian semi-lattices and then find the Jacobson radical members in proper Glivenko algebras and bounded Brouwerian semi-lattices. In the following, considering the ideal I and an element x of an L -algebra, we define the notion of I_x and, based on it, characterize the radical elements of that ideal in finite Brouwerian semi-lattices.

2. Preliminaries

In this section, we recall the known default notions and proven results that are used in other sections.

An algebraic structure $(L; \rightarrow, 1)$ of type $(2, 0)$ is called an L -algebra if it satisfies these conditions.

$$(L_1) \quad x \rightarrow x = x \rightarrow 1 = 1 \quad \text{and} \quad 1 \rightarrow x = x,$$

$$(L_2) \quad (x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z),$$

$$(L_3) \quad \text{if } x \rightarrow y = y \rightarrow x = 1, \text{ then } x = y,$$

for any $x, y, z \in L$.

Condition (L_1) states that 1 is a logical unit, and (L_2) is related to the quantum Yang-Baxter equation [8]. The logical unit is always unique. If

the binary operation $\rightarrow$ is taken as logical implicative, then there exists a partial order on L defined by

$$x \leq y \text{ if and only if } x \rightarrow y = 1. \quad (2.1)$$

You can see the proof in [18].

PROPOSITION 2.1 ([23]). Let $(L; \rightarrow, 1)$ be an L -algebra. Then $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$, for any $x, y, z \in L$.

PROPOSITION 2.2 ([23]). For an L -algebra $(L; \rightarrow, 1)$ and any $x, y, z \in L$, the following are equivalent.

- (i) $x \leq y \rightarrow x$,
- (ii) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$,
- (iii) $((x \rightarrow y) \rightarrow z) \rightarrow z \leq ((x \rightarrow y) \rightarrow z) \rightarrow ((y \rightarrow x) \rightarrow z)$.

DEFINITION 2.3 ([5]). An L -algebra $(L; \rightarrow, 1)$ which satisfies the following condition

$$x \rightarrow (y \rightarrow x) = 1, \quad (K)$$

for any $x, y \in L$, is called a KL -algebra.

Notation. If $(L; \rightarrow, 1)$ is a KL -algebra, then the equivalent statements of Proposition 2.2 hold.

DEFINITION 2.4 ([5]). A CKL -algebra is an L -algebra which satisfies the following condition

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \quad (C)$$

for any $x, y, z \in L$. In [19] bounded CKL -algebras are known as *Glivenko algebras*.

PROPOSITION 2.5 ([15]). Let $(L; \rightarrow, 1)$ be a CKL -algebra. Then the following properties hold for all $x, y, z \in L$.

- (i) $x \leq y \rightarrow z$ if and only if $y \leq x \rightarrow z$.
- (ii) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Note. Every CKL -algebra is a KL -algebra, since for any $x, y \in L$, we have

$$x \rightarrow (y \rightarrow x) = y \rightarrow (x \rightarrow x) = y \rightarrow 1 = 1. \quad (2.2)$$

DEFINITION 2.6 ([18]). A subset I of an L -algebra $(L; \rightarrow, 1)$ is called an *ideal* of L , if it satisfies the following conditions:

$$(I_1) \quad 1 \in I,$$

$$(I_2) \quad \text{if } x \in I \text{ and } x \rightarrow y \in I, \text{ then } y \in I,$$

$$(I_3) \quad \text{if } x \in I, \text{ then } (x \rightarrow y) \rightarrow y \in I,$$

$$(I_4) \quad \text{if } x \in I, \text{ then } y \rightarrow x \in I \text{ and } y \rightarrow (x \rightarrow y) \in I,$$

for any $x, y \in L$.

Clearly, $\{1\}$ and L are two trivial ideals of L . An ideal I of L is called a *proper ideal* if $I \neq L$. The set of all ideals of L is denoted by $\mathcal{I}_d(L)$ and the set of all proper ideals of L is denoted by $p\mathcal{I}_d(L)$. It is easy to see that $(\mathcal{I}_d(L), \subseteq)$ and $(p\mathcal{I}_d(L), \subseteq)$ are poset.

An L -algebra L is called *simple* if $\mathcal{I}_d(L) = \{\{1\}, L\}$.

PROPOSITION 2.7 ([5]). Suppose L is a CKL -algebra and $\emptyset \neq I \subseteq L$. Then $I \in \mathcal{I}_d(L)$ if and only if (I_1) and (I_2) hold.

Let L be an L -algebra and $\emptyset \neq X \subseteq L$. Then the smallest ideal of L containing X is called *the generated ideal by X* and it is denoted by $\langle X \rangle$. Clearly, $\langle X \rangle = \bigcap_{\substack{I \in \mathcal{I}_d(L) \\ X \subseteq I}} I$.

PROPOSITION 2.8 ([14]). Suppose that L is a CKL -algebra, $I \in \mathcal{I}_d(L)$ and $x \in L \setminus I$. Then

$$\langle I \cup \{x\} \rangle = \{a \in L \mid \exists n \in \mathbb{N} \text{ s.t. } x \xrightarrow{n} a \in I\}.$$

where

$$x \xrightarrow{0} a = a, x \xrightarrow{1} a = x \rightarrow a, \dots, x \xrightarrow{n} a = x \rightarrow (x \xrightarrow{n-1} a).$$

Note. Assume $(X; \leq)$ is a poset. For every $Y \subseteq X$, we define

$$\uparrow Y = \{x \in X \mid \exists y \in Y \text{ s.t. } y \leq x\} \text{ and } \downarrow Y = \{x \in X \mid \exists y \in Y \text{ s.t. } x \leq y\}.$$

The set Y is called *upset* if $\uparrow Y = Y$ and is called *downset* if $\downarrow Y = Y$. If $y \in X$, then $\downarrow \{y\}$ is denoted by $\downarrow y$.

If $(L; \rightarrow, 1)$ is an L -algebra and $I \in \mathcal{I}_d(L)$, then I is upset.

Suppose that $I \in p\mathcal{I}_d(L)$. Then I is called a *maximal ideal* of L if it is not included in any other proper ideal of L . The set of all maximal ideals of L is denoted by $\mathcal{Max}(L)$.

PROPOSITION 2.9 ([14]). Assume $M \in p\mathcal{I}_d(L)$. Then the following statements are equivalent.

- (i) $M \in \mathcal{Max}(L)$.
- (ii) $\langle M \cup \{x\} \rangle = L$, for any $x \in L \setminus M$.

PROPOSITION 2.10 ([14]). Let L be a bounded L -algebra and $I \in p\mathcal{I}_d(L)$. Then there exists $M \in \mathcal{Max}(L)$ such that $I \subseteq M$.

Note. From now on, we assume that $(L; \rightarrow, 1)$ or simply L is an L -algebra unless otherwise stated.

3. Radical and Jacobson radical in CKL -algebras

This section commences with the definitions of the radical of an ideal and the Jacobson radical in L -algebras. Next, we introduce proper Glivenko algebras and identify the Jacobson radical elements within them. Finally, we conclude with a conjecture regarding the elements of the Jacobson radical in finite L -algebras.

DEFINITION 3.1 ([14]). Let $I \in p\mathcal{I}_d(L)$. Then the intersection of all maximal ideals of L containing I is called a *radical* of I and it is denoted by $Rad(I)$. So

$$Rad(I) = \bigcap_{\substack{M \in \mathcal{Max}(L) \\ I \subseteq M}} M.$$

Clearly, $\mathcal{R}ad(I) \in p\mathcal{I}_d(L)$.

In the following, we recall the notion of a closure operator and show that $\mathcal{R}ad$ is a closure operator.

DEFINITION 3.2 ([1, 3]). Suppose P is an ordered set. A map $C : P \rightarrow P$ is called a *closure operator* if it satisfies

- (i) $x \leq C(x)$,
- (ii) $C(C(x)) = C(x)$,
- (iii) $x \leq y$ implies $C(x) \leq C(y)$,

for any $x, y \in P$. An element x of P is called a *closed* if $C(x) = x$.

PROPOSITION 3.3. The map $\mathcal{R}ad$ is a closure operator on $p\mathcal{I}_d(L)$.

PROOF: It is clear that $\mathcal{R}ad : p\mathcal{I}_d(L) \rightarrow p\mathcal{I}_d(L)$ is a map. Let $I, J \in p\mathcal{I}_d(L)$. Clearly, $I \subseteq \mathcal{R}ad(I)$. Furthermore, $\mathcal{R}ad(\mathcal{R}ad(I)) = \mathcal{R}ad(I)$ (See [14]). Suppose $I \subseteq J$ and $t \in \mathcal{R}ad(I)$. Then $t \in M$, for any $I \subseteq M \in \mathcal{M}ax(L)$. Since $I \subseteq J$, any maximal ideal containing J also contains I . Thus, if $J \subseteq S \in \mathcal{M}ax(L)$, then $t \in S$. Hence, $t \in \bigcap_{\substack{S \in \mathcal{M}ax(L) \\ J \subseteq S}} S = \mathcal{R}ad(J)$.

So, $\mathcal{R}ad(I) \subseteq \mathcal{R}ad(J)$. □

DEFINITION 3.4. The intersection of all maximal ideals of L is called *Jacobson radical* of L and it is denoted by $\mathcal{J}(L)$. In other words,

$$\mathcal{J}(L) = \bigcap_{M \in \mathcal{M}ax(L)} M.$$

Clearly, $\mathcal{J}(L) = \mathcal{R}ad(\{1\})$. Now, we verify these previously discussed notions when L is a chain.

PROPOSITION 3.5. If L is a bounded chain, then it has a unique maximal ideal.

PROOF: Since $\{1\} \in p\mathcal{I}_d(L)$, by Proposition 2.10, there exists $M \in \mathcal{M}ax(L)$ such that $\{1\} \subseteq M$. So $\mathcal{M}ax(L)$ is not empty. Now, we show that $p\mathcal{I}_d(L)$ is a chain. To see this, consider $I, J \in p\mathcal{I}_d(L)$ with $J \not\subseteq I$,

indicating there exists $y \in J$ which is not in I . For every $x \in I$, since L is a chain, we have either $x \leq y$ or $y \leq x$. Let $x \leq y$. Since I is upset, we get $y \in I$ which is a contradiction. Hence, $y \leq x$. Since J is upset, we get $x \in J$. So $I \subseteq J$. Thus, $p\mathcal{J}_d(L)$ is a chain and M is the maximal element of $p\mathcal{J}_d(L)$, implying $\mathcal{M}ax(L) = \{M\}$. $\square$

COROLLARY 3.6. If L is a bounded chain and $I \in p\mathcal{J}_d(L)$, then $\mathcal{R}ad(I) = \mathcal{J}(L)$.

PROOF: The proof is clear by Proposition 3.5. $\square$

COROLLARY 3.7. If L is a bounded chain and $I \in p\mathcal{J}_d(L)$, then I is closed with respect to the closure operator $\mathcal{R}ad$ if and only if it is a maximal ideal.

Now, we want to identify the elements of $\mathcal{J}(L)$, and the following proposition provides insight into $\mathcal{J}(L)$.

PROPOSITION 3.8. If L is a Glivenko algebra, then

- (i) $I \subseteq \{t \in L \mid \exists y \leq t \text{ such that } (t \xrightarrow{n} y) \rightarrow t \in I, \text{ for any } n \in \mathbb{N}\} \subseteq \mathcal{R}ad(I)$.
- (ii) If $t, y \in L$ satisfy $y \leq t$ and t is an upper bound for the sequence $\{t \xrightarrow{n} y\}_{n \in \mathbb{N}}$, then $t \in \mathcal{J}(L)$.

PROOF: (i) Let $A = \{t \in L \mid \exists y \leq t \text{ such that } (t \xrightarrow{n} y) \rightarrow t \in I, \text{ for any } n \in \mathbb{N}\} \not\subseteq \mathcal{R}ad(I)$. Then there exists $t \in A$ such that $t \notin \mathcal{R}ad(I)$. So, there exists $M \in \mathcal{M}ax(L)$ where $I \subseteq M$ and $t \notin M$. By Proposition 2.9, we get $\langle M \cup \{t\} \rangle = L$. Therefore, $y \in \langle M \cup \{t\} \rangle$. Using Proposition 2.8, there exists $n \in \mathbb{N}$ such that $t \xrightarrow{n} y \in M$. By the assumption, $(t \xrightarrow{n} y) \rightarrow t \in I \subseteq M$. So by (I_2) , we conclude $t \in M$, which is a contradiction. Hence, $t \in \mathcal{R}ad(I)$. Now, we show that $I \subseteq A$. Let $t \in I$. Then $t \leq t$ and $(t \xrightarrow{n} t) \rightarrow t = t \in I$, for any $n \in \mathbb{N}$. So, $t \in A$ and $I \subseteq A$. (ii) Assume that $M \in \mathcal{M}ax(L)$. Since t is an upper bound for this sequence, we have $t \xrightarrow{n} y \leq t$, for any $n \in \mathbb{N}$. So $(t \xrightarrow{n} y) \rightarrow t = 1 \in M$, for each $n \in \mathbb{N}$. By (i), we conclude that $t \in \mathcal{R}ad(M)$. Hence, $t \in M$ for any $M \in \mathcal{M}ax(L)$. Thus, $t \in \mathcal{J}(L)$. $\square$

DEFINITION 3.9. [21] Let L be an L -algebra. An element $x \in L$ is called *dense* if there exists $y \leq x$ such that $x \rightarrow y = y$. The set of all dense elements of L is denoted by $\mathcal{D}(L)$.

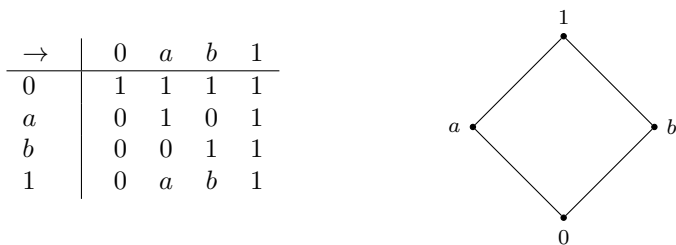
Using Proposition 3.8 and the concept of dense elements, we conclude that in Glivenko algebras, all dense elements belong to the Jacobson radical.

COROLLARY 3.10. If L is a Glivenko algebra, then $\mathcal{D}(L) \subseteq \mathcal{J}(L)$.

PROOF: Consider $t \in \mathcal{D}(L)$. Then there exists $y \in L$ such that $y \leq t$ and $t \rightarrow y = y$. For every $n \in \mathbb{N}$, we have $t \xrightarrow{n} y = y \leq t$, making t is an upper bound for the sequence $\{t \xrightarrow{n} y\}_{n \in \mathbb{N}}$. Applying Proposition 3.8 (ii), it follows that $t \in \mathcal{J}(L)$. $\square$

In the following example, we show that the condition (C) in the definition of Glivenko algebra in Corollary 3.10, is necessary.

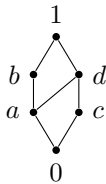
Example 3.11. Assume $(L = \{0, a, b, 1\}; \leq)$ is a poset with the following Hasse diagram. Consider the operation $\rightarrow$ on L as follows.



Then $(L; \rightarrow, 1)$ is an L -algebra. Since $a \not\leq b \rightarrow a = 0$, we conclude that L is not a CKL -algebra. On the other hand, L is a simple L -algebra. Hence, $\mathcal{J}(L) = \{1\}$. Moreover, $\mathcal{D}(L) = \{a, b, 1\}$. Hence, $\mathcal{D}(L) \not\subseteq \mathcal{J}(L)$.

DEFINITION 3.12. A Glivenko algebra is called *proper* if all of its principal ideals, except for $\langle 0 \rangle$, are proper ideals.

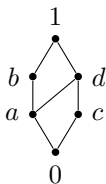
Example 3.13. Assume that $(L = \{0, a, b, c, d, 1\}; \leq)$ is a poset with the following Hasse diagram. Consider the operation $\rightarrow$ on L as defined below.



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	c	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	0	b	b	c	1	1
1	0	a	b	c	d	1

Then $(L; \rightarrow, 1)$ is a *CKL*-algebra. Moreover, $\mathcal{J}_d(L) = \{\langle 1 \rangle = \{1\}, \langle b \rangle = \{1, b\}, \langle d \rangle = \{1, d\}, \langle c \rangle = \{1, c, d\}, \langle a \rangle = \{a, b, d, 1\}, \langle 0 \rangle = L\}$. Therefore, L is a proper Glivenko algebra.

(ii) Let $(L = \{0, a, b, c, d, 1\}; \leq)$ be a poset with the following Hasse diagram. Define the operation $\rightarrow$ on L as follows.



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(L; \rightarrow, 1)$ is a *CKL*-algebra. Since $\langle a \rangle = L$, we conclude that L is not a proper Glivenko algebra.

We are now ready to state the main result of this section, which is the characterization of the Jacobson radical elements in the proper Glivenko algebras.

THEOREM 3.14. *If L is a proper Glivenko algebra, then $\mathcal{J}(L) = \mathcal{D}(L)$.*

PROOF: By Corollary 3.10, since L is a Glivenko algebra, it follows that $\mathcal{D}(L) \subseteq \mathcal{J}(L)$. Let $t \in \mathcal{J}(L)$ and $s \in L \setminus \{0\}$ such that $t \rightarrow 0 = s$. If $\langle s \rangle \neq L$, then by Proposition 2.10, there exists $T \in \mathcal{M}ax(L)$ such that $\langle s \rangle \subseteq T$. Therefore, $s \in T$. Since $t \in \bigcap_{M \in \mathcal{M}ax(L)} M$, we have $t \in T$. By (I_2) , we conclude $0 \in T$ and this is a contradiction. So, $\langle t \rightarrow 0 \rangle = L$. By the hypothesis, all of the principal ideals of L (except for $\langle 0 \rangle$) are

proper. On the other hand, $\langle s \rangle \notin p\mathcal{J}_d(L)$. Therefore, $s = 0$. Thus, $\mathcal{J}(L) \subseteq \{t \in L \mid t \rightarrow 0 = 0\} \subseteq \mathcal{D}(L)$ and $\mathcal{J}(L) = \mathcal{D}(L)$. $\square$

Question. What are the elements of $\mathcal{J}(L)$ in Glivenko algebras?

After examining a large number of examples, we arrived at this conjecture.

Open problem. We conjecture that if L is a Glivenko algebra, then

$$\mathcal{J}(L) = \mathcal{D}(L).$$

4. Radical and Jacobson radical in finite Brouwerian semi-lattices

This section aims to identify the elements of the radical of an ideal in finite Brouwerian semi-lattices. To achieve this, first we introduce the concept of I_x , for every element x and ideal I of L , and then we investigate its properties. We also characterize the Jacobson radical in bounded Brouwerian semi-lattices.

Notation. Let L be an L -algebra, $I \in \mathcal{J}_d(L)$ and $x \in L$. Considering

$$I_x := \{y \in L \mid x \rightarrow y \in I\} \quad \text{and} \quad I_L := \{I_x \mid x \in L\}$$

Clearly, $x \in I_x$ and $I_1 = I$.

Example 4.1. Consider Example 3.13. If $I = \{1, d\}$ and $J = \{b, 1\}$, then $I_b = \{a, b, d, 1\} = J_d$ and $J_b = \langle b \rangle = \{b, 1\}$.

DEFINITION 4.2. [17] An L -algebra L is called a *Hilbert algebra* if its binary operation $\rightarrow$ is left distributive on itself, that is for any $x, y, z \in L$:

$$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z),$$

Every Hilbert algebra is a CKL -algebra (See [14]).

PROPOSITION 4.3. If L is a Hilbert algebra, $I \in \mathcal{J}_d(L)$ and $a \in L$, then $I_a \in \mathcal{J}_d(L)$.

PROOF: Since any Hilbert algebra is a CKL -algebra, it is enough to show

that (I_1) and (I_2) hold. As $a \rightarrow 1 = 1 \in I$, we conclude $1 \in I_a$. Thus, (I_1) holds. Now, let $x, x \rightarrow y \in I_a$. Then $a \rightarrow x \in I$ and $a \rightarrow (x \rightarrow y) \in I$. Since L is a Hilbert algebra, we have $(a \rightarrow x) \rightarrow (a \rightarrow y) \in I$. Hence, $a \rightarrow y \in I$ and so, $y \in I_a$. Therefore, (I_2) holds and $I_a \in \mathcal{I}_d(L)$. $\square$

The Hilbert condition in Proposition 4.3 is fundamental. The following example demonstrates that if L is not a Hilbert algebra, then I_a may not necessarily be an ideal.

Example 4.4. Let $L = \{1 = \eta_0, \eta_1, \eta_2, \eta_3, \dots\}$ be a chain such that $\dots < \eta_3 < \eta_2 < \eta_1 < 1$ and for every $m, n \in \mathbb{N}$,

$$\eta_m \rightarrow \eta_n = \begin{cases} 1 & n \leq m \\ \eta_{n-m} & \text{otherwise.} \end{cases}$$

Then L is an L -algebra. Moreover,

$$1 = \eta_2 \rightarrow \eta_2 = \eta_2 \rightarrow (\eta_1 \rightarrow \eta_3) \neq (\eta_2 \rightarrow \eta_1) \rightarrow (\eta_2 \rightarrow \eta_3) = 1 \rightarrow \eta_1 = \eta_1.$$

So, L is not a Hilbert algebra. Consider the row corresponding to η_1 in the binary operation $\rightarrow$.

$\rightarrow$	$\dots$	η_4	η_3	η_2	η_1	1
$\dots$						
η_4	$\dots$	1	1	1	1	1
η_3	$\dots$	η_1	1	1	1	1
η_2	$\dots$	η_2	η_1	1	1	1
η_1	$\dots$	η_3	η_2	η_1	1	1
1	$\dots$	η_4	η_3	η_2	η_1	1

If $I = \{1\}$, then $I_{\eta_1} = \{1, \eta_1\}$. Since $\eta_1 \in I_{\eta_1}$, $\eta_1 \rightarrow \eta_2 = \eta_1 \in I_{\eta_1}$ and $\eta_2 \notin I_{\eta_1}$, we conclude I_{η_1} is not an ideal of L .

In the following, we see that the concept of I_x allows us to define a closure operator on $\mathcal{I}_d(L)$.

Remark 4.5. If L is a KL -algebra, $I \in \mathcal{I}_d(L)$ and $x \in L$, then, $I \subseteq I_x$. To see this, take $y \in I$. As $y \leq x \rightarrow y$, we have $x \rightarrow y \in I$. It implies $y \in I_x$.

PROPOSITION 4.6. If L is a Hilbert algebra, $I, J \in \mathcal{J}_d(L)$ and $x \in L$, then

- (i) $I \subseteq I_x$.
- (ii) $(I_x)_x = I_x$.
- (iii) $I \subseteq J$ implies $I_x \subseteq J_x$.

PROOF: (i) Since L is a CKL -algebra and consequently a KL -algebra, applying Remark 4.5, the proof is obvious.

(ii) Let $y \in (I_x)_x$. Then $x \rightarrow y \in I_x$. Consequently, $x \rightarrow (x \rightarrow y) \in I$. Since L is a Hilbert algebra, we have

$$x \rightarrow y = 1 \rightarrow (x \rightarrow y) = (x \rightarrow x) \rightarrow (x \rightarrow y) = x \rightarrow (x \rightarrow y) \in I.$$

Thus, $y \in I_x$, which implies $(I_x)_x \subseteq I_x$. Additionally, from (i), we know $I_x \subseteq (I_x)_x$. Therefore, $(I_x)_x = I_x$.

(iii) Suppose $I \subseteq J$ and $y \in I_x$. So, $x \rightarrow y \in I \subseteq J$ and $y \in J_x$. Hence, $I_x \subseteq J_x$. $\square$

COROLLARY 4.7. If L is a Hilbert algebra, then for any $x \in L$, $-_x : \mathcal{J}_d(L) \rightarrow \mathcal{J}_d(L)$ by $I \mapsto I_x$, for any $I \in \mathcal{J}_d(L)$, is a closure operator.

PROPOSITION 4.8. Assume L is a Hilbert algebra, $I \in \mathcal{J}_d(L)$ and $x \in L$. Then I is closed with respect to the closure operator $-_x$ if and only if $x \in I$.

PROOF: Let $x \in I$. We prove that $I_x = I$. By Proposition 4.6, $I \subseteq I_x$. Now, if $y \in I_x$ then $x \rightarrow y \in I$, leading to $y \in I$ by (I_2) . Thus, $I_x \subseteq I$, which implies $I_x = I$. For the converse, suppose that $I_x = I$. Since $x \in I_x$, we conclude $x \in I$. $\square$

PROPOSITION 4.9. Suppose that L is a Hilbert algebra, $I, J \in \mathcal{J}_d(L)$, and $x, y \in L$. Then

- (i) $I \subseteq \langle x \rangle$ if and only if $I_x = \langle x \rangle$.
- (ii) $(I \cap J)_x = I_x \cap J_x$.
- (iii) If $x \leq y$, then $I_y \subseteq I_x$.

PROOF: (i) Let $I \subseteq \langle x \rangle$. As $x \in I_x$ and $\langle x \rangle$ is the smallest ideal of L containing x , we have $\langle x \rangle \subseteq I_x$. Since $I \subseteq \langle x \rangle$, using Propositions 4.6 (iii)

and 4.8, we get $I_x \subseteq \langle x \rangle_x = \langle x \rangle$. Hence, $I_x = \langle x \rangle$. For the converse, assume $I_x = \langle x \rangle$ and $y \in I$. Since $I \subseteq I_x$, we have $y \in I_x$. Therefore, $I \subseteq \langle x \rangle$.

(ii) It is obvious.

(iii) Suppose that $t \in I_y$. Then $y \rightarrow t \in I$. As $x \leq y$, by Proposition 2.2, we conclude $y \rightarrow t \leq x \rightarrow t$. Therefore, $t \in I_x$, implying $I_y \subseteq I_x$. $\square$

In an L -algebra L , for any two elements a and b , we denote the greatest lower bound (if it exists) by $a \wedge b$. Now, we introduce the concept of Brouwerian semi-lattice and then we characterize the elements of $\mathcal{Rad}(I)$, for the ideals of finite Brouwerian semi-lattices.

DEFINITION 4.10. An L -algebra L is called a Brouwerian semi-lattice, if for every $a, b, c \in L$, $a \wedge b$ exists and

$$a \wedge b \leq c \iff a \leq b \rightarrow c.$$

Example 4.11. (i) Consider the chain $(L = \{0, a, 1\}; \leq)$. Define the binary operation $\rightarrow$ on L as follows.

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then $(L; \rightarrow, 1)$ is a Brouwerian semi-lattice. But the L -algebra introduced in Example 3.11, is not a Brouwerian semi-lattice. This is because $a \wedge b = 0 \leq a$ while $a \not\leq 0 = b \rightarrow a$.

(ii) Let $L = [0, 1]$. Define the operation $\rightarrow$ on L as follows

$$x \rightarrow y = \begin{cases} 1 & x \leq y \\ y & \text{otherwise.} \end{cases}$$

Then $(L; \rightarrow, 1)$ is an L -algebra. Since it is a chain, $x \wedge y \in L$ for any $x, y \in L$. Also, it is straightforward to see that L is an infinite Brouwerian semi-lattice.

(iii) Now, we define the operation $\rightarrow$ on L as follows

$$x \rightarrow y = \begin{cases} 1 & x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

With this operation, L becomes an infinite L -algebra that is not a Brouwerian semi-lattice. For example, we have $0.31 \leq 0.32 \rightarrow 0.3$, while $0.31 \not\leq 0.3$.

From [21] and [17], we have the following remark.

Remark 4.12. (i) Every Brouwerian semi-lattice is a KL -algebra. To show this, let L be a Brouwerian semi-lattice and $a, b \in L$. Since $a \wedge b \leq a$, we have $a \leq b \rightarrow a$.

(ii) Every Brouwerian semi-lattice is a Hilbert algebra. To see this, let L be a Brouwerian semi-lattice. Applying [11, Page 105-Rule 1.6], for any $a, b, c \in L$, we have

$$a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c = (a \rightarrow b) \rightarrow (a \rightarrow c).$$

As a result of Proposition 4.3 and Remark 4.12, we have the following corollary.

COROLLARY 4.13. If L is a Brouwerian semi-lattice, $I \in \mathcal{I}_d(L)$ and $a \in L$, then $I_a \in \mathcal{I}_d(L)$.

Inspired by the notion of an atom in Boolean algebras ([13]), we define the concept of an atom for L -algebras.

DEFINITION 4.14. An element a of a bounded L -algebra L is said to be an atom if $0 < a$ and there does not exist an element $c \in L$ such that $0 < c < a$. We denote the set of all atoms of L by $\text{Atom}(L)$.

Example 4.15. Consider the chain ($L = \{0, a, 1\}; \leq$) in Example 4.11 (i). Then a is an atom of L .

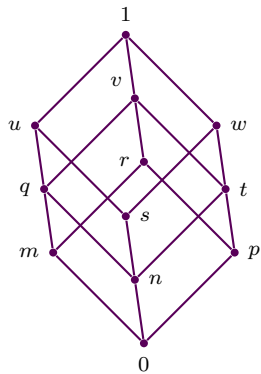
PROPOSITION 4.16. Let L be a bounded Brouwerian semi-lattice and $a \in \text{Atom}(L)$. Then $\langle a \rangle \in \mathcal{Max}(L)$.

PROOF: Assume that $J \in p\mathcal{I}_d(L)$ and $\langle a \rangle \subsetneq J$. Then there exists $0 \neq b \in J$ such that $b \notin \langle a \rangle$. Since L is a Brouwerian semi-lattice, we have $b \leq a \rightarrow (a \wedge b)$. Also, as $b \in J$, we get $a \rightarrow (a \wedge b) \in J$. Since $a \in \langle a \rangle \subseteq J$, by (I_2) , we have $a \wedge b \in J$. On the other hand, $a \wedge b \leq a$ and a is an atom

of L . Since $b \notin \langle a \rangle$, thus $a \wedge b \neq a$. So $a \wedge b = 0$ and $0 \in J$. Hence, $J = L$ and this is a contradiction. Therefore, $\langle a \rangle = J$ and $\langle a \rangle \in \mathcal{Max}(L)$. $\square$

In the following example, we show that the condition of Proposition 4.16, is necessary.

Example 4.17. Let $(L = \{0, m, n, p, q, r, s, t, u, v, w, 1\}; \leq)$ be a poset with the following Hasse diagram.



Define the operation $\rightarrow$ on L as follows.

$\rightarrow$	0	m	n	p	q	s	r	t	u	v	w	1
0	1	1	1	1	1	1	1	1	1	1	1	1
m	s	1	s	s	1	s	1	s	1	1	s	1
n	r	r	1	r	1	1	r	1	1	1	1	1
p	s	s	s	1	s	s	1	1	s	1	1	1
q	0	r	s	0	1	s	r	s	1	1	s	1
s	r	r	v	r	v	1	r	v	1	v	1	1
r	s	u	s	w	u	s	1	w	u	1	w	1
t	0	0	s	r	s	s	r	1	s	1	1	1
u	0	r	n	0	v	s	r	n	1	v	s	1
v	0	m	s	p	u	s	r	w	u	1	w	1
w	0	0	n	r	n	s	r	v	s	v	1	1
1	0	m	n	p	q	s	r	t	u	v	w	1

Then $(L; \rightarrow, 1)$ is an L -algebra. On the other hand, since $m \wedge p = 0 \leq n$, but $m \not\leq p \rightarrow n = s$, it follows that L is not a Brouwerian semi-lattice. Notice that $m \in \text{Atom}(L)$, but the ideal generated by it is L and is not a maximal ideal.

THEOREM 4.18. *Suppose that L is a finite Brouwerian semi-lattice and $I, J \in \mathcal{J}_d(L)$. Then*

$$J \in I_L \text{ if and only if } I \subseteq J.$$

PROOF: If L is a finite Brouwerian semi-lattice and $J \in I_L$ then, by Remarks 4.5 and 4.12, it follows that $I \subseteq J$. Conversely, take $I \subseteq J$. If $I = J$, then $J = I_1$. Let $I \subsetneq J$. Obviously, $J \setminus I \neq \emptyset$. Then $J \setminus I$ has the minimum element m . Because otherwise, if there exists an element $t \in J \setminus I$ such that $m \not\leq t$, then by the proof of Proposition 4.16, we have $m \wedge t \in J$. Since J is finite, this process stops at one point. So, $J \setminus I$ has the minimum element m . If $s \in J$ then $m \leq s$. Thus, $m \rightarrow s = 1 \in I$ and $s \in I_m$. Hence, $J \subseteq I_m$. For the converse, if $s \in I_m$ then $m \rightarrow s \in I \subseteq J$ and since $m \in J$, it follows from (I_2) that $s \in J$. Therefore, $I_m \subseteq J$, leading to $I_m = J$. Hence, $J \in I_L$. $\square$

COROLLARY 4.19. If L is a finite Brouwerian semi-lattice, then $I_L = \uparrow I$.

COROLLARY 4.20. If $I \in \mathcal{Max}(L)$ and $a \in L \setminus I$, then $I_a = L$.

PROOF: We know that $I \subseteq I_a$. Since I is a maximal ideal in L , we have either $I_a = I$ or $I_a = L$. However, since $a \in I_a$ and $a \notin I$, it follows that $I_a \neq I$. Therefore, $I_a = L$. $\square$

COROLLARY 4.21. Let L be a finite Brouwerian semi-lattice, $I \in \mathcal{J}_d(L)$ and $a \in \text{Atom}(L)$ such that $I \subseteq \uparrow a$. Then $I_a \in \mathcal{Max}(L)$.

PROOF: By Proposition 4.16, we have $\langle a \rangle \in \mathcal{Max}(L)$. Since $\langle a \rangle \subseteq I_a$, we conclude that $I_a = \langle a \rangle$ or $I_a = L$. If $I_a = L$, then $0 \in I_a$. So, $a \rightarrow 0 \in I$, and by the assumption, we have $a \rightarrow 0 \in \langle a \rangle$. By (I_2) , we get $0 \in \langle a \rangle$, which is a contradiction. Hence, $I_a = \langle a \rangle$. $\square$

Now, we are ready to present the main results of this section. Our main results concern finding the elements of radical of an ideal and Jacobson

radical in certain finite L -algebras.

THEOREM 4.22. *Let L be a finite Brouwerian semi-lattice and $I \in \mathcal{I}_d(L)$. Furthermore, let*

$$\Lambda = \bigcap_{x \in I} \downarrow x \setminus \{0\}, \quad \Delta = \{m \in \Lambda \mid m \text{ is an atom}\}.$$

Then

$$\mathcal{Rad}(I) = \{t \in L \mid m \rightarrow t \in I, \text{ for any } m \in \Delta\}.$$

PROOF: Suppose $m \in \Delta$. Using Proposition 4.9(ii), we have $I \subseteq I_m$. If $m \in \Delta$, then m is a lower bound of I and m is an atom. So, $I \subseteq \uparrow m$ and by Corollary 4.21, we conclude $I_m \in \mathcal{Max}(L)$. So,

$$\bigcap_{\substack{M \in \mathcal{Max}(L) \\ I \subseteq M}} M \subseteq \bigcap_{m \in \Delta} I_m.$$

On the other hand, if $I \subseteq M \in \mathcal{Max}(L)$, then by Theorem 4.18, there exists $m \in L$ such that $M = I_m$. Thus,

$$\bigcap_{m \in \Delta} I_m \subseteq \bigcap_{\substack{M \in \mathcal{Max}(L) \\ I \subseteq M}} M.$$

Hence,

$$\mathcal{Rad}(I) = \bigcap_{m \in \Delta} I_m = \{t \in L \mid m \rightarrow t \in I, \text{ for any } m \in \Delta\}.$$

□

THEOREM 4.23. *If L is a bounded Brouwerian semi-lattice, then $\mathcal{J}(L) = \mathcal{D}(L)$.*

PROOF: By Corollary 3.10, we have $\mathcal{D}(L) \subseteq \mathcal{J}(L)$. If $x \in L$ is not dense, then $x \rightarrow 0 > 0$. Hence, $\langle x \rightarrow 0 \rangle = \uparrow (x \rightarrow 0) \neq L$. By Proposition 2.10, there is a maximal ideal M with $x \rightarrow 0 \in M$. Thus $x \notin M$ (otherwise $0 \in M$), which yields $x \notin \mathcal{J}(L)$. □

5. Conclusion

In this paper, inspired by the concepts of the radical of an ideal and the Jacobson radical in Ring Theory, we have defined the radical of an ideal and the Jacobson radical in L -algebras. Our primary motivation has been to identify the elements of these radicals. To achieve this, we introduced the notion of I_a by considering an ideal I of an L -algebra L and an element $a \in L$. Additionally, we introduced specific types of L -algebras, such as proper Glivenko algebras, Brouwerian semi-lattices, and Hilbert algebras.

In this study, we successfully identified the elements of the radical of an ideal in finite Brouwerian semi-lattices and the elements of the Jacobson radical in proper Glivenko algebras and bounded Brouwerian semi-lattices. We aim to extend these results to characterize the elements of the radical of an ideal and the Jacobson radical in all L -algebras. Furthermore, we posed an open problem, whose resolution—whether through proof or counterexample—would provide a compelling direction for future research.

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Sławomir Przybyło 

DOUBLE EQUIVALENTIAL ALGEBRAS

Abstract

In this paper, we introduce the notion of double equivalential algebras, defined as subreducts of double Heyting algebras with respect to the equivalence and dual equivalence operations. We establish the fundamental properties of these structures and investigate several distinguished classes related to them, including the class of double equivalential subreducts of Boolean algebras and the variety generated by the three-element chain, which is shown to be semisimple.

Keywords: equivalential algebras, double Heyting algebras, linear order, semisimple.

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1. Introduction

This article introduces the concept of double equivalential algebras, which are specific subreducts of double Heyting algebras. Double Heyting algebras are Heyting algebras whose dual lattices also form Heyting algebras. In these structures, in addition to the lattice operations and the constants 0 and 1, there are also two distinct binary operations interpreted as relative pseudocomplements. Double Heyting algebras — also known as semi-Boolean algebras, bi-Heyting algebras, or Heyting–Brouwer algebras — first appeared in the early 1970s ([20, 21]) and were later studied by the following authors: R. Beazer [1], S. Ghilardi [6], L. Iturrioz [11], P. Köhler [13]. These structures are also related to other algebraic systems, such as dually pseudocomplemented Heyting algebras, which were introduced in [22]. Over the past decade, double Heyting algebras have been revisited in Taylor’s doctoral dissertation and publications ([28, 27, 29]). This line of research falls within the broader field of studying various subreducts of Heyting algebras, such as Brouwerian semilattices and Hilbert algebras.

In this article, we investigate another class of subreducts, namely those equipped with two binary operations: one corresponding to equivalence and the other to dual equivalence. Equivalential subreducts of Heyting algebras are referred to as equivalential algebras. This notion was introduced by J. K. Kabziński and A. Wroński in [12] as an algebraic counterpart of the equivalential fragment of intuitionistic propositional logic. These algebras were subsequently studied in depth by K. Słomczyńska ([24, 25]). In recent years, structures closely related to equivalential algebras have also been examined, particularly those with additional operations ([15, 17, 18]).

Equivalential algebras constitute an important example of Fregean algebras, that is, algebras whose language contains a constant term 1 and which are both 1-regular and congruence orderable. Indeed, it turns out that every congruence permutable Fregean variety has a binary term that turns each of its algebras into an equivalential algebra [9, p. 608]. Fregean varieties have been studied primarily by P. Idziak, K. Słomczyńska and A. Wroński ([9, 10]). Other examples of Fregean varieties include the varieties of Boolean algebras, Heyting algebras, Hilbert algebras, Brouwerian

semilattices, and other algebras closely related to the algebraic counterparts of classical, intuitionistic, and intermediate logics. Congruence orderability and 1-regularity allow for the introduction of a natural partial order on the universe of every Fregean algebra.

The purpose of this paper is to introduce and investigate the basic properties of double equivalential algebras. These structures are $(\leftrightarrow, \circ)$ -subreducts of double Heyting algebras equipped with the operation $\leftrightarrow$, defined by $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$, and the operation $\circ$, defined by $x \circ y = (x \leftarrow y) \vee (y \leftarrow x)$, where $\leftarrow$ is the dual operation to $\rightarrow$.

The study of these structures presents several challenges. In particular, they are not Fregean, as they are not congruence orderable. Moreover, the class of double equivalential algebras is not locally finite. For this reason, in what follows, we consider varieties generated by finite double equivalential chains. This line of investigation is closely related to the study of the equivalential fragment of one of the most important intermediate logics: Gödel–Dummett (or chain) intermediate logic LC, which was introduced by Kurt Gödel in 1932 [7] and later axiomatized by Michael Dummett [3] by adding the linearity axiom $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ to intuitionistic propositional logic (see [25, p. 1341]).

In Section 2, we recall the basic concepts related to double Heyting algebras and equivalential algebras. Next, in Section 3, we introduce the notion of double equivalential algebras. We provide examples of these structures, outline the difficulties associated with their study, and describe their fundamental properties. Subsequently, we consider two classes of algebras related to double equivalential algebras. In Subsection 3.4, we focus on subreducts of Boolean algebras, and in Subsection 3.5, we examine the variety generated by the three-element chain.

2. Double Heyting algebras and equivalential algebras

First, we recall some fundamental notions and facts about double Heyting algebras and equivalential algebras.

DEFINITION 2.1 ([27, p. 6]). An algebra $\mathbf{A} = (A, \vee, \wedge, \rightarrow, \leftarrow, 0, 1)$ is called a **double Heyting algebra** if $(A, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra and

$(A, \vee, \wedge, \leftarrow, 0, 1)$ is a dual Heyting algebra. More precisely, $\mathbf{A}$ is a double Heyting algebra if $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and the operations $\rightarrow$ and $\leftarrow$ satisfy the following equivalences:

$$x \wedge y \leq z \iff y \leq x \rightarrow z,$$

$$x \vee y \geq z \iff y \geq z \leftarrow x.$$

Basic information about Heyting algebras can be found in [19] and [4]. Below, we present some of their most important properties.

LEMMA 2.2 ([19, pp. 59–60]). *Let $\mathbf{L}$ be a Heyting algebra and let $x, y, z \in L$. Then:*

1. $x \leq y$ if and only if $x \rightarrow y = 1$,
2. $y \leq x \rightarrow y$,
3. $x \wedge (x \rightarrow y) = x \wedge y$,
4. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
5. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
6. if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $x \rightarrow z \geq y \rightarrow z$.

We denote the class of double Heyting algebras by $\mathcal{DH}$.

DEFINITION 2.3. By an **equivalential algebra** we mean an algebra $(A, \leftrightarrow)$ that is a $(\leftrightarrow)$ -subreduct of a Heyting algebra, where the operation $\leftrightarrow$ is defined by:

$$x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x).$$

We adopt the convention of associating to the left and replacing with ‘.’ (or ignoring) the symbol of the equivalence operation.

In 1975, Kabziński and Wroński proved that the class of all equivalential algebras is equationally definable by the following identities: $xyx = y$, $xyzxz = xz(yz)$, $xy(xzz)(xzz) = xy$, and thus forms a variety (see [12]). We denote this class by $\mathcal{E}$.

Remark 2.4 ([12, p. 420]). Let $\mathbf{A} \in \mathcal{E}$. For all $a, b, c \in A$, we have

1. $ab = ba$,
2. $aa = bb$,
3. $abbb = ab$,
4. $abbaa = abb$,
5. $abcc = acc(bcc)$,
6. $abbcc = accbb$,
7. $abb(bc)(bc) = abbcc$,
8. $abcb = a(bc)b$.

We can define the unit element by $1 := aa$, for an arbitrary $a \in A$ (see [24, p. 526]). We note that $xy = 1$ if and only if $x = y$. If A is an algebra equipped with a binary operation satisfying the conditions of Definition 2.3, then we denote the equivalential reduct of $\mathbf{A}$ by $\mathbf{A}^e$.

LEMMA 2.5 ([28, p. 4]). *Let $\mathbf{H}$ be a Heyting algebra and let $x, y \in H$. Then $x \cdot y$ is the greatest $z \in H$ such that $x \wedge z = y \wedge z$. That is:*

$$x \wedge z = y \wedge z \iff z \leq x \cdot y$$

Equivalential algebras are important examples of Fregean algebras.

DEFINITION 2.6 ([9, p. 597]). An algebra $\mathbf{A}$ with a distinguished constant term 1 is called **Fregean** if $\mathbf{A}$ is:

1. **1-regular**, i.e., $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \text{Con } \mathbf{A}$,
2. **congruence orderable**, i.e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$.

A variety $\mathcal{V}$ is said to be Fregean if all its algebras are Fregean.

These two properties of congruences allow us to introduce a natural partial order on the universe of every Fregean algebra $\mathbf{A}$ in the following

way: $a \leq b$ iff $\Theta_{\mathbf{A}}(1, b) \subseteq \Theta_{\mathbf{A}}(1, a)$. Clearly, 1 is the greatest element in this order. From 1-regularity it follows that the Fregean varieties are congruence modular (see [8]).

DEFINITION 2.7 ([26, p. 2]). An algebra $(A, \cdot, 0)$ with one binary operation $\cdot$ and one constant 0 is called an **equivalential algebra with zero** if $(A, \cdot)$ is an equivalential algebra and $\mathbf{A}$ satisfies the identity:

$$x00yy = x00.$$

Dually, we introduce the following definition and notation:

DEFINITION 2.8. By a **dual equivalential algebra** we mean an algebra $(A, \circ)$ that is a $(\circ)$ -subreduct of a dual Heyting algebra, where the operation $\circ$ is defined by:

$$x \circ y := (x \leftarrow y) \vee (y \leftarrow x).$$

As with the operation $\cdot$, we adopt the convention of associating to the left, that is, instead of $(x \circ y) \circ z$ we use the more concise notation: $x \circ y \circ z$. This convention is also adopted when the operations $\cdot$ and $\circ$ appear together. Moreover, we assume that the operators $\cdot$ and $\circ$ bind more strongly than the standard operations in a Heyting algebra (i.e., $\wedge, \vee$ and $\rightarrow$).

We can define the zero element by $0 := a \circ a$, for an arbitrary $a \in A$. Furthermore:

LEMMA 2.9. *If $\mathbf{H}$ is a dual Heyting algebra and $x, y \in H$, then $x \circ y$ is the smallest $z \in H$ such that $x \vee z = y \vee z$, that is:*

$$x \vee z = y \vee z \iff x \circ y \leq z.$$

Dually to Definition 2.7, we define a **dual equivalential algebra with zero** (in this case, the role of zero is played by one) as an algebra $(A, \circ, 1)$, with one binary operation $\circ$ and one constant 1, such that $(A, \circ)$ is a dual equivalential algebra and A satisfies the identity: $x \circ 1 \circ 1 \circ y \circ y = x \circ 1 \circ 1$.

We say that $x, y \in A$ are **orthogonal** if $xyy = x$ and $yx x = y$. Dually, we say that $x, y \in A$ are **dually orthogonal** if $x \circ y \circ y = x$ and $y \circ x \circ x = y$.

We now introduce the following notation:

$$d(x) := x \circ 1 \cdot 0,$$

$$g(x) := x \cdot 0 \circ 1.$$

LEMMA 2.10. *Let $\mathbf{H}$ be a double Heyting algebra and let $x, y \in H$. Then:*

1. $x \cdot 0 = x \rightarrow 0 = \max\{c \in H : c \wedge x = 0\}$, that is, $x \cdot 0$ is the pseudocomplement of x in H ,
2. $x \circ 1 = 1 \leftarrow x = \min\{c \in H : c \vee x = 1\}$, that is, $x \circ 1$ is the dual pseudocomplement of x in H ,
3. $x \cdot 0 \leq x \circ 1$,
4. $d(x) \leq x \circ 1 \circ 1 \leq x \leq x \cdot 0 \cdot 0 \leq g(x)$,
5. $x \cdot 0 = x \circ 1$ if and only if $d(x) = x$,
6. $x \cdot y \circ 1 \geq x \circ y$ and $x \circ y \cdot 0 \leq x \cdot y$.

PROOF: Items (1–2) follow from the definitions of $\cdot$ and $\circ$, items (3–5) follow from [28, p. 5], and item (6) from [28, p. 74]. $\square$

2.1. Filters in equivalential algebras

Let $\Theta(a, b)$ denote the congruence generated by the pair (a, b) . Let $\mathbf{A} \in \mathcal{E}$. By a **filter**, we mean a non-empty subset $F \subseteq A$ such that for all $a, b, x \in A$: (i) if $a \in F$, then $axx \in F$ and (ii) if $a \in F$ and $ab \in F$, then $b \in F$. It follows from [12, p. 421] that there is a one-to-one correspondence between the filters of $\mathbf{A}$ and its congruence relations. Namely, if $F \subseteq A$, then F is a filter if and only if the relation φ given by $a\varphi b \iff ab \in F$ is a congruence on A . Moreover, $\Theta(1, ab) = \Theta(a, b)$ (see [9, Theorem 3.8]).

Let $x \in A$. We denote by $[x]_e$ the smallest filter containing x . The relation $\leq_e$ defined by $x \leq_e y$ whenever $y \in [x]_e$ is a partial order on $\mathbf{A}$. We also introduce the relation $<<$ on A , defined by $x << y$ if $xy = x$ (i.e., $yx = 1$). Its restriction to $A \setminus \{1\}$ is a strict order, while $<< \cup \text{Id}$ is a partial order on A . We note that if $\mathbf{A}$ is a Heyting algebra and $x << y$, then $x \rightarrow y = 1$.

Dually, by a **dual filter** we mean a non-empty subset $F \subseteq A$ such that for all $a, b, x \in A$: (i) if $a \in F$, then $a \circ x \circ x \in F$ and (ii) if $a \in F$ and $a \circ b \in F$, then $b \in F$. Let $x \in A$. We denote by $(x)_d$ the smallest dual filter containing x . The relation $\leq_d$, defined by $x \leq_d y$ whenever $x \in (y)_d$ is a partial order on A . We also introduce the relation $<<_d$ on A defined by $x <<_d y$ if $x \circ y = y$ (i.e., $x \circ y \circ y = 0$).

3. Double equivalential algebras

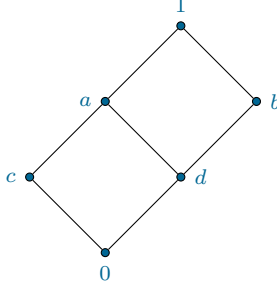
3.1. Definition and examples

DEFINITION 3.1. Let $\mathbf{A} = (A, \cdot, \circ, 1, 0)$ be a $(\cdot, \circ)$ -subreduct of a double Heyting algebra. Then $\mathbf{A}$ is called a **double equivalential algebra**.

We denote the class of double equivalential algebras by $\mathcal{DE}$. This class forms a quasivariety (see [14]). Note that if $\mathbf{A} = (A, \cdot, \circ, 1, 0) \in \mathcal{DE}$, then $(A, \cdot, 0)$ is an equivalential algebra with zero and $(A, \circ, 1)$ is a dual equivalential algebra with zero. Moreover, $1 = x \cdot x$ and $0 = x \circ x$ for all $x \in A$. Every algebra in $\mathcal{DE}$ has permutable congruences, since $(xyz)(xzzx)$ is a Mal'cev term for equivalential algebras (see [10, p. 332]). Consequently, every variety generated by algebras from $\mathcal{DE}$ is congruence permutable. Both the variety of equivalential algebras and the variety of dual equivalential algebras are Fregean – that is, they are congruence orderable, with the former being 1-regular and the latter 0-regular. Therefore, the class $\mathcal{DE}$ is both 1-regular and 0-regular (since point regularity is preserved under expansions of a language, see [9, p. 600]). However, the class $\mathcal{DE}$ is not Fregean, as it is not congruence orderable. Let us consider the following example.

Example 3.2. Let $\mathbf{A}$ be the $(\cdot, \circ)$ -reduct of the double Heyting algebra $\mathbf{3} \times \mathbf{2}$ (see Figure 1). It is easily seen that $\Theta(1, a) = \Theta(1, c)$ and $\Theta(1, 0) = A^2$. Therefore, $\text{Con } \mathbf{A}$ is a four-element lattice (in addition to the aforementioned ones, it also contains Id and $\Theta(1, b)$). Since $\Theta(1, a) = \Theta(1, c)$, the algebra $\mathbf{A}$ is not congruence orderable.

This presents a significant challenge in the analysis of these structures,

Figure 1: Double Heyting algebra $\mathbf{3} \times \mathbf{2}$.

specifically due to the lack of an inherent order. Although it is possible to define orderings such as $\leq_e$, $\leq_d$, $<<$, and $<<_d$ on the algebra $A \in \mathcal{DE}$ — and in the case where $\mathbf{A}$ is a Heyting algebra, also the standard lattice order (denoted by $\leq$) — these orderings do not coincide and fail to satisfy many of the properties that hold for the lattice order in double Heyting algebras. Moreover, in the case where $\mathbf{A}$ is a Boolean algebra, no two distinct elements of $A \setminus \{0, 1\}$ are comparable with respect to any of these orderings (except for $\leq$).

The following diagrams (Figure 2) illustrate the aforementioned orderings for the algebra presented in Example 3.2 (from left to right: $\leq_e$, $<<$, $\leq_d$, and $<<_d$). Note that even in this simple example, some of the laws satisfied by the lattice order in double Heyting algebras do not hold.

1. $d(x) \leq_e x$ (counterexample: $d(d) = 0 \not\leq_e d$),
2. $x \leq_e y \implies d(x) \leq_e d(y)$ (counterexample: $0 \leq_e a$, but $0 = d(0) \not\leq_e d(a) = c$; a similar situation occurs if we replace $\leq_e$ with $<<$),
3. $x \leq_e y \implies y0 \leq_e x0$ (counterexample: $c \leq_e a$, but $a0 = 0 \not\leq_e b = c0$; a similar situation occurs if we replace $\leq_e$ with $<<$).

Moreover, we have the following theorem.

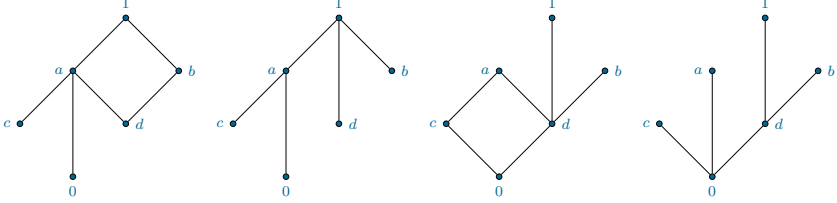


Figure 2: Different orderings for the algebra from Example 3.2 (from left to right: $\leq_e$, $<<$, $\leq_d$, and $<<_d$).

THEOREM 3.3. $\mathcal{DE}$ is not locally finite.

PROOF: We proceed analogously to [16, p. 142]. Let us consider the free Heyting algebra over one generator x , i.e., the Rieger–Nishimura lattice $\mathbf{R}$ (see Figure 3). First, observe that $\mathbf{R}$ is a double Heyting algebra. The universe of $\mathbf{R}$ has the form $R = \{0, 1\} \cup \{r_k : k \in \mathbb{N}\}$, where $r_1 = x$, $r_2 = x \cdot 0$, $r_{2n+1} = r_{2n} \circ r_{2n-1}$, $r_{2n+2} = r_{2n+1} \cdot r_{2n-1}$ for $n = 1, 2, \dots$. Therefore, the $(\cdot, \circ)$ -reduct of $\mathbf{R}$ is a one-generated infinite double equivalential algebra, hence $\mathcal{DE}$ is not locally finite. $\square$

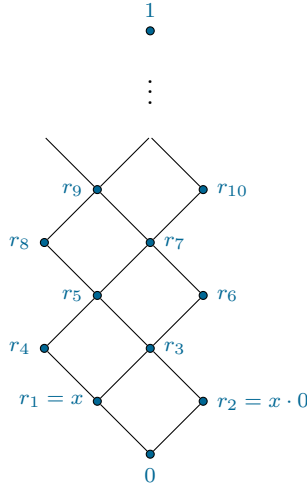
3.2. Basic properties

LEMMA 3.4. Let $\mathbf{H} \in \mathcal{DH}$ and let $a, b \in H$. Then:

1. $a \cdot b \geq a \wedge b$ and $a \circ b \leq a \vee b$,
2. $a0 \circ a = a0 \vee a$ and $(a \circ 1)a = (a \circ 1) \wedge a$.

PROOF: **Ad 1.** This follows from Lemmas 2.5 and 2.9.

Ad 2. From Lemma 2.9, we have: $(a0 \circ a) \vee a = (a0 \circ a) \vee a0$. Taking the meet of both sides with $a0$, we obtain: $((a0 \circ a) \vee a) \wedge a0 = ((a0 \circ a) \vee a0) \wedge a0 = a0$. By distributivity, $((a0 \circ a) \wedge a0) \vee (a \wedge a0) = a0$, which implies that $(a0 \circ a) \wedge a0 = a0$, and hence $a0 \circ a \geq a0$. Next, taking the meet of both sides

Figure 3: Rieger–Nishimura lattice $\mathbf{R}$.

of the original equality $(a0 \circ a) \vee a = (a0 \circ a) \vee a0$ with a , and proceeding analogously, we obtain $a0 \circ a \geq a$. Therefore, $a0 \circ a \geq a0 \vee a$. Since, by item (1), we know that $a0 \circ a \leq a0 \vee a$, it follows that $a0 \circ a = a0 \vee a$. The second equality is proved analogously. $\square$

THEOREM 3.5. *Let $\mathbf{A} \in \mathcal{DE}$ and let $a, b \in A$. Then:*

1. $(a \circ 1)aa = a \circ 1$, $a0 \circ a \circ a = a0$,
2. $a(a \circ 1)(a \circ 1) = a$, $a \circ (a0) \circ (a0) = a$,
3. $(a0 \circ a) \cdot 0 = 0$, $(a \circ 1 \cdot a) \circ 1 = 1$,
4. $a0 \circ a \cdot a = a00$, $a \circ 1 \cdot a \circ a = a \circ 1 \circ 1$,
5. $(a0) \cdot (a \circ 1) = a \cdot d(a)$, $(a0) \circ (a \circ 1) = a \circ g(a)$,
6. if $a << b$, then $b0 = 0$ and $g(b) = 1$.

PROOF: If a given item contains two dual identities, we prove the first one (the second, being its dual, can be shown analogously). It suffices to show that the given identities hold in double Heyting algebras.

Ad 1. Using item (2) from Lemma 3.4, we know that $(a \circ 1)aa \geq a \circ 1 \geq (a \circ 1) \wedge a = (a \circ 1)a$. From this and Lemma 2.5, it follows that $(a \circ 1)aa \wedge a = (a \circ 1)aa \wedge (a \circ 1)a = (a \circ 1)a$. Since $a \circ 1 \leq (a \circ 1)aa$, by modularity we obtain:

$$((a \circ 1) \vee a) \wedge (a \circ 1)aa = (a \circ 1) \vee (a \wedge (a \circ 1)aa). \quad (3.1)$$

Because $(a \circ 1) \vee a = 1$, the left-hand side of (3.1) equals $(a \circ 1)aa$. On the other hand, the right-hand side of (3.1), by the previously established equality, can be rewritten as $(a \circ 1) \vee (a \circ 1)a = a \circ 1$. Hence, the claim follows.

Ad 2. The proof proceeds analogously to item (1).

Ad 3. Using item (2) from Lemma 3.4 and the properties of operations in Heyting algebras, we obtain: $(a0 \circ a) \cdot 0 = (a0 \vee a) \cdot 0 = [(a0 \vee a) \rightarrow 0] \wedge [0 \rightarrow (a0 \vee a)] = (a0 \vee a) \rightarrow 0 = (a0 \rightarrow 0) \wedge (a \rightarrow 0) = (a00) \wedge a0 = 0$.

Ad 4. $a0 \circ a \cdot a = (a0 \vee a) \cdot a = [(a0 \vee a) \rightarrow a] \wedge [a \rightarrow (a0 \vee a)] = (a0 \vee a) \rightarrow a = (a0 \rightarrow a) \wedge (a \rightarrow a) = a0 \rightarrow a = (a \rightarrow 0) \rightarrow a$.

Since $axx = (a \rightarrow x) \rightarrow a$, the final expression equals $a00$.

Ad 5. This follows from Lemma 2.10 (item (5)).

Ad 6. From the assumption that $a \cdot b = a$, it follows that $b \rightarrow a = a$. Thus, $b0 = b \rightarrow 0 = b \rightarrow (a \wedge 0) = (b \rightarrow a) \wedge (b \rightarrow 0) = a \wedge (b \rightarrow 0)$. Hence, $b \rightarrow 0 \leq a \leq b$, and therefore $b \rightarrow 0 = (b \rightarrow 0) \wedge b = b \wedge 0 = 0$. $\square$

Note that, based on items (1)–(2) of Theorem 3.5, it follows that the elements a and $a \circ 1$ are orthogonal, while the elements a and $a0$ are dually orthogonal.

DEFINITION 3.6. Let $\mathbf{A} \in \mathcal{DE}$, and let $a \in A$. The element a is called **central** if $d(a) = a$.

THEOREM 3.7. *Let $\mathbf{H} \in \mathcal{DH}$ and let $a, b \in H$. Then, if $a < b$, it follows that $b0 << (a0) \circ a$.*

PROOF: Assume that $a < b$. Then $a0 \geq b0$. Since, by Lemma 3.4, item (2), we have $a0 \vee a = a0 \circ a$, it remains to show that

$$(a0 \vee a) \cdot (b0) = b0. \quad (3.2)$$

We first show that $[(a0 \vee a) \cdot (b0)] \wedge a = 0$. From the properties of operations in a Heyting algebra, we have:

$$[(a0 \vee a) \cdot (b0)] \wedge a = [((a0 \vee a) \rightarrow b0) \wedge (b0 \rightarrow (a0 \vee a))] \wedge a. \quad (3.3)$$

From the assumption, we have $a0 \vee a \geq b0$, so $b0 \rightarrow (a0 \vee a) = 1$. Therefore, expression (3.3) is equal to $[(a0 \vee a) \rightarrow b0] \wedge a = (a \rightarrow b0) \wedge (a0 \rightarrow b0) \wedge a = [(a \rightarrow b0) \wedge a] \wedge (a0 \rightarrow b0) = (a \wedge b0) \wedge (a0 \rightarrow b0) = a \wedge b0$. Since $b \wedge b0 = 0$ and $a < b$, it follows from Lemma 2.5 that expression (3.3) equals 0. By Lemma 2.5, we obtain $(a0 \vee a) \cdot (b0) \leq a0$.

Once again, by Lemma 2.5, we have:

$$[(a0 \vee a) \cdot (b0)] \wedge (a0 \vee a) = [(a0 \vee a) \cdot (b0)] \wedge b0. \quad (3.4)$$

From the previously proven inequality, the left-hand side of (3.4) equals $(a0 \vee a) \cdot (b0)$. As for the right-hand side, since $a0 \geq b0$, and by Lemma 3.4, item (1), we obtain $(a0 \vee a) \cdot (b0) \geq (a0 \vee a) \wedge (b0) \geq b0$. Therefore, the right-hand side of (3.4) is equal to $b0$, and we conclude that: $(a0 \vee a) \cdot (b0) = b0$. $\square$

COROLLARY 3.8. *Let $\mathbf{H} \in \mathcal{DH}$ and let $c, x \in H$. Then, if $c < x0$, it follows that $x << (c \cdot 0) \circ c$.*

PROOF: By Theorem 3.7 and the fact that $x \leq x00$, the claim follows. $\square$

Let $\mathbf{A} \in \mathcal{DE}$, and let $x \in A$. The element x is called **regular** if $x00 = x$, and **dense** if $x00 = 1$.

Remark 3.9. Let $\mathbf{A} \in \mathcal{DE}$, and let $d \in A$. If d is a dense element, then $g(d) = 1$. Moreover, if there exist elements $x, d \in A$ such that $x << d$, then d is dense.

PROOF: From the density of d , we have $d0 = d000 = 1 \cdot 0 = 0$. Hence, $g(d) = d \cdot 0 \circ 1 = 0 \circ 1 = 1$. The second part follows from item (6) of Theorem 3.5. $\square$

THEOREM 3.10. *Let $\mathbf{A} \in \mathcal{DE}$. For any $a \in A$, the element $(a \cdot 0) \circ a$ is dense, and $g((a \cdot 0) \circ a) = 1$. Dually, $d((a \circ 1) \cdot a) = 0$.*

PROOF: If $a = 1$, then clearly $(a \cdot 0) \circ a$ is dense and $g(a) = 1$. Let us therefore assume that $a \neq 1$. Then $a << 1$. It follows from Theorem 3.7 that $1 \cdot 0 = 0 << (a \cdot 0) \circ a$. By Remark 3.9, we conclude that $(a \cdot 0) \circ a$ is dense and $g((a \cdot 0) \circ a) = 1$. $\square$

3.3. Congruences of double equivalential algebras

THEOREM 3.11. *Let $\mathbf{A} \in \mathcal{DE}$, and let $\varphi \in \text{Con } \mathbf{A}$. Then, for all $x, y \in A$:*

1. $x \in 1/\varphi \text{ iff } x \circ 1 \in 0/\varphi$,
2. $xy \in 1/\varphi \text{ iff } x \circ y \in 0/\varphi \text{ iff } (x \circ y) \cdot 0 \in 1/\varphi$,
3. $x \in 1/\varphi \text{ iff } d(x) \in 1/\varphi$,
4. $0/\varphi = \{x \in A : x0 \in 1/\varphi\}$.

PROOF: Let $\varphi \in \text{Con } \mathbf{A}$. Observe that $\text{Con } \mathbf{A} = \text{Con } \mathbf{A}^e \cap \text{Con } \mathbf{A}^\circ$, where $\mathbf{A}^\circ$ denotes the $(\circ)$ -reduct of $\mathbf{A}$. Hence, $(x, y) \in \varphi$ if and only if $(1, xy) \in \varphi$, and this is equivalent to $(0, x \circ y) \in \varphi$.

Ad 1. This follows from the equivalences:

$$x \in 1/\varphi \iff (x, 1) \in \varphi \iff (x \circ 1, 0) \in \varphi \iff x \circ 1 \in 0/\varphi.$$

Ad 2. The following equivalences hold:

$$\begin{aligned} xy \in 1/\varphi &\iff (1, xy) \in \varphi \iff (x, y) \in \varphi \iff (x \circ y, 0) \in \varphi \iff \\ &\iff (x \circ y) \cdot 0 \in 1/\varphi. \end{aligned}$$

Ad 3. From item (1) and its dual, it follows that:

$$x \in 1/\varphi \iff x \circ 1 \in 0/\varphi \iff d(x) = x \circ 1 \cdot 0 \in 1/\varphi.$$

Ad 4. Since $x = x \cdot 1 = x \circ 0$, item (2) implies that $x \in 0/\varphi$ if and only if $x \circ 0 \in 1/\varphi$. $\square$

The following theorem, whose proof is due to Katarzyna Słomczyńska, specifies the conditions under which an equivalential congruence is also a double equivalential congruence.

THEOREM 3.12 ([23]). *Let $\mathbf{A} \in \mathcal{DE}$, and let $\alpha \in \text{Con } \mathbf{A}^e$. Denote $F := 1/\alpha$. Then, $\alpha \in \text{Con } \mathbf{A}$ iff the following conditions are satisfied:*

1. $ab \in F$ iff $(a \circ b) \cdot 0 \in F$,
2. if $a0 \in F$, then $(a \circ z) \cdot z \in F$ for all $z \in A$.

PROOF: It suffices to show that α is compatible with $\circ$. Let $(a, b) \in \alpha$ and $z \in A$. Then $ab \in F$, so by condition (1), $(a \circ b) \cdot 0 \in F$. By condition (2), we have $(a \circ b \circ z) \cdot z \in F$, and again, by condition (1), $((a \circ b \circ z) \circ z) \cdot 0 \in F$. Hence, $((a \circ z) \circ (b \circ z)) \cdot 0 \in F$. Finally, applying condition (1) once more yields $(a \circ z) \cdot (b \circ z) \in F$, i.e., $(a \circ z, b \circ z) \in \alpha$. $\square$

3.4. Double equivalential subreducts of Boolean algebras

Recall that for $(A, \cdot) \in \mathcal{E}$, the operation $\cdot$ is associative if and only if the identity $abb = a$ holds for all $a, b \in A$.

Let $\mathbb{S}\mathbb{B}^r$ denote the class of all $(\cdot, \circ)$ -subreducts of Boolean algebras.

THEOREM 3.13. *$(A, \cdot, \circ, 1, 0) \in \mathbb{S}\mathbb{B}^r$ if and only if $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, $x \circ y = x \cdot y \cdot 0$, and the following quasi-identity holds: $0 = 1 \implies x = y$.*

PROOF: The implication from left to right is straightforward. So assume that $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, that $x \circ y = x \cdot y \cdot 0$, and that the quasi-identity $0 = 1 \implies x = y$ holds. By [26, Theorem 3.13], it follows that $(A, \cdot, 0)$ can be embedded into an algebra $\mathbf{B}$ belonging to the quasivariety of $(\cdot, 0)$ -subreducts of Heyting algebras. Since the operation $\cdot$ is associative, it follows from the proof of Theorem 3.13 in [26] that $\mathbf{B}$ is a

Boolean algebra. Moreover, because $x \circ y = x \cdot y \cdot 0$, this embedding also preserves the operation $\circ$. $\square$

Note that in the above theorem, instead of assuming that $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, we could have assumed only that they are equiv-
alential algebras with zero. The associativity of both operations follows
from the condition $x \circ y = x \cdot y \cdot 0$, since in that case we have, for any
element $x \in A$: $x00 = x \circ 0 = x$. However, the converse does not hold, as
demonstrated by the following example.

Example 3.14. Let $(A, \cdot, \circ, 1, 0)$ be a structure where the operations $\cdot$ and $\circ$
are defined by the following tables:

$\cdot$	1	a	b	c	d	e	f	0
1	1	a	b	c	d	e	f	0
a	a	1	d	e	b	c	0	f
b	b	d	1	f	a	0	c	e
c	c	e	f	1	0	a	b	d
d	d	b	a	0	1	f	e	c
e	e	c	0	a	f	1	d	b
f	f	0	c	b	e	d	1	a
0	0	f	e	d	c	b	a	1

$\circ$	1	a	b	c	d	e	f	0
1	0	d	e	f	a	b	c	1
a	d	0	c	b	1	f	e	a
b	e	c	0	a	f	1	d	b
c	f	b	a	0	e	d	1	c
d	a	1	f	e	0	c	b	d
e	b	f	1	d	c	0	a	e
f	c	e	d	1	b	a	0	f
0	1	a	b	c	d	e	f	0

Thus, $(A, \cdot)$ is the $(\cdot)$ -reduct of the Boolean algebra shown in Figure 4
on the left, while $(A, \circ)$ is the $(\circ)$ -reduct of the Boolean algebra depicted
in Figure 4 on the right (observe that this algebra differs from the first one
by swapping the positions of the elements d and f).

In this case, $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, but $(A, \cdot, \circ, 1, 0)$ is not
a $(\cdot, \circ)$ -subreduct of a Boolean algebra, since the condition $x \circ y = x \cdot y \cdot 0$
does not hold (for example, $a \cdot f \cdot 0 = 1$, whereas $a \circ f = e$).

From Theorem 3.13, we obtain the following corollary.

COROLLARY 3.15. If $(A, \cdot, \circ, 1, 0) \in \mathbf{SBB}^r$, then

$$\text{Con}(A, \cdot, \circ, 1, 0) = \text{Con}(A, \cdot, 0) = \text{Con}(A, \circ, 1).$$

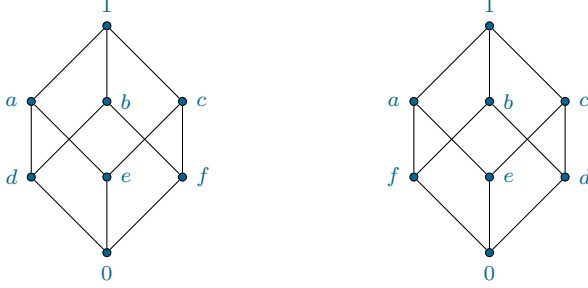


Figure 4: Boolean algebras whose reducts are $(A, \cdot)$ and $(A, \circ)$, respectively.

Now consider the case where the condition $x \circ y = x \cdot y$ holds.

Remark 3.16. Let $(A, \cdot, \circ, 1, 0) \in \mathcal{DE}$ and suppose that $x \circ y = x \cdot y$ for all $x, y \in A$. Then $0 = 1$ and $(A, \cdot)$ is a Boolean group.

PROOF: Notice that $1 = 1 \cdot 1 = 1 \circ 1 = 0$. Furthermore, for any $x, y \in A$, we have $xyy = x11yy = x00yy = x00 = x11 = x$. $\square$

3.5. $V(\mathbf{3})$

Let $n \in \mathbb{N}$. We denote by $\mathbf{n}$ the $(\cdot, \circ)$ -reduct of an n -element double Heyting algebra whose underlying set forms a chain.

In particular, $\mathbf{3} = (\{0 < * < 1\}, \cdot, \circ, 1, 0)$ and $\mathbf{2} = (\{0 < 1\}, \cdot, \circ, 1, 0)$. Let $V(\mathbf{n})$ denote the variety generated by $\mathbf{n}$.

Remark 3.17. If $\mathbf{A} \in V(\mathbf{n})$, then the following identities hold:

1. $(x \circ y) \cdot 0 = (x \cdot y) \circ 1 \cdot 0$,
2. $(x \cdot y) \circ 1 = (x \circ y) \cdot 0 \circ 1$.

PROOF: These identities can be readily verified in $\mathbf{n}$ and hence hold in $V(\mathbf{n})$. $\square$

Using the above remark, Theorem 3.12 can be simplified for $\mathbf{A} \in V(\mathbf{n})$, taking the following form.

THEOREM 3.18. *Let $\mathbf{A} \in V(\mathbf{n})$, and let $\alpha \in \text{Con } \mathbf{A}^e$. Denote $F := 1/\alpha$. Then, $\alpha \in \text{Con } \mathbf{A}$ iff the following conditions are satisfied:*

1. $a \in F$ iff $d(a) \in F$,
2. if $a0 \in F$, then $(a \circ z) \cdot z \in F$ for all $z \in A$.

PROOF: If $A \in V(\mathbf{n})$, then, using Remark 3.17, we can rewrite condition (1) from Theorem 3.12 as follows:

$$ab \in F \text{ iff } (a \circ b) \cdot 0 \in F \text{ iff } (a \cdot b) \circ 1 \cdot 0 \in F.$$

Since $a = a \cdot 1$, the assertion follows. $\square$

Let $2_{0=1}$ denote the two-element algebra $(A, \cdot, \circ, 1, 0)$ in which the constants 0 and 1 have the same interpretation. Note that, similarly to [26, p. 3], $2_{0=1} \in H(\mathbf{2} \times \mathbf{2})$, and therefore $2_{0=1} \in V(\mathbf{3})$. From Remark 3.17, it follows that if $x, y \in 2_{0=1}$, then $x \cdot y = x \circ y$.

Observe that in the case of $\mathbf{3}$, we have $d(*) = 0$. Hence, Theorem 3.18 implies the following remark.

Remark 3.19. The algebras $\mathbf{3}$, $\mathbf{2}$, and $2_{0=1}$ are simple.

We will use the following theorem:

THEOREM 3.20 (Foster–Pixley [5]). *Let $\mathbf{S}_1, \dots, \mathbf{S}_n$ be simple algebras in a congruence-permutable variety V . If*

$$C \leq \mathbf{S}_1 \times \cdots \times \mathbf{S}_n$$

is a subdirect product, then

$$C \cong \mathbf{S}_{i_1} \times \cdots \times \mathbf{S}_{i_k}$$

for some $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$.

THEOREM 3.21. *Let $\mathbf{A} \in \text{SP}(\mathbf{3}, \mathbf{2}, 2_{0=1})$ with $1 < |A| < \infty$. Then $\mathbf{A} \in \text{P}(\mathbf{3}, \mathbf{2}, 2_{0=1})$.*

PROOF: Suppose $\mathbf{A} \in \mathbf{S}(\mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m)$ for some $n, k, m \in \mathbb{N}$. Since the algebras $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$ are simple, and all of their nontrivial subalgebras are also simple, it follows that $\mathbf{A}$ is a subdirect product of a finite algebra from $\mathbf{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$. By the Foster–Pixley Theorem, $\mathbf{A} \cong \mathbf{3}^i \times \mathbf{2}^j \times \mathbf{2}_{0=1}^l$ for some $i, j, l \in \mathbb{N}$ such that $i \leq n$, $j \leq k$, and $l \leq m$. $\square$

We now describe the structure of homomorphic images of algebras of the form $\mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$.

Clearly, $\text{Con } \mathbf{A} \subseteq \text{Con } \mathbf{A}^e$. Using the fact that, by Theorem 3.18, $a \in 1/\varphi$ if and only if $d(a) \in 1/\varphi$, we can effectively restrict the set of congruences in $\text{Con } \mathbf{A}^e$ to those that also belong to $\text{Con } \mathbf{A}$.

Let $\mathbf{A} = \mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$, for some $n, k, m \in \mathbb{N}$, and let $x \in A$. We note that x is central if and only if $x_i \in \{0, 1\}$ for all $i \in \{1, \dots, n+k+m\}$. Moreover, for each such i , we have $d(x)_i = 1$ if $x_i = 1$, and $d(x)_i = 0$ if $x_i \in \{0, *\}$. Observe that the neutral element of the operation $\circ$, that is, 0, has the following form: $0_i = 0$ for all $i \in \{1, \dots, n+k\}$, and $0_i = 1$, for all $i \in \{n+k+1, \dots, n+k+m\}$. By a double equivalential filter of $\mathbf{A}$ we mean an equivalential filter F whose corresponding congruence belongs to $\text{Con } \mathbf{A}$.

THEOREM 3.22. *Let $\mathbf{A} = \mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$, for some $n, k, m \in \mathbb{N}$, and let $x \in A$. Then:*

1. $F = \{1, x, d(x)\}$, where $x_l = *$ for exactly one $l \in \{1, \dots, n\}$, and $x_i = 1$ for all $i \in \{1, \dots, n+k+m\}$ with $i \neq l$, is a double equivalential filter.
2. $H = \{1, x\}$, where $x \neq 1$ is such that $x_i = 1$ for all $i \in \{1, \dots, n\}$, and $x_j \in \{0, 1\}$ for all $j \in \{n+1, \dots, n+k+m\}$, is a double equivalential filter.

PROOF: First, note that both F and H are equivalential filters. Obviously, F and H satisfy condition (1) of Theorem 3.18. We will show that they also satisfy condition (2) of Theorem 3.18.

Ad 1. Without loss of generality, we may assume that $l = 1$. Let $a0 \in F$. If $a0 = 1$, then $a = 0$, and in this case, $a \circ z \cdot z = 0 \circ z \cdot z = z \cdot z = 1 \in F$,

for all $z \in A$. The case $a0 = x$ is impossible. Thus, let $a0 = d(x)$. Then $a_1 \in \{*, 1\}$, $a_i = 0$ for all $i \in \{2, \dots, n+k\}$, and $a_i = 1$ for all $i \in \{n+k+1, \dots, n+k+m\}$. Let $z \in A$. Then $a_i \circ z_i \cdot z_i = 0 \circ z_i \cdot z_i = z_i \cdot z_i = 1$ for all $i \in \{2, \dots, n+k\}$, and $a_i \circ z_i \cdot z_i = 1 \circ z_i \cdot z_i = z_i \cdot z_i = 1$ for all $i \in \{n+k+1, \dots, n+k+m\}$. Hence, $a \circ z \cdot z \in F$.

Ad 2. Let $a0 \in H$. As before, if $a0 = 1$, then $a \circ z \cdot z = 1 \in H$. Suppose now that $a0 = x$, and define $K = \{j \in \{n+1, \dots, n+k\} : x_j = 1\}$ and $M = \{j \in \{n+k+1, \dots, n+k+m\} : x_j = 0\}$. Then a is such that $a_i = 0$ for all $i \in \{1, \dots, n\} \cup K \cup M$, and $a_j = 1$ for all $j \in \{n+1, \dots, n+k+m\} \setminus (K \cup M)$. Let $z \in A$. Then $a_i \circ z_i \cdot z_i = 0 \circ z_i \cdot z_i = 1 = x_i$ for all $i \in \{1, \dots, n\} \cup K$, and $a_i \circ z_i \cdot z_i = 0 \circ z_i \cdot z_i = 0 = x_i$ for all $i \in M$. Now let $j \in \{n+1, \dots, n+k+m\} \setminus (K \cup M)$. Two cases are possible:

1) If $z_j = 1$, then $a_j \circ z_j \cdot z_j = 1 \circ 1 \cdot 1 = 0 = x_j$ for all $j \in \{n+1, \dots, n+k\} \setminus K$, and $a_j \circ z_j \cdot z_j = 1 \circ 1 \cdot 1 = 1 = x_j$ for all $j \in \{n+k+1, \dots, n+k+m\} \setminus M$,
 2) If $z_j = 0$, then $a_j \circ z_j \cdot z_j = 1 \circ 0 \cdot 0 = 0 = x_j$ for all $j \in \{n+1, \dots, n+k\} \setminus K$, and $a_j \circ z_j \cdot z_j = 1 \circ 0 \cdot 0 = 1 = x_j$ for all $j \in \{n+k+1, \dots, n+k+m\} \setminus M$.
 Hence, $a_j \circ z_j \cdot z_j = x_j$ for all $j \in \{1, \dots, n+k+m\}$. Therefore, $a \circ z \cdot z = x \in H$, which shows that condition (2) of Theorem 3.18 is satisfied. $\square$

THEOREM 3.23. *Let $\mathbf{A} \in \text{HSP}(\mathbf{3})$ with $1 < |A| < \infty$. Then $\mathbf{A} \in \text{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$.*

PROOF: Let $\mathbf{A} \in \text{HSP}(\mathbf{3})$ be such that $1 < |A| < \infty$. By Theorem 3.21, it follows that $\mathbf{A} \in \text{HP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$. By the Second Isomorphism Theorem, it suffices to show that if $\mathbf{B} = \mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$ for some $n, k, m \in \mathbb{N}$, and α is an atom in $\text{Con } \mathbf{B}$, then $\mathbf{B}/\alpha \in \text{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$. We will show that the only atoms in $\text{Con } \mathbf{B}$ are those congruences corresponding to the filters specified in Theorem 3.22. By item (1) of Theorem 3.18, it follows that these are indeed atoms in $\text{Con } \mathbf{B}$.

Assume, for the sake of contradiction, that there exists another filter G corresponding to a congruence α which is an atom in $\text{Con } \mathbf{B}$. From item (1) of Theorem 3.18, it follows that there exists a central element y such that $y \in G$. If y is such that $y_i = 1$ for all $i \in \{1, \dots, n\}$ and $y_j \in \{0, 1\}$ for all $j \in \{n+1, \dots, n+k+m\}$, then G contains a subset corresponding to the filter H from Theorem 3.22.

We will show that otherwise there exists an element $x \in G$ such that $x_l = *$ for exactly one $l \in \{1, \dots, n\}$ and $x_i = 1$ for all $i \in \{1, \dots, n+k+m\}$ with $i \neq l$. Let $l \in \{1, \dots, n\}$ be the smallest index such that $y_l = 0$. Notice that if x is such that $x_l = *$ and $x_j = 1$ for all $j \in \{1, \dots, n+k+m\} \setminus \{l\}$, then $y << x$. Hence, $x \in G$, so such an element x indeed exists, as desired. From item (1) of Theorem 3.18, it follows that $d(x)$ also belongs to G . Consequently, G contains a subset corresponding to the filter F from Theorem 3.22. Therefore, G is not an atom in the lattice of filters — a contradiction.

Hence, the only atoms in $\text{Con } \mathbf{B}$ are the congruences corresponding to the filters described in Theorem 3.22.

Finally, observe that if F is as in item (1) of Theorem 3.22, then $B/F \cong \mathbf{3}^{n-1} \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$. If H is as in item (2) of Theorem 3.22, then $B/H \cong \mathbf{3}^n \times \mathbf{2}^i \times \mathbf{2}_{0=1}^j$, where $i+j = k+m-1$. $\square$

COROLLARY 3.24. The only finite subdirectly irreducible algebras in $V(\mathbf{3})$ are $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$.

From Quackenbush's Theorem (see [2, p. 226]), we obtain the following corollary:

COROLLARY 3.25. The only subdirectly irreducible algebras in $V(\mathbf{3})$ are $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$.

THEOREM 3.26. *The variety $V(\mathbf{3})$ is:*

1. *semisimple,*
2. *directly representable (i.e., it is finitely generated and has only finitely many finite directly indecomposable members).*

Moreover, if $\mathbf{A} \in V(\mathbf{3})$ and $|A| < \infty$, then $\mathbf{A}$ is congruence uniform (i.e., for every $\varphi \in \text{Con } \mathbf{A}$ and all $a, b \in A$, we have $|a/\varphi| = |b/\varphi|$).

PROOF: Item (1) follows from Corollary 3.25 and Remark 3.19. Item (2) follows from Theorem 3.23. Congruence uniformity follows from McKenzie's Theorem [2, p. 188]. $\square$

Remark 3.27. Let $n \geq 2$. Then $\mathbf{n} = (\{0 < x_1 < x_2 < \dots < x_{n-2} < 1\}, \cdot, \circ, 1, 0)$ is a simple algebra. Moreover, for any $a, b \in \mathbf{n}$, and any $k \in$

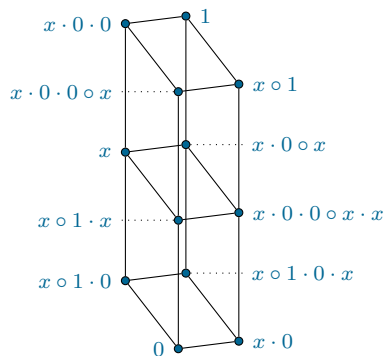


Figure 5: Double Heyting algebra $\mathbf{3} \times \mathbf{2}^2$ with generator x .

$\{1, \dots, n-2\}$, we have $a \cdot b = x_k$ if and only if $a = x_k$ and $b > x_k$ (or vice versa), and $a \circ b = x_k$ if and only if $a = x_k$ and $b < x_k$ (or vice versa). Consequently, the element x_k can be generated only by using itself. Therefore, the results and theorems from this subsection can be generalized from the case $n = 3$ to any $n \geq 2$.

Example 3.28. Let $\mathcal{F}_{\mathbf{n}}(k)$ be the k -generated free algebra in the variety $V(\mathbf{n})$. Then, for every $n \geq 3$, the algebra $\mathcal{F}_{\mathbf{n}}(1)$ is the $(\cdot, \circ)$ -reduct of the double Heyting algebra $\mathbf{3} \times \mathbf{2}^2$, with generator x . This algebra is illustrated in Figure 5.

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ON CERTAIN COMPLETENESS RESULTS FOR S4.2 MODAL LOGIC AND ITS APPLICATIONS IN MODAL LOGICS OF CLASSES OF STRUCTURES

Abstract

In this paper, we obtain new classes of Kripke frames complete for the modal logic S4.2, through the means of the so-called *control statements*. We provide applications of the theorem in the field of modal logics of classes of structures.

Keywords: control statements, modal logic, S4.2, modal logics of classes of structures.

1. Introduction

The modal logic S4.2 is a well-known logic. It is a normal modal logic axiomatised by the following formulas:

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- K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$,
- T: $\Box p \rightarrow p$,
- 4: $\Box p \rightarrow \Box \Box p$,
- .2: $\Diamond \Box p \rightarrow \Box \Diamond p$.

It is widely known to be characterised by the class of frames that are directed partial preorders (quasiorders), that is, preorders satisfying the condition

$$wRu \ \& \ wRu' \Rightarrow \exists z(uRz \ \& \ u'Rz). \quad (1.1)$$

In [2] Hamkins and Löwe provided another characterisation. First, we need the following definition:

DEFINITION 1.1. A *pre-Boolean algebra* is a preorder $\langle P, \leq \rangle$ such that its quotient $\langle P/\sim, \leq/\sim \rangle$ by a relation $\sim$ defined

$$w \sim u \Leftrightarrow wRu \ \& \ uRw,$$

is a Boolean lattice.

We can similarly understand the concept of a pre-lattice. Now we can understand the following theorem:

THEOREM 1.2 ([2, Theorem 10]). *Modal logic S4.2 is sound and complete with respect to the class of finite pre-Boolean algebras.*

The completeness proof utilises the notion of a *control statements*, namely *buttons* and *switches*. The authors used this result in order to obtain a method for proving that S4.2 is an upper bound on modal validities in a certain class of structures, connected by a model-theoretic relation to construct a certain Kripke model, where possible worlds are first-order structures.

This paper aims to provide other completeness results (countably many) of the logic S4.2 by means of another type of control statements introduced by Hamkins and Löwe, which are *ratchets*. This leads also to another method for proving the completeness of Kripke models of first-order structures.

2. Modal logic

The language we use is that of standard modal logic: let Var be a set of denumerably many propositional variables and $\mathcal{L} = Var \cup \{\wedge, \neg, \Box\}$. The symbols $\vee, \rightarrow, \leftrightarrow, \Diamond$ are defined as usual. The set of modal propositional formulas $Fm_{\mathcal{L}}$ is defined in the standard way.

In this paper only the modal logic S4.2 concerns us – we provided its axioms in the introduction. As a normal modal logic, it is closed under the rules of necessitation ($\frac{\alpha}{\Box\alpha}$), modus ponens and substitution.

Let us quickly review the semantics we use here. It is that of standard Kripke models: a Kripke model is a triple $\mathcal{M} = \langle W, R, v \rangle$, where W is a nonempty set, a collection of so called *possible worlds*, $R \subseteq W \times W$ is an *accessibility relation*, and $v: Var \rightarrow \mathcal{P}(W)$ is a valuation. A Kripke frame is a tuple $\mathcal{F} = \langle W, R \rangle$ consisting only of the first two elements of the Kripke model. Satisfaction in a Kripke model is defined in a standard way:

If $w \in W$, then:

$$\begin{array}{ll}
 w \Vdash p & \iff w \in v(p), \\
 w \Vdash \alpha \wedge \beta & \iff w \Vdash \alpha \text{ and } w \Vdash \beta, \\
 w \Vdash \neg\alpha & \iff w \not\Vdash \alpha, \\
 w \Vdash \Box\alpha & \iff wRu \Rightarrow u \Vdash \alpha, \text{ for all } u \in W.
 \end{array}$$

We say that a formula $\alpha \in Fm_{\mathcal{L}}$ is valid in the model $\mathcal{M}$ iff it is true in every $w \in W$. Similarly, we say that α is valid in a frame $\mathcal{F}$ iff it is valid in all models built on that frame.

The control statements in standard Kripke models are an easy concept: they are special kinds of formulas. In a Kripke model $\mathcal{M}$, in a world $w \in W$ a formula β is a *button* iff $w \Vdash \Diamond\Box\beta$. Similarly, a formula σ is a *switch* in w iff $w \Vdash \Box(\Diamond\sigma \wedge \Diamond\neg\sigma)$. Furthermore, we say that a button β in w is *pushed*, if $w \Vdash \beta$, otherwise it is *unpushed*. Similarly, we say that a switch σ is *on* in w , if $w \Vdash \sigma$, otherwise it is *off*. It is useful to take such formulas into sets, which then can be called sets of independent control statements. A set X of control statements is *independent* in w iff:

- for any $B \subseteq X$ of unpushed buttons there exists $u \in W$ such that wRu and $u \Vdash \alpha$, for $\alpha \in B$ and $u \not\Vdash \beta$, for any button $\beta \in X \setminus B$ that is unpushed in w ,
- for any $S \subseteq X$ of switches there exists $u \in W$ such that wRu and $u \Vdash \sigma \wedge \neg\psi$, for $\sigma \in S$ and switches $\psi \in X \setminus S$.

This framework allowed Hamkins and Löwe to obtain the following result:

THEOREM 2.1 ([2, Lemma 9]). *A class of frames $\mathbb{F}$ is sound and complete for $S4.2$ if and only if every frame in $\mathbb{F}$ is a $S4.2$ frame and there are Kripke models with frames in $\mathbb{F}$ for arbitrarily large finite independent families of buttons and switches.*

3. Completeness

In order to provide our result we need to present one special set of control statements introduced by Hamkins and Löwe, called ratchets (volume control in [2], ratchets in [1] and later works). A *ratchet* $\{\beta_1, \dots, \beta_n\}$ in a world w is a finite sequence of buttons, such that:

- $w \Vdash \Diamond(\beta_i \wedge \neg\beta_k)$, for $k > i$,
- $w \Vdash \Box\beta_m \rightarrow \beta_j$, for $j \leq m$.

We say that a ratchet's volume at w is i iff $w \Vdash \Box\beta_i$.

A ratchet is not an independent set of control statements, but it can be independent of another set of independent control statements S in w , if:

- pushing buttons in S and turning switches on or off does not imply increasing ratchet's volume,
- increasing ratchet's volume does not imply that any buttons in S that were unpushed are pushed, nor that the pattern of switches is changed.

One more definition is in order:

DEFINITION 3.1. A modal logic Λ is consistent with a set S of control statements, iff there is a Kripke model M for Λ and there is $w \in M$ such that S is an independent set of control statements in w .

Let us establish the following connection between ratchets and buttons:

THEOREM 3.2. *Suppose Λ is a modal logic and $n \geq 2$. The following conditions are equivalent:*

1. Λ is consistent with arbitrarily large finite families of independent buttons,
2. Λ is consistent with n arbitrarily large, finite, independent ratchets,
3. Λ is consistent with 2 arbitrarily large, finite, independent ratchets.

PROOF: (1) $\Rightarrow$ (2): Suppose Λ is consistent with arbitrarily large families of independent buttons. That means that there are models of Λ with worlds admitting any finite-sized families of independent buttons. For n ratchets of length m , we take a model M , a world $w \in M$ and B , a family of $n \cdot m$ buttons in w . Divide it in any way into disjoint subsets of cardinality m . For each such subset B_0 , the set $\{\beta_1, \beta_1 \wedge \beta_2, \dots, \bigwedge_{1 \leq j \leq m} (\beta_j)\}$ forms a ratchet in w of length m (for $\beta_j \in B_0$), independent of other ratchets.

(2) $\Rightarrow$ (3): Let $B_i = \{\beta_{i_k} : 0 \leq k \leq m\}$ be the i th independent ratchet of length $m+1$. The sequences $\{\bigwedge_{1 \leq i \leq n} (\beta_{i_0}), \bigwedge_{1 \leq i \leq n} (\beta_{i_0} \wedge \beta_{i_1}), \dots, \bigwedge_{1 \leq i \leq n} (\bigwedge_{1 \leq j \leq m} (\beta_{i_j}))\}$ for even $i \leq n$ form one ratchet, while for odd $i \leq n$ form the other.

(3) $\Rightarrow$ (1): Let n be any positive natural number. We will construct a set of n independent buttons. Let R_{n^2} and L_{n^2} be two ratchets of length n^2 . Let r_k be a sentence that says that the volume of the ratchet R is k . We analogously define l_m . Let $A = \{r_k \wedge l_m : k + m = n^2\}$. Let us define sets of independent buttons of cardinality $2 \leq i \leq n$ inductively:

- $(r_k \wedge l_m), (r_{k'} \wedge l_{m'}) \in A$ are two independent buttons if $k \neq k'$,
- if $\varphi_1, \dots, \varphi_j$ are independent buttons, then take sentences $\psi_1, \dots, \psi_{2j} \in A$ such that $\neg(\Box \bigwedge_{s \leq 2j} (\psi_s) \rightarrow \varphi_i)$, for all $1 \leq i \leq 2j$, and if

$\psi_i = r_k \wedge l_m$, $\psi_{i'} = r_{k'} \wedge l_{m'}$ and $i \leq i'$, then $k \leq k'$. Since the ratchets R and L can be arbitrarily large, we will find such sentences. The sentences $\varphi_i \vee \psi_{2i}$ together with the sentence $\bigvee_{2n+1=k \leq 2j} (\psi_k)$ form a set of independent buttons, and its cardinality is $n + 1$. It is easy to check that these sentences are buttons and are independent of each other. $\square$

COROLLARY 3.3. Let $m \geq 2$ be a natural number greater than 1. The logic S4.2 is sound and complete with respect to the following classes of frames:

$$\langle \prod_{1 \leq i \leq m} (C_i) \times \mathcal{P}(S_k), \leq^* \rangle,$$

for all m -sized sets of finite chains C_i and all k , where S_k is a set of cardinality k , and the order is defined as $\langle a_1, \dots, a_m, S \rangle \leq^* \langle b_1, \dots, b_m, Z \rangle \iff a_1 \leq b_1 \ \& \ \dots \ \& \ a_m \leq b_m$.

The intuition here is that these frames are direct products of linearly ordered lattices, where each node is replaced with a cluster of mutually accessible worlds (the last parameter, S , is responsible for that).

PROOF: By the theorem by Hamkins and Löwe [2] it suffices to show that each such frame is an S4.2 frame and that there are frames for arbitrarily large families of independent buttons and switches. For any $m \geq 2$ it is easy to see that these frames are directed (in the sense of condition (1.1)) preorders, hence S4.2 is valid in them.

To prove the second part, by the Theorem 3.2, it suffices to find two arbitrarily long ratchets independent of each other and of arbitrarily large sets of independent switches in models built on those frames. Let us show it for $m = 2$, but this works for all $m \geq 2$. Let C_i be a linear order of length i . For a given frame $\langle C_i \times C_j \times \mathcal{P}(S_k), \leq^* \rangle$, let us assume without loss of generality that $C_i = \langle \{0, \dots, i-1\}, \leq \rangle$, $C_j = \langle \{0, \dots, j-1\}, \leq \rangle$, $S_k = \{0, \dots, k\}$. Let $P = \{p_0, \dots, p_i\}$, $Q = \{q_0, \dots, q_j\}$ and $R = \{r_0, \dots, r_k\}$ be sets of distinct variables. Any valuation v satisfying the following condition:

$$(a, b, S) \in v(p) \iff p \leq p_a \text{ or } p \leq q_b \text{ or } p \in S$$

is a valuation that allows us to find two independent ratchets of length i (the set P) and j (the set Q), as well as a set (R) of k independent switches. Since in a class there are arbitrarily large finite frames of this sort, we are able to produce arbitrarily large ratchets and sets of switches, which concludes the proof. $\square$

This result provides us with infinitely many completeness theorems for S4.2, for all $m \in \mathbb{N}$. The frames are pre-lattices, and the lattices obtained by making the quotient algebra by the relation $\sim$ (defined as in Definition 1.1) are distributive.

4. Applications

The authors of [2] used their result (Theorem 1.2) in order to prove the following:

THEOREM 4.1 ([2, Theorem 14]). *If a world w in a Kripke model $\mathcal{M}$ has arbitrarily large families of independent buttons and switches, then its modal logic is contained within S4.2.*

This comes in handy when working in a setting of classes of first-order structures. This setting originates in works on arithmetic and set theory, and considers an ordered tuple $\langle \mathcal{C}, \mathcal{R} \rangle$, something similar to a Kripke frame – $\mathcal{C}$ is a certain (typically proper) class of first-order structures, and $\mathcal{R}$ is a model-theoretic relation connecting members of $\mathcal{C}$. This intends to work as a frame, whose modal logic we investigate. To achieve that goal, one needs to define valuations accordingly: let L be a standard first-order language. Any function $t: Var \rightarrow Fm_L$ we call a L -translation. The valuations are then defined in the following way: $v_t(p) = \{M \in \mathcal{C} : M \models t(p)\}$. Only such valuations are considered in order to avoid metamathematical issues.

It might be seen as arbitrary to pick the first-order language for such translations, especially strange since many interesting model-theoretic relations would be inexpressible in it. Thanks to the results by Saveliev and Shapirovsky [3] the modal logic obtained via these valuations will be the stronger than in any stronger language. Furthermore, if we intend to use the Theorem 4.1 or similar in order to prove that the lower bound is the

same as the upper bound on modal validities in a given class, we obtain the result that this logic is the robust logic of this class, i.e. the logic of this class does not depend on the choice of language.

In order to understand this method in the mentioned framework we need to understand what does it mean to be a button or a switch in a first-order structure. A first-order sentence $\varphi \in Fm_L$ is called a button in M iff for all translations t and all $p \in Var$ such that $t(p) = \varphi$ $M \Vdash \Diamond \Box p$. Switch is understood similarly through the means of translation.

Due to Theorem 3.2 we can state the theorem in terms of ratchets:

COROLLARY 4.2. *If a world w in a Kripke model $\mathcal{M}$ has two arbitrarily large independent ratchets independent of arbitrarily large families of switches, then its modal logic is contained within S4.2.*

As an example we will provide a tuple $\langle \mathcal{C}, \mathcal{R} \rangle$ such that its modal logic in the above sense is exactly S4.2.

Let $L_{<}$ be a first-order language together with the set $\{<, c_0, c_1\}$, where $<$ is a binary relational symbol and c_0, c_1 are constant symbols. Let $T =$ theory of linear orders together with the following two axioms:

1. $\forall x \exists y (y > x)$,
2. $c_0 < c_1$.

THEOREM 4.3. *The robust modal logic of the frame $\langle \mathcal{C}, \mathcal{R} \rangle$, for $\mathcal{C} = Mod(T)$ (class of models of T) and $\mathcal{R} = \subseteq$ (embeddability relation) is S4.2.*

PROOF: Soundness is easy, as it suffices to show that condition (1.1) is met by the class $\mathcal{C}$ under $\mathcal{R}$. In $\mathcal{C}$ even a stronger property is exhibited, as any two linear orders have a common extension (take, for example, a direct lexicographical product); hence S4.2 is valid.

For completeness, we will find two independent ratchets independent of an arbitrary length, as well as arbitrarily large sets of independent switches. Let $M \in \mathcal{C}$ be any model of T . Let $S = \{\sigma_n : n \in \mathbb{N}\}$, where σ_n is a sentence that says "there are two objects greater than c_1 such that they do not have an immediate predecessor, between them are no such objects, and are n objects without an immediate successor". These are our switches: one can switch them on in any extension of M where this is satisfied, then

one can make them off in any discrete extension they embed into. For ratchets, take sequences φ_m, ψ_k for $m, k \in \mathbb{N}$, where φ_m is a sentence that says "there are at least m elements smaller (with respect to $<$) than c_0 " and ψ_k is a sentence that says "there are at least k elements between c_0 and c_1 ". It is straightforward that these are buttons in M and those with bigger indexes imply those with smaller, as well as the fact that one can find $N \supsetneq M$ in which any desired button is true, and the one with greater index is not. Since sentences σ_n concern what happens in the top part of linear order, sentences ψ_k what is true in its middle, and sentences φ_m what happens at the bottom, they don't interfere with one another, and are independent. So there are two independent ratchets of arbitrary length and arbitrary large families of independent switches, so from Corollary 4.3 follows completeness. $\square$

Note that proving completeness by the means of Theorem 4.1 would be significantly more complicated.

This result brings the following corollary:

COROLLARY 4.4. Let T_n , for $n \geq 3$ be theory in language with $<$ binary relational symbol and $c_0, c_1, \dots, c_{n-1}$ constant symbols, extending T and having as additional axioms $c_i < c_j$, for $i < j$. Robust modal logic of $\langle \text{Mod}(T_n), \supsetneq \rangle$ is S4.2.

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PERFECT AND LOCAL BOUNDED EQUALITY ALGEBRAS

Abstract

In this paper, by considering the notion of bounded equality algebra, we introduce the concepts of perfect, integral and local bounded equality algebras and we state and prove some related results. Specially, we define the notions of some filters such as primary and perfect filters and we get the relation between perfect filters with perfect bounded equality algebras and primary filters in the local bounded equality algebras. Finally, by defining the concepts of radical, locally finite, and peculiar bounded equality algebra, we give a representation for any bounded p-equality and local bounded equality algebras.

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1. Introduction

Novák developed implication-based fuzzy type theory as a higher-order fuzzy logic, which corresponds to the fuzzy version of classical type theory in classical higher-order logic. While classical-type theory models rely on equality, with identity as the principal connective, this approach differs from the equality-based models. To address this diversity, Novák and Baets introduced a specific algebra of truth values for fuzzy type theory, known as EQ-algebras. Jenei [8] later introduced equality algebras motivated by EQ-algebras. This structure contains an equivalence, a meet operation, and a constant 1. A non-commutative version of this structure was also studied in [9]. Jenei [8] found a close connection between equality algebras and BCK-algebras. Ciungu [4] and then Dvurečenskij et. al [6] improved this result not only for equality algebras but also for pseudo equality algebras. Internal states, and states on equality algebras as well as special types of deductive systems on equality algebras were studied in [5], [21], and [22], respectively.

In recent times, various aspects of equality algebras have been studied, such as commutative equality and $\&$ -equality in [10], radical filter on equality algebras in [11], implicative equality, annihilators, right and left maps on equality in [1, 13], different representation of equality algebras in [12], the concept of involution in [3], Stone filters and MBJ-neutrosophic filters on equality algebras in [7, 19], prime filter, co-annihilators and ordinal sum on equality algebras in [14, 15, 16, 17, 20], the concept of derivation and state on equality algebras in [18, 21] and the relation between equality algebra and other logical algebra in [22].

In this paper, by considering the notion of bounded equality algebra, we introduce the concepts of perfect, integral and local bounded equality algebras and we state and prove some related results. Specially, we define the notions of some filters such as primary and perfect filters and we get the

relation between perfect filters with perfect bounded equality algebras and primary filters in the local bounded equality algebras. Finally, by defining the concepts of radical, locally finite, and peculiar bounded equality algebra, we give a representation for any bounded p-equality and local bounded equality algebras.

2. Preliminaries

In this section, we have some basic concepts about equality algebra that will be used in later sections. We only remind you of some definitions and results.

DEFINITION 2.1 ([8]). An algebraic structure $(E; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if it satisfies the following conditions, for all $x, y, z \in E$,

(E1) $(E, \wedge, 1)$ is a $\wedge$ -semilattice with the greatest element 1,

(E2) $x \sim y = y \sim x$,

(E3) $x \sim x = 1$,

(E4) $x \sim 1 = x$,

(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,

(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,

(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

An equality algebra $(E, \leq)$ with a binary relation defined by “ $x \leq y$ if and only if $x \wedge y = x$ ” is a poset. Also, we define the operation “ $\rightarrow$ ” on E as: $x \rightarrow y = x \sim (x \wedge y)$.

The set $\emptyset \neq X \subseteq E$ is called a *sub-equality algebra* if it is closed under all operations on E . It means that for any $x, y \in X$, $x \wedge y, x \sim y \in X$.

An equality algebra $(E, \sim, \wedge, 0, 1)$ is called a *bounded equality algebra* if for every $x \in E$, $0 \leq x$. In a bounded equality algebra E , we define the

negation operation “ $'$ ” by: $x' = x \rightarrow 0 = x \sim 0$, for all $x \in E$. If for any $x \in E$, $x'' = x$, then E is called an *involution equality algebra*. A lattice equality algebra is an equality algebra that is a lattice. An equality algebra E is called a positive equality algebra or p-equality if for any $x, y, z \in E$, $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$ (see [22]).

Note. From now on, let $(E; \wedge, \sim, 1)$ or E be an equality algebra, unless otherwise stated.

PROPOSITION 2.2 ([8, 22]). The following conditions hold, for any $x, y, z \in E$:

- (i) $x \leq (x \rightarrow y) \rightarrow y$,
- (ii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (iii) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
- (iv) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,
- (v) $x \rightarrow y \leq (x \wedge z) \rightarrow (y \wedge z)$,
- (vi) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

DEFINITION 2.3 ([9]). Let F be a non-empty subset of E . Then F is called a *filter* of E if for all $x, y \in E$, we have

- (i) $x \in F$ and $x \leq y$ imply $y \in F$,
- (ii) $x \in F$ and $x \sim y \in F$ imply $y \in F$.

The set of all filters of E is denoted by $\mathcal{F}(E)$.

PROPOSITION 2.4 ([5, 9]). Let $\emptyset \neq F \subseteq E$. Then $F \in \mathcal{F}(E)$ if and only if, for all $x, y \in E$, $1 \in F$, and if $x \in F$ and $x \rightarrow y \in F$, then $y \in F$.

Clearly, for any $F \in \mathcal{F}(E)$, $1 \in F$, and F is called a *proper filter* of E if $F \neq E$. If E is bounded, then $F \in \mathcal{F}(E)$ is proper if and only if it does not contain 0.

Let $F \in \mathcal{F}(E)$. Define the relation θ_F on E by

$$(x, y) \in \theta_F \quad \text{if and only if} \quad \{x \rightarrow y, y \rightarrow x\} \subseteq F.$$

The set of all congruence relations on E is denoted by $\text{Con}(E)$.

Let $\frac{E}{F} = \{[x] \mid x \in E\}$, where $[x] = \{y \in E \mid (x, y) \in \theta_F\}$. Then define the binary relation $\leq_F$ on $\frac{E}{F}$ by:

$$[x] \leq_F [y] \quad \text{if and only if} \quad x \rightarrow y \in F,$$

which is an order relation on $\frac{E}{F}$. For any $x, y \in E$, define

$$[x] \sim_F [y] = [x \sim y] \quad \text{and} \quad [x] \wedge_F [y] = [x \wedge y].$$

Then $\left(\frac{E}{F}, \sim_F, \wedge_F, 1_{\frac{E}{F}}\right)$ is called a *quotient equality algebra* and denoted by $\frac{E}{F}$, where $1_{\frac{E}{F}} = [1]_{\theta_F} = F$.

DEFINITION 2.5 ([17]). Let $\emptyset \neq A \subseteq E$. The smallest filter of E containing A is called *the generated filter by A in E* , which is denoted by $\langle A \rangle$. Indeed, $\langle A \rangle = \bigcap_{A \subseteq F \in \mathcal{F}(E)} F$.

PROPOSITION 2.6 ([17]). Let $\emptyset \neq A \subseteq E$. Then

$$\begin{aligned} \langle A \rangle &= \{x \in E \mid a_1 \rightarrow (a_2 \rightarrow (\dots \rightarrow (a_n \rightarrow x) \dots)) = 1, \text{ for some } n \in \mathbb{N} \\ &\quad \text{and } a_1, \dots, a_n \in A\}. \end{aligned}$$

In particular, for any element $a \in E$, we have $\langle a \rangle = \{x \in E \mid a \xrightarrow{n} x = 1, \text{ for some } n \in \mathbb{N}\}$, where $x \xrightarrow{0} y = y$ and $x \xrightarrow{n} y = x \rightarrow (x \xrightarrow{n-1} y)$.

DEFINITION 2.7 ([2]). Let F be a non-empty subset of E such that $1 \in F$ and $x, y, z \in E$. Then

- (i) F is called a *positive implicative filter* of E if $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$,

- (ii) F is called an *implicative filter* of E if $z \rightarrow ((x \rightarrow y) \rightarrow x) \in F$ and $z \in F$, then $x \in F$.
- (iii) F is called a *fantastic filter* of E , if $z \in F$ and $z \rightarrow (x \rightarrow y) \in F$ imply $((y \rightarrow x) \rightarrow x) \rightarrow y \in F$.

PROPOSITION 2.8 ([2]). Let E be bounded and $F \in \mathcal{F}(E)$. Then F is an implicative filter of E if and only if for any $x \in E$, $(x' \rightarrow x) \rightarrow x \in F$.

DEFINITION 2.9 ([2, 14, 17]). Suppose $F \in \mathcal{F}(E)$ is proper and $x, y \in E$. Then F is called

- (i) a *maximal filter* of E if it is not included in any other proper filter of E . The set of all maximal filters of E is denoted by $\mathcal{Max}(E)$.
- (ii) a *prime filter* of E if for any $x, y \in E$, $x \rightarrow y \in F$ or $y \rightarrow x \in F$. The set of all prime filters of E is denoted by $\mathcal{Prime}(E)$.
- (iii) a $\vee$ -*irreducible* if $x \vee y \in F$ implies $x \in F$ or $y \in F$, where E is a lattice. The set of all $\vee$ -irreducible filters of E is denoted by $\mathcal{Spec}(E)$.

PROPOSITION 2.10 ([14]). Suppose E is a lattice. Then

- (i) $F \in \mathcal{Spec}(E)$ if and only if $\frac{E}{F}$ is a chain.
- (ii) $\mathcal{Prime}(E) \subseteq \mathcal{Spec}(E)$.

PROPOSITION 2.11 ([14]).

- (i) Let E be bounded. $M \in \mathcal{Max}(E)$ if and only if $(x \notin M$ if and only if there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in M)$.
- (ii) Let E be bounded. If $F \in \mathcal{F}(E)$ is proper, then there exists $M \in \mathcal{Max}(E)$ such that $F \subseteq M$.
- (iii) Each maximal filter of E is prime. Indeed, $\mathcal{Max}(E) \subseteq \mathcal{Prime}(E)$.

Let F be a proper filter of E . The intersection of all maximal filters of E that contain F is called *the radical of F* and is denoted by $\text{Rad}(F)$. If

$F = \{1\}$, then $Rad(E)$ is the intersection of all maximal filters of E (see [11]). Also, we have $Rad'(E) = \{x \in E \mid x' \in Rad(E)\}$. (For more details, see [11]).

THEOREM 2.12 ([11]). *Consider $F \in \mathcal{F}(E)$. Then*

$$\begin{aligned} Rad(F) &= \{x \in E \mid (x \xrightarrow{n} 0) \rightarrow x \in F, \text{ for any } n \in \mathbb{N}\} \\ &\subseteq \{x \in E \mid x' \rightarrow x \in F\}. \end{aligned}$$

Note. From now on, consider that $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra, unless otherwise stated.

3. Perfect and integral equality algebras

In this section, we introduce the notion of order on bounded equality algebra and by using this concept we define different kinds of bounded equality algebras such as perfect and integral bounded equality algebras and investigate the relation between them.

DEFINITION 3.1. Let $x \in E$. Then the least natural number n such that $x \xrightarrow{n} 0 = 1$ is called the *order* of x and is denoted by $ord(x) = n$. If there is no natural number such that $x \xrightarrow{n} 0 = 1$, then $ord(x) = \infty$.

Remark 3.2. It is clear that $ord(0) = 1$ and $ord(1) = \infty$.

Example 3.3. (i) Let $(E = \{0, a, b, 1\}, \leq)$ be a chain where $0 \leq a \leq b \leq 1$. Define the operation “ $\sim$ ” on E as follows:

$\sim$	0	a	b	1	$\rightarrow$	0	a	b	1
0	1	a	0	0	0	1	1	1	1
a	a	1	a	a	a	a	1	1	1
b	0	a	1	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(E, \wedge, \vee, \sim, 0, 1)$ is a bounded equality algebra where $ord(a) = 2$ and $ord(b) = ord(1) = \infty$.

(ii) Let $E = \{0, a, b, c, d, 1\}$ be a poset where $0 \leq a, b \leq c \leq d \leq 1$. Define “ $\sim$ ” on E as follows:

$\sim$	0	a	b	c	d	1	$\rightarrow$	0	a	b	c	d	1
0	1	d	d	d	c	0	0	1	1	1	1	1	1
a	d	1	c	d	c	a	a	d	1	d	1	1	1
b	d	c	1	d	c	b	b	d	d	1	1	1	1
c	d	d	d	1	d	c	c	d	d	d	1	1	1
d	c	c	c	d	1	d	d	c	c	c	d	1	1
1	0	a	b	c	d	1	1	0	a	b	c	d	1

Then $(E, \wedge, \vee, \sim, 0, 1)$ is a bounded equality algebra. Clearly, $\text{ord}(a) = \text{ord}(b) = \text{ord}(c) = 2$, $\text{ord}(d) = 3$ and $\text{ord}(1) = \infty$.

(iii) Let $E = [0, 1]$ and $x, y \in E$. Define

$$x \rightarrow y = \begin{cases} 1 & x \leq y \\ (1-x) \vee y & x > y \end{cases} \quad \text{and} \quad x \sim y = (x \rightarrow y) \wedge (y \rightarrow x).$$

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra where for any $x \leq \frac{1}{2}$, $\text{ord}(x) = 2$ and for any $x > \frac{1}{2}$, $\text{ord}(x) = \infty$.

PROPOSITION 3.4. Let $x, y \in E$.

- (i) $\langle x \rangle$ is proper if and only if $\text{ord}(x) = \infty$.
- (ii) If $x \leq y$ and $\text{ord}(y) < \infty$, then $\text{ord}(x) < \infty$.
- (iii) If $x \leq y$ and $\text{ord}(x) = \infty$, then $\text{ord}(y) = \infty$.
- (iv) Let $M \in \mathcal{Max}(E)$. Then $M \subseteq \{x \in E \mid \text{ord}(x) = \infty\}$.

PROOF: (i) Suppose $\langle x \rangle$ is proper and $\text{ord}(x) < \infty$, by the contrary. Then there is $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 = 1$. According to the definition of $\langle x \rangle$, $0 \in \langle x \rangle$ and so $\langle x \rangle = E$, which is a contradiction. Conversely, suppose $\text{ord}(x) = \infty$. Then for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \neq 1$. Thus $0 \notin \langle x \rangle$. Hence, $\langle x \rangle$ is a proper filter of E .

(ii) Suppose $x \leq y$ and $\text{ord}(y) < \infty$. Then there is $n \in \mathbb{N}$ such that $y \xrightarrow{n} 0 = 1$. By Proposition 2.2(iii), $y \rightarrow 0 \leq x \rightarrow 0$. Also, by Proposition 2.2(iii), $y \rightarrow (y \rightarrow 0) \leq y \rightarrow (x \rightarrow 0)$. Again, by Proposition 2.2(iii), $y \rightarrow (x \rightarrow 0) \leq x \rightarrow (x \rightarrow 0)$. So we have $y \rightarrow (y \rightarrow 0) \leq x \rightarrow (x \rightarrow 0)$.

By repeating this method, we get $1 = y \xrightarrow{n} 0 \leq x \xrightarrow{n} 0$. Hence, $x \xrightarrow{n} 0 = 1$, and so $\text{ord}(x) = n$. Therefore, $\text{ord}(x) < \infty$.

(iii) Let $x \leq y$. Similar to the proof of (ii), $y \xrightarrow{n} 0 \leq x \xrightarrow{n} 0$. Since for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \neq 1$, we get that for any $n \in \mathbb{N}$, $y \xrightarrow{n} 0 \neq 1$. Hence, $\text{ord}(y) = \infty$.

(iv) Let $M \in \text{Max}(E)$ and $\mathcal{C} = \{x \in E \mid \text{ord}(x) = \infty\}$. We prove that $M \subseteq \mathcal{C}$. For this, suppose $x \in M$. If $\text{ord}(x) < \infty$, then there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 = 1$ and so $x \xrightarrow{n} 0 \in M$. Since $x \in M$ and $M \in \mathcal{F}(E)$, we get $0 \in M$, a contradiction. Therefore, $\text{ord}(x) = \infty$ and so $x \in \mathcal{C}$. $\square$

PROPOSITION 3.5. Assume E is a chain and $F \in \mathcal{F}(E)$. Then

(i) If $x \notin \text{Rad}(F)$, then $\text{ord}(x) < \infty$.

(ii) $\text{Rad}(F) = \{x \in E \mid \text{ord}(x) = \infty\}$.

PROOF: (i) Assume $x \notin \text{Rad}(F)$. Then by Proposition 2.12, there exists $n \in \mathbb{N}$ such that $(x \xrightarrow{n} 0) \rightarrow x \notin F$. So, $x \xrightarrow{n} 0 \not\leq x$. Since if $x \xrightarrow{n} 0 \leq x$, then $(x \xrightarrow{n} 0) \rightarrow x = 1 \in F$, a contradiction. By assumption, E is a chain, thus $x \leq x \xrightarrow{n} 0$, and so $x \xrightarrow{n+1} 0 = 1$. Hence, $\text{ord}(x) = n + 1 < \infty$.

(ii) Let $x \in \text{Rad}(F)$ and $\text{ord}(x) < \infty$. Since $x \in \text{Rad}(F)$, for any $M \in \text{Max}(E)$ such that $F \subseteq M$, $x \in M$. In addition, $\text{ord}(x) < \infty$, so there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 = 1 \in M$. Since $x \in M$ and $M \in \mathcal{F}(E)$, we get $0 \in M$, which is a contradiction. So $\text{ord}(x) = \infty$. Hence, by (i), the proof is clear. $\square$

In the following example we show that the condition “chain” in Proposition 3.5 is necessary.

Example 3.6. Let $E = \{0, a, b, 1\}$ be a poset where $0 \leq a, b \leq 1$. Define the operation “ $\sim$ ” on E as follows:

$\sim$	0	a	b	1	$\rightarrow$	0	a	b	1
0	1	b	a	0	0	1	1	1	1
a	b	1	0	a	a	b	1	b	1
b	a	0	1	b	b	a	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra. Clearly, $\text{Max}(E) = \{\{a, 1\}, \{b, 1\}\}$ and so $\text{Rad}(E) = \{1\}$. In addition, $a, b \notin \text{Rad}(E)$ and $\text{ord}(a) = \text{ord}(b) = \infty$.

DEFINITION 3.7. Assume $\emptyset \neq X \subseteq E$, is a sub-equality algebra of E . Then X is called *perfect* if for any $x \in X$,

$$\text{ord}(x) < \infty \text{ if and only if } \text{ord}(x') = \infty.$$

Example 3.8. (i) In Example 3.3(i), E is not perfect, since $\text{ord}(a) = \text{ord}(a') = 2 < \infty$.

(ii) Consider $(E = \{0, a, b, c, 1\}, \leq)$ to be a poset where $0 \leq c \leq a, b \leq 1$. The operation “ $\sim$ ” and the implication “ $\rightarrow$ ” on E are given by the following tables:

	$\sim$	0	a	b	c	1
	0	1	0	0	0	0
	a	0	1	c	b	a
	b	0	c	1	a	b
	c	0	b	a	1	c
	1	0	a	b	c	1
	$\rightarrow$	0	a	b	c	1
	0	1	1	1	1	1
	a	0	1	b	b	1
	b	0	a	1	a	1
	c	0	1	1	1	1
	1	0	a	b	c	1

Then $E = (E, \wedge, \vee, \sim, 0, 1)$ is a bounded equality algebra. Clearly, E is perfect.

(iii) According to Example 3.3(iii), E is perfect.

DEFINITION 3.9. Let E be a lattice. An element $a \in E$ is called the *complemented* if there exists an element $b \in E$ such that $a \vee b = 1$ and $a \wedge b = 0$. Then the element b is called a *complement of a*. The set of all complemented elements is denoted by $\mathcal{C}(E)$, i.e., $\mathcal{C}(E) = \{a \in E \mid \exists b \in E; a \vee b = 1 \text{ \& } a \wedge b = 0\}$.

Obviously, in any chain equality algebra, $\mathcal{C}(E) = \{0, 1\}$.

THEOREM 3.10. Suppose E has just one maximal filter such as M such that $M = E \setminus \{0\}$. Then the following statements hold:

(i) E is perfect and for any $x \in M$, $\text{ord}(x) = \infty$.

(ii) If E is Boolean, then $E = \{0, 1\}$.

PROOF: (i) By Proposition 3.4(iv), $M \subseteq \{x \in E \mid \text{ord}(x) = \infty\}$. Since $M = E \setminus \{0\}$, we get that for any $x \in E \setminus \{0\}$, $\text{ord}(x) = \infty$. Also, for $x \in E$, if $x = 0$, then $x' = 1$ and so $\text{ord}(x) = 1 < \infty$ and $\text{ord}(x') = \infty$. If $x \neq 0$, then $x \in M$ and so $\text{ord}(x) = \infty$. Clearly, $x' \notin M$, so $x' = 0$ and $\text{ord}(x') < \infty$. Therefore, E is perfect.

(ii) Let $x \in E \setminus \{0, 1\}$. Then $x \in M$ and so $x' \notin M$. Thus, $x' = 0$ for any $x \in E \setminus \{0\}$. Then $x \vee x' = x \vee 0 = x \neq 1$, so $x \notin \mathcal{C}(E)$. Hence, $\mathcal{C}(E) = \{0, 1\}$. Now, if E is Boolean, then $E = \mathcal{C}(E) = \{0, 1\}$. $\square$

Example 3.11. In Example 3.8(ii), $E \setminus \{0\} = M = \{a, b, c, 1\} \in \mathcal{Max}(E)$.

PROPOSITION 3.12. If $\text{Rad}(E) \cup \text{Rad}'(E)$ is a subalgebra of E , then it is perfect.

PROOF: Let $x \in \text{Rad}(E) \cup \text{Rad}'(E)$, then we have the following two cases:

Case 1: $x \in \text{Rad}(E)$. Then for any $M \in \mathcal{Max}(E)$, $x \in M$ and by Proposition 3.4(iv), $\text{ord}(x) = \infty$. In addition, since $x \in \text{Rad}(E)$, by Proposition 2.12, $x' \rightarrow x = 1$ and so $x' \leq x$. By Proposition 2.2(i),

$$1 = x \rightarrow (x' \rightarrow 0) \leq x' \rightarrow (x' \rightarrow 0) = x' \xrightarrow{2} 0,$$

and so $x' \xrightarrow{2} 0 = 1$. Hence, $\text{ord}(x') < \infty$.

Case 2: $x \in \text{Rad}'(E)$. Then $x' \in \text{Rad}(E)$, and so $\text{ord}(x') = \infty$. By Case 1, $\text{ord}(x'') < \infty$ and since $x \leq x''$, by Proposition 3.4(ii), we get $\text{ord}(x) < \infty$. Therefore, according to the above cases, $\text{Rad}(E) \cup \text{Rad}'(E)$ is perfect. $\square$

DEFINITION 3.13. Let $F \in \mathcal{F}(E)$ be proper and $x \in E$. Then the least $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in F$ is called *order of x respect to F* and denoted by $\text{ord}_F(x) = n$. If there is no $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in F$, then $\text{ord}_F(x) = \infty$.

Example 3.14. According to Example 3.3(i), let $F = \{b, 1\}$. Then $\text{ord}_F(a) = 2$ and $\text{ord}_F(b) = \text{ord}_F(1) = \infty$.

DEFINITION 3.15. A proper filter P of E is called a *perfect filter* if for any $x \in E$,

$$\text{ord}_F(x) < \infty \text{ if and only if } \text{ord}_F(x') = \infty.$$

Example 3.16. (i) According to Example 3.8(ii), the filter $F = \{a, 1\}$ is a perfect filter of E .

(ii) According to Example 3.3(i), $F = \{b, 1\}$ and $G = \{1\}$ are not perfect filters since $\text{ord}_F(a) = \text{ord}_F(a') < \infty$.

Remark 3.17. If $\{1\}$ is a perfect filter of E , then every filter of E is perfect. Since if $F \in \mathcal{F}(E)$ such that for $x \in E$, $\text{ord}_F(x) = \infty$, then for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin F$, we get $x \xrightarrow{n} 0 \neq 1 \notin \{1\}$. Thus, $\text{ord}_{\{1\}}(x) = \infty$. Moreover, $\{1\}$ is perfect, thus by definition we get $\text{ord}_{\{1\}}(x') < \infty$, and so there exists $m \in \mathbb{N}$ such that $x' \xrightarrow{m} 0 \in \{1\} \subseteq F$. Hence, $\text{ord}_F(x') < \infty$, and so F is perfect.

PROPOSITION 3.18. Let F be a perfect filter of E and $F \subseteq G$, where G is a proper filter of E . Then G is a perfect filter of E .

PROOF: Consider F is a perfect filter of E . If $G \in \mathcal{F}(E)$ such that for $x \in E$, $\text{ord}_G(x) = \infty$, then for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin G$, we get $x \xrightarrow{n} 0 \notin F$. Thus, $\text{ord}_F(x) = \infty$. Moreover, by assumption F is perfect, thus by definition we get $\text{ord}_F(x') < \infty$, and so there exists $m \in \mathbb{N}$ such that $x' \xrightarrow{m} 0 \in F \subseteq G$. Hence, $\text{ord}_G(x') < \infty$, and so G is perfect. $\square$

THEOREM 3.19. Let $P \in \mathcal{F}(E)$ be proper. Then $\frac{E}{P}$ is perfect if and only if P is a perfect filter of E .

PROOF: Let P be a perfect filter of E , and $[x] \in \frac{E}{P}$. If $\text{ord}([x]) < \infty$, then there exists $n \in \mathbb{N}$ such that $[x] \xrightarrow{n} [0] = [1]$, and so $[x \xrightarrow{n} 0] = [1]$. Then for $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \in P$. Thus, $\text{ord}_P(x) < \infty$ and by assumption, $\text{ord}_P(x') = \infty$. Then for any $m \in \mathbb{N}$, $x' \xrightarrow{m} 0 \notin P$ and so $[x'] \xrightarrow{m} [0] \neq [1]$. Hence, $\text{ord}([x']) = \infty$. Now, suppose $\text{ord}([x]) = \infty$. Then for any $n \in \mathbb{N}$, $[x] \xrightarrow{n} [0] \neq [1]$ and so $[x \xrightarrow{n} 0] \neq [1]$. Thus, for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin P$, and so $\text{ord}_P(x) = \infty$. Since P is perfect, we get $\text{ord}_P(x') < \infty$. Then there exists $m \in \mathbb{N}$ such that $x' \xrightarrow{m} 0 \in P$ and so this implies that $[x'] \xrightarrow{m} [0] = [1]$.

Hence $\text{ord}([x']) < \infty$. Therefore, $\frac{E}{P}$ is perfect.

Conversely, suppose $\frac{E}{P}$ is perfect and for $x \in E$, $\text{ord}_P(x) = \infty$. Thus, for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin P$. Hence, for any $n \in \mathbb{N}$, $[x] \xrightarrow{n} [0] \neq [1]$ and so $\text{ord}([x]) = \infty$. Since $\frac{E}{P}$ is perfect, $\text{ord}([x']) < \infty$. So there exists $m \in \mathbb{N}$ such that $[x'] \xrightarrow{m} [0] = [1]$. Hence, $x' \xrightarrow{m} 0 \in P$, and so $\text{ord}_P(x') < \infty$. The proof of the other side is similar. Therefore, P is a perfect filter. $\square$

PROPOSITION 3.20. E is perfect if and only if for any proper filter $F \in \mathcal{F}(E)$, F is a perfect filter of E .

PROOF: If for any $F \in \mathcal{F}(E)$, F is a perfect filter of E , then $\{1\}$ is a perfect filter of E and by Theorem 3.19, $\frac{E}{\{1\}}$ is perfect, and so E is perfect.

Conversely, suppose E is perfect, $F \in \mathcal{F}(E)$ and for $x \in E$, $\text{ord}_F(x) = \infty$. Then, for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin F$. Hence, $x \xrightarrow{n} 0 \neq 1$, and so $x \xrightarrow{n} 0 \notin \{1\}$. It means that $\text{ord}_{\{1\}}(x) = \infty$. Moreover, since E is perfect, by Theorem 3.19, we get $\{1\}$ is a perfect filter of E and so $\text{ord}_{\{1\}}(x') < \infty$. Thus, there exists $m \in \mathbb{N}$ such that $x' \xrightarrow{m} 0 \in \{1\}$. Hence, there exists $m \in \mathbb{N}$ such that $x' \xrightarrow{m} 0 \in F$, and so $\text{ord}_F(x') < \infty$. Therefore, F is a perfect filter of E . $\square$

DEFINITION 3.21. A proper filter P of E is called a *primary filter* of E , if for any $x, y \in E$, $x \rightarrow y' \in P$ implies that there is $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in P$ or $y \xrightarrow{n} 0 \in P$.

Example 3.22. (i) According to Example 3.3(i), $P = \{b, 1\}$ is primary.
(ii) According to Example 3.6, $F = \{1\}$ is not primary.

Open Problem: Let F be a primary filter of E and $F \subseteq G$, where G is a proper filter of E . Is G a primary filter of E ?

THEOREM 3.23. *Every perfect filter is a primary filter.*

PROOF: Suppose a proper filter P is perfect and $y \rightarrow x' \in P$. Let for

any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin P$. Then $\text{ord}_P(x) = \infty$. Since P is perfect, we get $\text{ord}_P(x') < \infty$, and so there exists $m \in \mathbb{N}$ such that $x' \xrightarrow{m} 0 \in P$. Hence, $[x'] \xrightarrow{m} [0] = [1]$. Since $y \rightarrow x' \in P$, we get $[y] \leq [x']$. By the proof of Proposition 3.4(ii), $[x'] \xrightarrow{m} [0] \leq [y] \xrightarrow{m} [0]$ and so $[y] \xrightarrow{m} [0] = [1]$. Hence, $y \xrightarrow{m} 0 \in P$. Therefore, P is a primary filter of E . $\square$

In the following example we show that the converse of Theorem 3.23 does not hold, in general.

Example 3.24. According to Example 3.3(i), $\{b, 1\}$ is a primary filter but is not perfect, since $\text{ord}(a) = \text{ord}(a') < \infty$.

DEFINITION 3.25. E is called *integral*, if for any $x, y \in E$, $x \leq y'$ implies $x' = 1$ or $y' = 1$.

Example 3.26. (i) In Example 3.3(i), E is not integral. Since $a \leq a' = a$ but $a' = a \neq 1$.

(ii) In Example 3.8(ii), E is integral.

THEOREM 3.27. Consider $P \in \mathcal{F}(E)$ to be proper. If $\frac{E}{P}$ is integral, then P is primary.

PROOF: Suppose $\frac{E}{P}$ is integral and for $x, y \in E$, $x \rightarrow y' \in P$. Then $[x] \rightarrow [y'] = [1]$, and so $[x] \leq [y']$. Since $\frac{E}{P}$ is integral, we get $[x'] = [1]$ or $[y'] = [1]$. Thus, $[x \rightarrow 0] = [1]$ or $[y \rightarrow 0] = [1]$. Hence, $x' \in P$ or $y' \in P$ (choose $n = 1$ in definition of primary). Therefore, P is primary. $\square$

The following example shows that the converse of Proposition 3.27 does not hold, in general.

Example 3.28. According to Example 3.3(i), the filter $\{b, 1\}$ is primary but since $a \leq a'$, we can not consequence that $a' = 1$.

THEOREM 3.29. Let E be p -equality. Then $\frac{E}{P}$ is integral if and only if P is primary.

PROOF: $(\Rightarrow)$ By Theorem 3.27, the proof is clear.

($\Leftarrow$) Assume $[x] \leq [y']$, then $x \rightarrow y' \in P$. Since P is primary, there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in P$ or $y \xrightarrow{n} 0 \in P$. Since E is a p-equality, we have $x \rightarrow 0 = x \xrightarrow{n} 0 \in P$ or $y \rightarrow 0 = y \xrightarrow{n} 0 \in P$, and so $x' \in P$ or $y' \in P$. Hence, $[x'] = [1]$ or $[y'] = [1]$. Therefore, $\frac{E}{P}$ is integral. $\square$

PROPOSITION 3.30. Let E be a chain and $F \in \mathcal{F}(E)$. If $F = \text{Rad}(F)$, then F is primary.

PROOF: Suppose $x \rightarrow y' \in F$, for $x, y \in E$. Then we have the following cases:

Case 1: If $x \in F$, then $y' \in F$, and so F is primary.

Case 2: If $y \in F$, then by Proposition 2.2(ii), $y \rightarrow x' = x \rightarrow y' \in F$, and so $x' \in F$. Hence, F is primary.

Case 3: If $x, y \notin F$, then $x, y \notin \text{Rad}(F)$. By Proposition 3.5(i), there exist $n, m \in \mathbb{N}$ such that $x \xrightarrow{n} 0 = 1 \in F$ and $y \xrightarrow{m} 0 = 1 \in F$. Hence, F is primary.

Therefore, in all the above cases we prove that F is primary. $\square$

THEOREM 3.31. If P is a positive implicative and maximal filter of E , then $\frac{E}{P}$ is integral and so P is primary.

PROOF: Suppose $[x] \leq [y']$. Then $x \rightarrow y' \in P$, and so we have the following cases:

Case 1: If $x \in P$ and $y \notin P$, since $x \rightarrow y' \in P$ and $P \in \mathcal{F}(E)$, then $y' \in P$, and so $[y'] = [1]$.

Case 2: If $y \in P$ and $x \notin P$, since by Proposition 2.2(ii), $x \rightarrow y' = y \rightarrow x' \in P$ and $P \in \mathcal{F}(E)$, then $x' \in P$, and so $[x'] = [1]$.

Case 3: If $x, y \in P$, then since by Proposition 2.2(ii), $x \rightarrow y' = y \rightarrow x' \in P$ and $P \in \mathcal{F}(E)$, then $[y'] = [1]$ and $[x'] = [1]$, and so $x', y' \in P$. From $x, y \in P$, we get $0 \in P$, which is a contradiction, since $P \in \text{Max}(E)$.

Case 4: If $x, y \notin P$, then since $P \in \text{Max}(E)$, by Proposition 2.11(iii), we get $P \in \text{Prime}(E)$, so for any $x, y \in E$, we get $x \rightarrow y \in P$ or $y \rightarrow x \in P$. If $x \rightarrow y \in P$, since $x \rightarrow y' \in P$ and P is positive implicative, then $x \rightarrow 0 \in P$ and so $x' \in P$. Hence, $[x'] = [1]$. If $y \rightarrow x \in P$, then since by Proposition

2.2(ii), $x \rightarrow y' = y \rightarrow x' \in P$ and P positive implicative, we get $y' \in P$ and so $[y'] = [1]$.

Therefore, in all above cases, $\frac{E}{P}$ is integral and by Theorem 3.27, P is primary. $\square$

DEFINITION 3.32. The set $\mathcal{D}_s(E)$ is called a *dense set* of E and is defined as follows:

$$\mathcal{D}_s(E) = \{x \in E \mid x' = 0\}.$$

Example 3.33. According to Example 3.3(i), $\mathcal{D}_s(E) = \{b, 1\}$ and in Example 3.8(ii), $\mathcal{D}_s(E) = \{a, b, c, 1\}$.

PROPOSITION 3.34. $\mathcal{D}_s(E) \in \mathcal{F}(E)$.

PROOF: Clearly, $1 \in \mathcal{D}_s(E)$, and so $\mathcal{D}_s(E) \neq \emptyset$. Consider $x, x \rightarrow y \in \mathcal{D}_s(E)$. Then $x' = 0 = (x \rightarrow y)'$. By Proposition 2.2(ii) and (vi), we have

$$0 = x' = x \rightarrow 0 = x \rightarrow (x \rightarrow y)' = (x \rightarrow y) \rightarrow x' \geq y \rightarrow 0 = y'.$$

Thus, $y' = 0$ and so $y \in \mathcal{D}_s(E)$. Therefore, $\mathcal{D}_s(E) \in \mathcal{F}(E)$. $\square$

PROPOSITION 3.35. Let $F \in \mathcal{F}(E)$. If $\frac{E}{F}$ is involutive, then $\mathcal{D}_s(E) \subseteq F$.

PROOF: Since $\frac{E}{F}$ is involutive, for any $[x] \in \frac{E}{F}$, we have $[x] = [x'']$. Suppose $x \in \mathcal{D}_s(E)$. Then $x' = 0$ and so $x'' = 1$. Since, $[x] = [x''] = [1]$, we have $x \in F$. Hence, $\mathcal{D}_s(E) \subseteq F$. $\square$

In the following example, we show that the converse of Proposition 3.35, may not be true in general.

Example 3.36. According to Example 3.3(ii), let $F = \{1\}$. Clearly, $\mathcal{D}_s(E) \subseteq F$, but $E \cong \frac{E}{F}$ is not involutive.

Note. We call E has *Gödel negation property* if $\mathcal{D}_s(E) = E \setminus \{0\}$.

Example 3.37. According to Example 3.8(ii), $\mathcal{D}_s(E) = \{a, b, c, 1\} = E \setminus \{0\}$. Hence, E has Gödel negation property.

PROPOSITION 3.38. Let $F \in \mathcal{F}(E)$ and E has Gödel negation property. Then $\frac{E}{F}$ is involutive if and only if $\mathcal{D}_s(E) \subseteq F$.

PROOF: Suppose $\mathcal{D}_s(E) \subseteq F$ and there exists $[x] \in \frac{E}{F}$ such that $[x] \neq [x'']$. By Proposition 2.2(i), we know $[x] \leq [x'']$, thus, $[x''] \not\leq [x]$. It means that $x'' \rightarrow x \notin F$. By assumption, $x'' \rightarrow x \notin \mathcal{D}_s(E)$, and so $(x'' \rightarrow x)' \neq 0$, which is a contradiction with Gödel negation property. The proof of the converse holds by Proposition 3.35. $\square$

PROPOSITION 3.39. Assume E has Gödel negation property. If F is a proper fantastic (implicative) filter, then $F = E \setminus \{0\}$.

PROOF: Since E has Gödel negation property, we get $\mathcal{D}_s(E) = E \setminus \{0\}$. By assumption, F is a proper filter, so $0 \notin F$, and so $F \subseteq E \setminus \{0\}$. Now, suppose $x \in E \setminus \{0\}$. From E has Gödel negation property, $x' = 0$.

If F is fantastic, since $1 \rightarrow (0 \rightarrow x) = 1 \in F$ and $1 \in F$, by assumption, we have $x'' \rightarrow x = ((x \rightarrow 0) \rightarrow 0) \rightarrow x \in F$. Since $x'' = 1 \in F$, we get $x \in F$. Hence, $F = E \setminus \{0\}$.

Now, if F is implicative, then by Proposition 2.8, for any $x \in E$, $(x' \rightarrow x) \rightarrow x \in F$ and so $x \in F$. Hence, $F = E \setminus \{0\}$. $\square$

COROLLARY 3.40. Consider E has Gödel negation property. Then every proper fantastic (implicative) filter of E is maximal.

DEFINITION 3.41. A proper filter F of E is called an *integral filter* of E if $x \rightarrow y' \in F$ implies $x' \in F$ or $y' \in F$.

Example 3.42. (i) According to Example 3.8(ii), the filter $F = \{b, 1\}$ is an integral filter of E .

(ii) In Example 3.22(ii), $F = \{1\}$ is not integral.

Remark 3.43. Every integral filter of E is primary. According to Example 3.3(i), $F = \{b, 1\}$ is a primary filter but is not integral, since $a \rightarrow a' = a \rightarrow a = 1$ but $a' = a'' = a \notin \{b, 1\}$. So the converse may not be true in general.

THEOREM 3.44. Assume E is involutive, F is an integral filter of E and $F \subseteq G$, where G is a proper filter of E . Then G is an integral filter of E .

PROOF: Let $x, y \in E$ such that $x \rightarrow y' \in G$. Since $(x \rightarrow y')' \rightarrow (x \rightarrow y')' = 1 \in F$ and F is an integral filter of E , we have $(x \rightarrow y')'' \in F$ or $(x \rightarrow y')' \in F$. If $(x \rightarrow y')'' \in F$, then since E is involutive, we get $x \rightarrow y' \in F$, and by assumption, $x' \in F$ or $y' \in F$. From $F \subseteq G$, we get $x' \in G$ or $y' \in G$. Hence, G is integral. If $(x \rightarrow y') \rightarrow 0 = (x \rightarrow y')' \in F$, since $F \subseteq G$ and $x \rightarrow y' \in G$, then $0 \in G$, a contradiction, since G is proper. Therefore, G is an integral filter of E . $\square$

PROPOSITION 3.45. Assume P is a proper filter of E . Then P is integral if and only if $\frac{E}{P}$ is integral.

PROOF: Suppose $\frac{E}{P}$ is integral and $x \rightarrow y' \in P$. Then $[x] \leq [y']$ and by assumption, $[x'] = [1]$ or $[y'] = [1]$. Hence, $x' \in P$ or $y' \in P$. Therefore, P is an integral filter of E . The proof of the converse is similar. $\square$

PROPOSITION 3.46. E is integral if and only if E has Gödel negation property.

PROOF: Let E be integral. Since $0' = 1$, clearly, $0 \notin \mathcal{D}_s(E)$. So $\mathcal{D}_s(E) \subseteq E \setminus \{0\}$. Suppose $x \in E \setminus \{0\}$. Since E is integral, we get $\frac{E}{\{1\}}$ is integral and by Proposition 3.45, $\{1\}$ is an integral filter of E . By Proposition 2.2(i), $x \rightarrow x'' = 1 \in \{1\}$. Since $\{1\}$ is integral, we have $x' = 1$ or $x'' = 1$. If $x' = 1$, by Proposition 2.2(i), $x \leq x'' = 0$, and so $x = 0$, a contradiction. If $x'' = 1$, by Proposition 2.2(iv), $x' = x''' = 0$, and so $x \in \mathcal{D}_s(E)$. Hence, $E \setminus \{0\} \subseteq \mathcal{D}_s(E)$. Therefore, $\mathcal{D}_s(E) = E \setminus \{0\}$. Hence, E has Gödel property.

Conversely, let E has Gödel negation property. For proving that E is integral it is enough to prove that $\{1\}$ is integral. For this, suppose $x \rightarrow y' \in \{1\}$, where $x, y \in E$.

If $x = y = 0$, then $x' = y' = 1$, and so $x \rightarrow y' = 1 \in \{1\}$ and clearly, $x', y' \in \{1\}$.

If $x \neq 0$ and $y = 0$, then $y' = 1$, and so $y' \in \{1\}$.

If $x = 0$ and $y \neq 0$, then $x' = 1 \in \{1\}$.

If $x, y \neq 0$, then by assumption $x' = y' = 0$, so $0 = x' = x \rightarrow 0 = x \rightarrow y' \in F$, a contradiction, since F is proper.

Hence, in all the above cases, $\{1\}$ is integral, and by Proposition 3.45, E is integral. $\square$

PROPOSITION 3.47. If E is a lattice and integral, then for any $x, y \in E$, $x \rightarrow y' = x' \vee y'$.

PROOF: By Proposition 3.46, since E is integral, we have $E \setminus \{0\} = \mathcal{D}_s(E)$. Now, let $x, y \in E$.

If $x = y = 0$, then $x \rightarrow y' = 1$ and $x' = y' = 1$, and so $x \rightarrow y' = x' \vee y'$.

If $x = 0$ and $y \neq 0$, then $x \rightarrow y' = 1$, and so $x \rightarrow y' = x' \vee y'$.

If $x \neq 0$ and $y = 0$, then $x \rightarrow y' = 1 = x' \vee y'$.

If $x, y \neq 0$, then $x' = y' = 0$ and so $x \rightarrow y' = 0 = x' \vee y'$.

Hence, in all above cases we have $x \rightarrow y' = x' \vee y'$. $\square$

PROPOSITION 3.48. If E is a chain, then E is integral if and only if for any $x, y \in E$, $x \rightarrow y' = x' \vee y'$.

PROOF: Let for any $x, y \in E$, $x \rightarrow y' = x' \vee y'$. It is enough to prove that E has Gödel negation property. Clearly, $\mathcal{D}_s(E) \subseteq E \setminus \{0\}$. Now, let $x \in E \setminus \{0\}$. Since $x \rightarrow x'' = 1$ and by assumption $x \rightarrow y' = x' \vee y'$, we get $x' \vee x'' = 1$. Since E is a chain, we get $x' = 1$ or $x'' = 1$. If $x' = 1$, then $x'' = 0$ and by Proposition 2.2(i), $x = 0$, a contradiction. If $x'' = 1$, then by Proposition 2.2(iv), $x' = x''' = 0$, and so $x \in \mathcal{D}_s(E)$. Hence, $\mathcal{D}_s(E) = E \setminus \{0\}$. Now, by Proposition 3.46, E is integral. The proof of the converse, by Proposition 3.47 is clear. $\square$

PROPOSITION 3.49. Consider E to be an integral involutive lattice. If F is an integral filter of E , then $F \in \text{Spec}(E)$.

PROOF: Let F be an integral filter of E and $x, y \in E$ such that $x \vee y \in F$. Then by Proposition 3.47 and assumption, $x' \rightarrow y'' = x'' \vee y'' = x \vee y \in F$. Since F is an integral filter of E , we have $x = x'' \in F$ or $y = y'' \in F$. Hence, $F \in \text{Spec}(E)$. $\square$

Clearly, converse of Proposition 3.49 does not hold in general. Since according to Example 3.3(i), $F = \{b, 1\} \in \text{Spec}(E)$ but is not integral, since $a \rightarrow a' = a \rightarrow a = 1$ but $a' = a'' = a \notin F$.

THEOREM 3.50. *E is an integral equality algebra if and only if $\text{Rad}(E) = E \setminus \{0\}$.*

PROOF: Consider E to be an integral equality algebra. Then by Proposition 3.46, $\mathcal{D}_s(E) = E \setminus \{0\}$. Let $x \in \text{Rad}(E)$. Then by Proposition 2.12, for any $n \in \mathbb{N}$, $(x \xrightarrow{n} 0) \rightarrow x = 1$, and so for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \leq x$. If $x = 0$, then $1 = x \xrightarrow{n} 0 \leq x = 0$, which is a contradiction, so $\text{Rad}(E) \subseteq E \setminus \{0\}$. Now, suppose $x \in E \setminus \{0\}$. Then $x \in \mathcal{D}_s(E)$, and so $x' = 0$. Thus,

$$x \xrightarrow{n} 0 = x \xrightarrow{n-1} (x \rightarrow 0) = x \xrightarrow{n-1} 0 = \cdots = x \rightarrow 0 = 0.$$

Hence, for any $x \in E \setminus \{0\}$, $x \xrightarrow{n} 0 \leq x$ and so $(x \xrightarrow{n} 0) \rightarrow x = 1 \in F$. Hence, $x \in \text{Rad}(E)$. Therefore, $\text{Rad}(E) = E \setminus \{0\}$.

Conversely, by Proposition 3.46, it is enough to prove that $\mathcal{D}_s(E) = E \setminus \{0\}$. Let $x \in E \setminus \{0\} = \text{Rad}(E)$. If $x' \neq 0$, then $x' \in \text{Rad}(E)$ and so $x, x' \in \text{Rad}(E)$ which is a contradiction. Thus, $x' = 0$, and so $x \in \mathcal{D}_s(E)$. Hence, $E \setminus \{0\} \subseteq \mathcal{D}_s(E) \subseteq E \setminus \{0\}$. Therefore, $\mathcal{D}_s(E) = E \setminus \{0\}$. $\square$

COROLLARY 3.51. Let E be integral and $F \in \mathcal{F}(E)$. Then

(i) $\text{Rad}(F) = E \setminus \{0\}$.

(ii) $\text{Rad}(F) \in \text{Max}(E)$.

PROOF: (i) Let $F \in \mathcal{F}(E)$. Since $\{1\} \subseteq F \subseteq E$. By [11, Proposition 3.14(ii)], $\text{Rad}(\{1\}) \subseteq \text{Rad}(F) \subseteq \text{Rad}(E)$. Then by Theorem 3.50, we have $E \setminus \{0\} \subseteq \text{Rad}(F) \subseteq E \setminus \{0\}$. Hence, $\text{Rad}(F) = E \setminus \{0\}$.

(ii) By (i) the proof is clear. $\square$

Example 3.52. According to Example 3.3(i), $F = \{b, 1\} \in \text{Max}(E)$. Clearly, $F \neq E \setminus \{0\}$. In addition, E is not integral.

PROPOSITION 3.53. Let E be integral, $F, G \in \mathcal{F}(E)$ and $F \subseteq G$ such that G be proper. If $\text{Rad}(F) = F$, then $\text{Rad}(G) = G$.

PROOF: By Corollary 3.51(i), $Rad(F) = E \setminus \{0\}$. Since $\{1\} \subseteq F$, we have $Rad(\{1\}) \subseteq Rad(F)$ and so $E \setminus \{0\} \subseteq Rad(F) = F \subseteq G \subset E$. Thus, $E \setminus \{0\} = F = Rad(F) = G$. Since $F \subseteq G$, by [11, Proposition 3.14(ii)], $Rad(F) \subseteq Rad(G)$. Hence, $Rad(G) = G$. $\square$

4. Local equality algebras

In this section, we define the notion of local bounded equality algebra and investigate the relation between local and other kinds of bounded equality algebras. We prove that if E is a bounded lattice local equality algebra, then $\mathcal{C}(E) = \{0, 1\}$. In addition, we show that every local bounded equality algebra is perfect or locally finite or peculiar.

DEFINITION 4.1. E is called *local*, if for any $x \in E$, $ord(x) < \infty$ or $ord(x') < \infty$.

Example 4.2. (i) According to Example 3.8(ii), clearly, $ord(1) = ord(c) = ord(a) = ord(b) = \infty$ and $ord(c') = ord(a') = ord(b') = ord(0) = 1$. Hence, E is local.

(ii) According to Example 3.3(iii), E is local.

(iii) According to Example 3.3(i), E is local.

(iv) In Example 3.22(ii), E is a bounded equality algebra which is not local since $ord(a) = ord(a') = \infty$.

THEOREM 4.3. *If E has a unique maximal filter, then E is local.*

PROOF: Suppose $x \in E$ such that $ord(x) = ord(x') = \infty$. By Proposition 3.4(i), $\langle x \rangle$ and $\langle x' \rangle$ are two proper filters of E . By Proposition 2.11(ii) and assumption, $\langle x \rangle, \langle x' \rangle \subseteq M$ and so $x, x' = x \rightarrow 0 \in M$. Thus, $0 \in M$, which is a contradiction. So, $ord(x) < \infty$ or $ord(x') < \infty$. Hence, E is local. $\square$

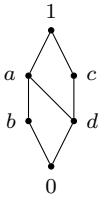
THEOREM 4.4. *Consider E to be a p -equality algebra. Then E is local if and only if E has a unique maximal filter.*

PROOF: Suppose E is local and $M \in \mathcal{Max}(E)$. By Proposition 3.4(iv), $M \subseteq \{x \in E \mid ord(x) = \infty\}$. Now, assume $x \notin M$ and $ord(x) = \infty$. By Proposition 2.11(i), there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in M$. Since E is

p-equality and $x \rightarrow 0 = x \xrightarrow{n} 0$, we get $x \rightarrow 0 \in M$. Thus, by Proposition 3.4(iv), $ord(x \rightarrow 0) = \infty$. Since E is local, $ord((x \rightarrow 0) \rightarrow 0) < \infty$ and by Propositions 3.4(ii) and 2.2(i), $ord(x) < \infty$, which is a contradiction. Thus, for any $M \in \mathcal{Max}(E)$, $M = \{x \in E \mid ord(x) = \infty\}$. Hence, E has a unique maximal filter. The proof of the converse by Theorem 4.3 is hold. $\square$

In the following example we show that the condition p-equality in Theorem 3.7 is necessary.

Example 4.5. Consider $E = \{0, a, b, c, d, 1\}$ is a lattice with the next Hasse diagram. Define the operation “ $\sim$ ” on E as follows:



$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then $(E, \sim, \wedge, 1)$ is an equality algebra which is not a p-equality algebra, since

$$c = a \rightarrow d = a \rightarrow (a \rightarrow 0) \neq (a \rightarrow a) \rightarrow (a \rightarrow 0) = 1 \rightarrow d = d.$$

Also, E is not local, since $ord(b) = ord(b') = \infty$.

THEOREM 4.6. *Let $P \in \mathcal{F}(E)$ be proper. Then $\frac{E}{P}$ is local if and only if P is primary.*

PROOF: Suppose $\frac{E}{P}$ is local, for any $x, y \in E$, $x \rightarrow y' \in P$ and for any $n \in \mathbb{N}$, $x \xrightarrow{n} 0 \notin P$. Then $[x] \xrightarrow{n} [0] \neq [1]$. Since $x \rightarrow y' \in P$, we have $[x] \leq [y']$ and similar to the proof of Proposition 3.4(ii), we have $[y'] \xrightarrow{n} [0] \leq [x] \xrightarrow{n} [0] \neq [1]$, for any $n \in \mathbb{N}$. Thus, $[y'] \xrightarrow{n} [0] \neq [1]$, and so $[y'] \xrightarrow{n} 0 \neq [1]$. Hence, $\text{ord}([y']) = \infty$. Since $\frac{E}{P}$ is local, we have $\text{ord}([y']) < \infty$. By Proposition 2.2(i), $\text{ord}([y]) < \infty$. So there exists $k \in \mathbb{N}$ such that $[y] \xrightarrow{k} [0] = [1]$ and so $y \xrightarrow{k} 0 \in P$.

Conversely, by Proposition 2.2(i), $x \rightarrow x'' = 1 \in P$. Since P is primary, there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in P$ or $x' \xrightarrow{n} 0 \in P$. If $\text{ord}([x]) = \infty$, then for any $n \in \mathbb{N}$, $[x] \xrightarrow{n} [0] \neq [1]$ and so $[x \xrightarrow{n} 0] \neq [1]$. Hence, $x \xrightarrow{n} 0 \notin P$, thus, $x' \xrightarrow{n} 0 \in P$, and so $[x'] \xrightarrow{n} [0] = [1]$. Hence, $\text{ord}([x']) < \infty$. The proof of the other case is similar. Therefore, $\frac{E}{P}$ is local $\square$

PROPOSITION 4.7. If E is integral, then E is local.

PROOF: Since E is integral, we get $\frac{E}{\{1\}}$ is integral. Then by Theorem 3.27, $\{1\}$ is primary. By Theorem 4.6, $\frac{E}{\{1\}}$ is local. Hence, E is local. $\square$

According to Example 3.3(i), E is local but E is not integral, since $a \leq a' = a$ and so $a = a' \neq 1$.

COROLLARY 4.8. Let $F \in \mathcal{F}(E)$. If $F = \text{Rad}(F)$, then $\frac{E}{F}$ is local.

PROOF: By Proposition 3.30 and Theorem 4.6, the proof is clear. $\square$

COROLLARY 4.9. $\{1\}$ is a primary filter of E if and only if E is local.

PROPOSITION 4.10. If E is perfect and p-equality, then $E = \text{Rad}(E) \cup \text{Rad}'(E)$.

PROOF: Since E is perfect, we get $\frac{E}{\{1\}}$ is perfect, and by Theorem 3.19, $\{1\}$ is perfect. By Theorem 3.23, $\{1\}$ is primary and by Corollary 4.9, E is local. Hence, E has a unique maximal filter M that contains all $x \in E$ such that $\text{ord}(x) = \infty$. Suppose $x \in E$. If $\text{ord}(x) = \infty$, then $x \in \text{Rad}(E)$. If $\text{ord}(x) < \infty$, since E is perfect, then $\text{ord}(x') = \infty$ and so $x' \in \text{Rad}(E)$. Thus, $x \in \text{Rad}'(E)$. Hence, $E = \text{Rad}(E) \cup \text{Rad}'(E)$. $\square$

COROLLARY 4.11. Consider E to be a p-equality algebra. A proper filter $P \subseteq E$ is primary if and only if there is a unique $M \in \text{Max}(E)$ such that $P \subseteq M$.

PROOF: By Proposition 4.6, P is primary if and only if $\frac{E}{P}$ is local if and only if by Theorem 3.7, $\frac{E}{P}$ has only one maximal filter, such as $\frac{M}{P}$. Since there is a one-to-one correspondence between the filters of E and $\frac{E}{P}$, we get there is a unique $M \in \text{Max}(E)$ such that $P \subseteq M$. $\square$

COROLLARY 4.12. If E is local, then for any $G \in \mathcal{F}(E)$, $\frac{E}{G}$ is local, too.

PROPOSITION 4.13. Consider E to be a local p-equality. Then E is perfect if and only if $\frac{E}{\text{Rad}(E)} = \{[0], [1]\}$.

PROOF: Since E is perfect, we get $\frac{E}{\{1\}}$ is perfect, and by Theorem 3.19, $\{1\}$ is perfect. By Theorem 3.23, $\{1\}$ is primary and by Corollary 4.9, E is local. Thus, it has just one maximal filter M such that $\text{Rad}(E) = M$. In addition, since E is perfect, by Proposition 4.10, $E = \text{Rad}(E) \cup \text{Rad}'(E)$. Let $x \in E$. If $x \in \text{Rad}(E)$, then $[x] = [1]$. If $x \in \text{Rad}'(E)$, then $x' \in \text{Rad}(E)$, and so $[x'] = [1]$. Thus, $[x''] = [0]$. By Proposition 2.2(i), $[x] \leq [x'']$, and so $[x] = [0]$. Therefore, $\frac{E}{\text{Rad}(E)} = \{[0], [1]\}$.

Conversely, let $x \in E$. Then $[x] \in \frac{E}{\text{Rad}(E)} = \{[0], [1]\}$. If $[x] = [1]$, then $x \in \text{Rad}(E)$. If $[x] = [0]$, then $[x'] = [1]$, and so $x' \in \text{Rad}(E)$. Hence,

$x \in \text{Rad}'(E)$. Thus, $\text{ord}(x) = \infty$ or $\text{ord}(x') = \infty$. If $\text{ord}(x) = \infty$ and $\text{ord}(x') = \infty$, then by Proposition 3.4(i), $\langle x \rangle$ and $\langle x' \rangle$ are two proper filters of E , and by Proposition 2.11(i) and assumption, $x, x' = x \rightarrow 0 \in M$, where $M \in \text{Max}(E)$, so $0 \in M$, which is a contradiction. Hence, E is perfect. $\square$

COROLLARY 4.14. Every chain equality algebra is local.

Note. Set

$$\mathcal{D}(E) = \{x \in E \mid \text{ord}(x) = \infty\}$$

and

$$\mathcal{D}^*(E) = \{x \in E \mid \exists y \in \mathcal{D}(E) \text{ s.t. } x \leq y'\}.$$

THEOREM 4.15. Assume E is local. Then E is perfect if and only if $E = \mathcal{D}(E) \cup \mathcal{D}^*(E)$.

PROOF: Suppose $x \in E$. If $\text{ord}(x) = \infty$, then $x \in \mathcal{D}(E)$. If $\text{ord}(x) < \infty$, then $x \notin \mathcal{D}(E)$. Since E is perfect, we get $\text{ord}(x') = \infty$, and so $x' \in \mathcal{D}(E)$. By Proposition 2.2(i), we know $x \leq x''$ and by definition of $\mathcal{D}^*(E)$, we get $x \in \mathcal{D}^*(E)$. Hence, $E \subseteq \mathcal{D}(E) \cup \mathcal{D}^*(E)$. The proof of the other side is clear. Therefore, $E = \mathcal{D}(E) \cup \mathcal{D}^*(E)$.

Conversely, let $E = \mathcal{D}(E) \cup \mathcal{D}^*(E)$ and $x \in E$. If $\text{ord}(x) < \infty$, then $x \notin \mathcal{D}(E)$. By assumption, $x \in \mathcal{D}^*(E)$. So, there is $y \in \mathcal{D}(E)$ such that $x \leq y'$. By Proposition 2.2(i) and (iii), $y \leq y'' \leq x'$. Since $y \in \mathcal{D}(E)$, we get $\text{ord}(y) = \infty$ and by Proposition 3.4(iii), $\text{ord}(x') = \infty$. Thus, in this case E is perfect. Now, if $\text{ord}(x) = \infty$, then $x \in \mathcal{D}(E)$. If $\text{ord}(x') = \infty$, the E is not local, which is a contradiction. Thus, $\text{ord}(x') < \infty$. Therefore, in both cases, E is perfect. $\square$

PROPOSITION 4.16. Consider E to be a p-equality algebra and $\mathcal{N}(E) = \{x \in E \mid \exists n \in \mathbb{N} \text{ s.t. } x \xrightarrow{n} 0 = 1\}$. Then

- (i) $E = \mathcal{N}(E) \cup \text{Rad}(E)$ if and only if E is local.
- (ii) If E is a lattice and local, then $\mathcal{C}(E) = \{0, 1\}$.

PROOF: (i) Suppose E is local. Then by Theorem 3.7, E has a unique maximal filter such as M and so $\text{Rad}(E) = M$. Let $x \in E$. Then by

assumption $\text{ord}(x) < \infty$ or $\text{ord}(x') < \infty$. If $\text{ord}(x) < \infty$, then clearly $x \in \mathcal{N}(E)$. If $\text{ord}(x) = \infty$, then by Theorem 3.7, $x \in M = \text{Rad}(E)$. Hence, $E = \mathcal{N}(E) \cup \text{Rad}(E)$.

Conversely, suppose $M_1, M_2 \in \text{Max}(E)$ such that $M_1 \neq M_2$. Then there are $x \in M_1 \setminus M_2$. Since $x \in E$, by assumption, $x \in \mathcal{N}(E)$ or $x \in \text{Rad}(E)$. If $x \in \mathcal{N}(E)$, then there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 = 1 \in M_1$, and so $0 \in M_1$, which is a contradiction. If $x \in \text{Rad}(E)$, then according to definition of $\text{Rad}(E)$, $x \in M_2$, which is a contradiction. The proof of the other case is similar. Hence, E is local.

(ii) Suppose E is local. Then by Theorem 3.7, E has a unique maximal filter such as M . Let $x \in \mathcal{C}(E)$. Then $x \vee x' = 1 \in M$. Since by Propositions 2.10(ii) and 2.11(iii) $M \in \text{Spec}(E)$, we get $x \in M$ or $x' \in M$. Thus, $\text{ord}(x) = \infty$ or $\text{ord}(x') = \infty$. By Proposition 3.4(i), if $\text{ord}(x) = \infty$ and $\text{ord}(x') = \infty$, then by Proposition 3.19(ii), $x, x' \in M$, and so $0 \in M$, a contradiction. If $\text{ord}(x) = \infty$, then $\text{ord}(x') < \infty$, so there exists $n \in \mathbb{N}$ such that $x' \xrightarrow{n} 0 = 1$. Since E is p-equality algebra, we get $x' \rightarrow 0 = 1$ and so $x'' = 1$. Thus, by Proposition 2.2(v), $x' = x''' = 0$. Since $x \vee x' = 1$, we get $x = 1$. If $\text{ord}(x) < \infty$, then there exists $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 = 1$, and so $x \rightarrow 0 = 1$. Hence, $x' = 1$. Thus, by Proposition 2.2(iv), $x \leq x'' = 0$. Thus, $x = 0$. Therefore, $\mathcal{C}(E) = \{0, 1\}$. $\square$

DEFINITION 4.17. An equality algebra E is called *locally finite* if for any $x \in E \setminus \{1\}$, $\text{ord}(x) < \infty$.

Example 4.18. Let $E = \{0, a, b, c, 1\}$ be a chain where $0 \leq a \leq b \leq c \leq 1$. Define the operation “ $\sim$ ” on E as follows:

$\sim$	0	a	b	c	0	$\rightarrow$	0	a	b	c	1
0	1	c	b	b	1	0	1	1	1	1	1
a	c	1	c	c	a	a	c	1	1	1	1
b	b	c	1	c	b	b	b	c	1	1	1
c	b	c	c	1	c	c	b	c	c	1	1
1	0	a	b	c	1	1	0	a	b	c	1

Then $(E, \wedge, \sim, 0, 1)$ is a locally finite equality algebra.

DEFINITION 4.19. E is called a *peculiar equality algebra* if it is local and there are $x, y \in E \setminus \{0, 1\}$ such that $\text{ord}(x) = \infty$, $\text{ord}(y) < \infty$ and $\text{ord}(y') < \infty$.

Example 4.20. (i) In Example 3.3(i), E is a peculiar equality algebra.

(ii) In Example 3.3(ii), E is a peculiar equality algebra.

(iii) In Example 3.8(ii), E is not a peculiar equality algebra.

THEOREM 4.21. *Let E be local. Then E is perfect or locally finite or peculiar.*

PROOF: We have the following cases:

Case 1: Suppose that E is not locally finite and peculiar. We prove that E is perfect. Since E is not locally finite, there exists $x \in E \setminus \{1\}$ such that $\text{ord}(x) = \infty$. Since E is local, we have $\text{ord}(x') < \infty$. Moreover, since $\text{ord}(x) = \infty$, if there exists $y \in E$ such that $\text{ord}(y), \text{ord}(y') < \infty$, then E is peculiar, which is a contradiction. Hence, for all $y \in E$, $\text{ord}(y) = \infty$ or $\text{ord}(y') = \infty$. If there exists $y \in E$ such that $\text{ord}(y) = \text{ord}(y') = \infty$, then E is not local, which is a contradiction. Therefore, E is perfect.

Case 2: Suppose that E is not perfect and locally finite. We prove that E is peculiar. By assumption, E is local. Since E is not locally finite, there exists $x \in E \setminus \{1\}$ such that $\text{ord}(x) = \infty$. Also, E is not perfect. Then there exists $y \in E$ such that $\text{ord}(y) = \text{ord}(y') = \infty$ or $\text{ord}(y), \text{ord}(y') < \infty$. If $\text{ord}(y) = \text{ord}(y') = \infty$, then E is not local, which is a contradiction. Hence, there exists $y \in E$ such that $\text{ord}(y), \text{ord}(y') < \infty$. Therefore, E is peculiar.

Case 3: Suppose that E is not perfect and peculiar. We prove that E is locally finite. Since E is not perfect, there exists $y \in E$ such that $\text{ord}(y) = \text{ord}(y') = \infty$ or $\text{ord}(y), \text{ord}(y') < \infty$. If $\text{ord}(y) = \text{ord}(y') = \infty$, then E is not local, which is a contradiction. Thus, there exists $y \in E$ such that $\text{ord}(y), \text{ord}(y') < \infty$. Now, let x be an arbitrary element of $E \setminus \{1\}$. If $x = y$ or $x = y'$, then $\text{ord}(x) < \infty$. Suppose that $x \neq y$. If $\text{ord}(x) = \infty$, then E is peculiar, which is a contradiction. Thus, $\text{ord}(x) < \infty$. Hence, for all $x \in E \setminus \{1\}$, $\text{ord}(x) < \infty$. Therefore, E is locally finite. $\square$

5. Conclusion

By considering the notion of bounded equality algebra, the concepts of local and perfect bounded equality algebras are defined and some related results are proved. Specially, the notions of order of elements in bounded equality algebras and some filters such as primary and perfect filters are introduced and the relation between perfect filters with perfect bounded equality algebras and primary filter with the local bounded equality algebras are investigated. In addition, by defining the concept of locally finite bounded equality algebra, a relation between local and perfect bounded equality algebras are studied. In addition, it was shown that every integral bounded equality algebra is local and every local bounded equality algebra is perfect or locally finite or peculiar.

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VALUATIONS ON L -ALGEBRAS

Abstract

The purpose of this paper is to define the notion of valuation maps on L -algebras and introduce its properties. For this, the concepts of valuation and 1-valuation maps are introduced and some characterizations and related properties are investigated. Using a valuation map, a congruence relation is obtained and proved that this congruence relation is equivalent to the congruence relation made by an ideal. Also, showed that the image and inverse image of any valuation is a valuation. Finally, by using a 1-valuation map ξ on L -algebra $\mathcal{L}$, an L -algebra $(\tilde{\mathcal{L}}, *, \{1\})$ and a metric space $(\tilde{\mathcal{L}}, d_\xi)$ has been obtained.

Keywords: L -algebra, valuation map, 1-valuation map, topology, metric space.

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1. Introduction

L -algebras, which are connected to algebraic logic and quantum structures, were first introduced by Rump [11]. Numerous examples have demonstrated the usefulness of L -algebras. Yang and Rump [13] characterized pseudo-MV-algebras and Bosbach's noncommutative bricks as L -algebras. In their work, Wu and Yang [18] provided several approaches to show that orthomodular lattices belong to a particular class of L -algebras. Wu et al. [17] further established that every lattice-ordered effect algebra possesses a natural L -algebra structure.

Several other mathematicians have also investigated the relationship between basic algebras and L -algebras [3]. They showed that a basic algebra satisfying

$$(b \oplus \neg q) \oplus \neg(v \oplus \neg q) = (b \oplus \neg v) \oplus \neg(q \oplus \neg v)$$

can be transformed into an L -algebra. Conversely, if an L -algebra with 0 satisfying certain conditions can be structured as an involutive bounded lattice, then it is necessarily a lattice-ordered effect algebra (see [17]).

Recently, researchers have studied L -algebras from different perspectives. For further information, see [2, 1, 4, 6, 7, 8, 10, 14, 15, 19, 20].

Song, Roh, and Jun [16] introduced the concepts of quasi-valuation maps in BCK/BCI -algebras based on subalgebras and ideals. They also investigated various properties of these maps and established relationships between quasi-valuation maps based on subalgebras and those based on ideals. In another study [9], the continuity of operations in a BCK -algebra was examined using a topology induced by a pseudo-valuation. Moreover, the authors showed that the product of a finite number of a pseudo metric space associated with a pseudo-valuation. They further proved that if a BCK -algebra X possesses a pseudo-valuation, then every quotient space of X admits a pseudometric. The completion of these spaces was also investigated.

In addition, in [5], the concepts of S -quasi-valuation maps and F -quasi-valuation maps were introduced based on subalgebras and filters in hoops, and their associated properties were studied. Relationships between S -quasi-valuation maps and F -quasi-valuation maps were established. Using

the F -quasi-valuation map, the authors introduced a (pseudo)metric space and demonstrated that all operations in a hoop are uniformly continuous.

The aim of this paper is to define valuations on L -algebras and investigate their properties. We provide characterizations of valuations and 1-valuation maps and study their associated properties. Using valuation maps, we define a congruence relation and prove that it is equivalent to a congruence relation induced by an ideal. Furthermore, we show that the image and inverse image of a valuation under a homomorphism are also valuations. Finally, we introduce a metric space associated with a valuation map and establish that the operation “ $*$ ” in an L -algebra is uniformly continuous.

2. Preliminaries

This section lists the known definitions and results that will be utilized later.

DEFINITION 2.1 ([7]). An L -algebra is an algebraic structure $(\mathcal{L}; *, 1)$ of type $(2, 0)$ that satisfies the following conditions:

- (L1) $q * q = q * 1 = 1$ and $1 * q = q$,
- (L2) $(q * v) * (q * b) = (v * q) * (v * b)$,
- (L3) if $q * v = v * q = 1$, then $q = v$,

for any $q, v, b \in \mathcal{L}$. Condition (L1) asserts that 1 is a logical unit, while (L2) is linked to the quantum Yang-Baxter equation, it is the so-called cycloid equation. It is important to note that a logical unit is always unique. We have to notice that all axioms are needed to ensure that the following relation is a partial order relation:

$$q \leq v \text{ if and only if } q * v = 1, \quad (2.1)$$

(L1) ensures reflexivity, (L2) implies transitivity and (L3) is used to prove that the relation is antisymmetric. So $(\mathcal{L}, \leq)$ is a poset. If $\mathcal{L}$ has a smallest element 0, then it is called a *bounded L -algebra*.

PROPOSITION 2.2 ([13]). Consider an L -algebra $(\mathcal{L}; *, 1)$. If $q \leq v$, then for any $q, v, b \in \mathcal{L}$, it follows that $b * q \leq b * v$.

PROPOSITION 2.3 ([13]). The statements below are equivalent for an L -algebra $\mathcal{L}$:

- (i) $q \leq v * q$,
- (ii) if $q \leq b$, then $b * v \leq q * v$,
- (iii) $((q * v) * b) * b \leq ((q * v) * b) * ((v * q) * b)$,

for any $q, v, b \in \mathcal{L}$.

DEFINITION 2.4 ([12]). An L -algebra $\mathcal{L}$, that satisfying:

$$q * (v * q) = 1, \quad (K)$$

for any $q, v \in \mathcal{L}$, is called a KL -algebra. A CKL -algebra, is an L -algebra that satisfies the equation:

$$q * (v * b) = v * (q * b), \quad (C)$$

for any $q, v, b \in \mathcal{L}$ (see [12]).

Clearly, every CKL -algebra is a KL -algebra, since for any $q, v \in \mathcal{L}$, we have

$$q * (v * q) = v * (q * q) = v * 1 = 1.$$

Note. We have to notice that the notion of KL -algebras concerns the properties in Proposition 2.3.

PROPOSITION 2.5 ([4]). Assume $(\mathcal{L}, *, 1)$ is a CKL -algebra. For any $q, v, b \in \mathcal{L}$, the following properties hold:

- (i) $q * (v * q) = 1$, i.e., $q \leq v * q$,
- (ii) $q \leq (q * v) * v$,
- (iii) $q \leq v * b$ if and only if $v \leq q * b$,
- (iv) if $q \leq v$, then $v * b \leq q * b$,
- (v) $((q * v) * b) * b \leq ((q * v) * b) * ((v * q) * b)$,

- (vi) $b * v \leq (v * q) * (b * q)$,
- (vii) $b * v \leq (q * b) * (q * v)$.

DEFINITION 2.6 ([1]). An L -algebra $(\mathcal{L}, *, 1)$ is a positive L -algebra, if for any $q, v, b \in \mathcal{L}$, the following condition is satisfied:

$$q * (v * b) = (q * v) * (q * b). \quad (P)$$

THEOREM 2.7 ([1]). Every positive L -algebra is a CKL -algebra.

DEFINITION 2.8 ([11]). A subset $\mathbb{I}$ of an L -algebra $(\mathcal{L}, *, 1)$ is termed an ideal of $\mathcal{L}$, if it meets the following conditions, for all $q, v \in \mathcal{L}$.

- (I_1) $1 \in \mathbb{I}$,
- (I_2) if $q \in \mathbb{I}$ and $q * v \in \mathbb{I}$, then $v \in \mathbb{I}$,
- (I_3) if $q \in \mathbb{I}$, then $(q * v) * v \in \mathbb{I}$,
- (I_4) if $q \in \mathbb{I}$, then $v * q \in \mathbb{I}$ and $v * (q * v) \in \mathbb{I}$.

The collection of all ideals of $\mathcal{L}$, is represented by $\mathcal{Id}(\mathcal{L})$.

PROPOSITION 2.9 ([4]). Every ideal of $\mathcal{L}$ is an upset.

If $(\mathcal{L}, *, 1)$ is a CKL -algebra, then for proving that $\mathbb{I} \in \mathcal{Id}(\mathcal{L})$, the conditions (I_3) and (I_4) can be omitted. In fact, for any $q \in \mathbb{I}$, using (C) and (I_1), we have:

$$q * ((q * v) * v) = (q * v) * (q * v) = 1 \in \mathbb{I},$$

for any $v \in \mathcal{L}$. It follows from (I_2) that $(q * v) * v \in \mathbb{I}$. Therefore, (I_3) holds. Furthermore, if $q \in \mathbb{I}$, then for any $v \in \mathcal{L}$, using (K) we have $q * (v * q) = 1 \in \mathbb{I}$ and by (I_2), $v * q \in \mathbb{I}$.

THEOREM 2.10 ([11]). Consider the L -algebra $(\mathcal{L}, *, 1)$. In this context, every ideal $\mathbb{I}$ of $\mathcal{L}$ establishes a congruence relation on $\mathcal{L}$, for any q and v within $\mathcal{L}$, where

$$q \sim_{\mathbb{I}} v \Leftrightarrow q * v, v * q \in \mathbb{I}. \quad (2.2)$$

Conversely, every congruence relation $\sim$ defines an ideal $\mathbb{I} = \{q \in \mathcal{L} \mid q \sim 1\}$.

Note. According to Theorem 2.10, the correspondence between ideals and congruences is one-to-one.

Note. From now on, we let $(\mathcal{L}, *, 1)$ or $\mathcal{L}$, for short, be an L -algebra.

3. Valuations on L -algebras

In this section, our objective is to define valuations on L -algebras and introduce their properties. Additionally, we will provide characterizations of valuations.

DEFINITION 3.1. A real-valued function $\xi : \mathcal{L} \rightarrow \mathbb{R}$ is referred to as a *valuation map on $\mathcal{L}$* , if it fulfills the following conditions:

$$(VL_1) \quad \xi(1) = 0,$$

$$(VL_2) \quad \xi(q * v) \geq \xi(v) - \xi(q), \text{ for any } q, v \in \mathcal{L}.$$

The valuation map ξ is classified as a *1-valuation map*, if it satisfies the following condition, for any $q \in \mathcal{L}$:

$$\xi(q) = 0 \text{ implies } q = 1.$$

The collection of all valuation maps on $\mathcal{L}$ is denoted by $\mathcal{V}(\mathcal{L})$. When considering $\xi \in \mathcal{V}(\mathcal{L})$, the combination $(\mathcal{L}, \xi)$ is referred to as a *valued L -algebra*.

Example 3.2. Consider $\mathcal{L} = \{0, f, y, 1\}$ as a poset, where $0 < f, y < 1$. In this case, $(\mathcal{L}, *, 0, 1)$ forms a bounded L -algebra, where:

$*$	0	f	y	1
0	1	1	1	1
f	y	1	y	1
y	f	f	1	1
1	0	f	y	1

Let us define the map $\xi : \mathcal{L} \rightarrow \mathbb{R}$ as follows: $\xi(1) = 0$, $\xi(0) = \xi(f) = \alpha$, and $\xi(y) = \beta$, where $\alpha \geq \beta$. It can be observed that $\xi \in \mathcal{V}(\mathcal{L})$.

Remark 3.3. Consider $\xi \in \mathcal{V}(\mathcal{L})$. It follows that $\xi(q * v) + \xi(v * q) \geq 0$, for any $q, v \in \mathcal{L}$. This is due to the fact that, for all $q, v \in \mathcal{L}$, we have the following relationship:

$$\xi(q * v) + \xi(v * q) \geq \xi(v) - \xi(q) + \xi(q) - \xi(v) = 0.$$

Hence, $\xi(q * v) + \xi(v * q) \geq 0$.

PROPOSITION 3.4. Consider $\xi \in \mathcal{V}(\mathcal{L})$. For any $q, v, b \in \mathcal{L}$, the following relations hold:

- (i) ξ is order reversing.
- (ii) For any $q \in \mathcal{L}$, $\xi(q) \geq 0$.
- (iii) If $q \leq v * b$, then $\xi(b) \leq \xi(q) + \xi(v)$.
- (iv) If $\mathcal{L}$ is a KL -algebra, then $\xi(q * v) \leq \xi(q) + \xi(v)$.

If $\mathcal{L}$ is a CKL -algebra, then

- (v) $\xi(q * b) \leq \xi(q * v) + \xi(v * b)$.
- (vi) $\xi((q * (v * b)) * b) \leq \xi(q) + \xi(v)$.
- (vii) $\xi(q * (v * b)) \leq \xi((q * v) * b)$.

PROOF: (i) Assume $q, v \in \mathcal{L}$ such that $q \leq v$. According to the property (3.1), it follows that $q * v = 1$. Therefore, by applying the property (VL_2) , we obtain:

$$0 = \xi(1) = \xi(q * v) \geq \xi(v) - \xi(q).$$

Hence, $\xi(v) \leq \xi(q)$.

(ii) This follows directly from (VL_1) and (i).

(iii) Assume $q, v, b \in \mathcal{L}$. If $q \leq v * b$, then by (i), we have $\xi(v * b) \leq \xi(q)$. Additionally, by (VL_2) , we find that $\xi(v * b) \geq \xi(b) - \xi(v)$. Hence, $\xi(b) - \xi(v) \leq \xi(q)$. Therefore, $\xi(b) \leq \xi(q) + \xi(v)$.

(iv) As $\mathcal{L}$ is a KL -algebra, it follows that for any $q, v \in \mathcal{L}$, we have $v \leq q * v$. By (i), we have $\xi(q * v) \leq \xi(v)$. Also, by (VL_2) , $\xi(v) - \xi(q) \leq \xi(q * v)$, and so $\xi(v) \leq \xi(q) + \xi(q * v)$. Hence,

$$\xi(q * v) \leq \xi(v) \leq \xi(q) + \xi(q * v) \leq \xi(q) + \xi(v).$$

Therefore, $\xi(q * v) \leq \xi(q) + \xi(v)$.

(v) Let $q, v, b \in \mathcal{L}$. By Proposition 2.5(vi), we have $q * v \leq (v * b) * (q * b)$. Utilizing property (i), we get $\xi((v * b) * (q * b)) \leq \xi(q * v)$. Furthermore, by (VL_2) , we find that $\xi(q * b) - \xi(v * b) \leq \xi(q * v)$. Therefore, we obtain $\xi(q * b) \leq \xi(q * v) + \xi(v * b)$.

(vi) As $\mathcal{L}$ is a CKL -algebra, according to Proposition 2.5(ii), we have $q \leq (q * (v * b)) * (v * b)$. Consequently, using property (i), we can continue as follows:

$$\xi((q * (v * b)) * (v * b)) \leq \xi(q). \quad (3.1)$$

Additionally, based on Proposition 2.5(ii), we have $v \leq (v * b) * b$. By (i), we obtain:

$$\xi((v * b) * b) \leq \xi(v). \quad (3.2)$$

Hence,

$$\begin{aligned} \xi((q * (v * b)) * b) &\leq \xi((q * (v * b)) * (v * b)) + \xi((v * b) * b) \quad \text{by (v)} \\ &\leq \xi(q) + \xi((v * b) * b) \quad \text{by (3.1)} \\ &\leq \xi(q) + \xi(v). \quad \text{by (3.2)} \end{aligned}$$

Hence, $\xi((q * (v * b)) * b) \leq \xi(q) + \xi(v)$.

(vii) Consider $q, v, b \in \mathcal{L}$. Proposition 2.5(i) implies that $v \leq q * v$. Furthermore, by Proposition 2.5(iv), we find that $(q * v) * b \leq v * b$. Therefore, using property (i), we can conclude that $\xi(v * b) \leq \xi((q * v) * b)$. Additionally, based on Proposition 2.5(i), we have $v * b \leq q * (v * b)$. Applying property (i) once again, we find that $\xi(q * (v * b)) \leq \xi(v * b)$. Hence, $\xi(q * (v * b)) \leq \xi((q * v) * b)$. $\square$

Set $\ker \xi = \{q \in \mathcal{L} \mid \xi(q) = 0\}$.

Remark 3.5. (i) Consider $\xi \in \mathcal{V}(\mathcal{L})$. It follows that $\tau\xi \in \mathcal{V}(\mathcal{L})$ and $\ker(\tau\xi) = \ker \xi$, for any $\tau \in \mathbb{R}$ with $\tau > 0$. Here, $\tau\xi : \mathcal{L} \rightarrow \mathbb{R}$ is defined as $(\tau\xi)(q) = \tau\xi(q)$, for all $q \in \mathcal{L}$.

(ii) Assuming $\xi_1, \xi_2 \in \mathcal{V}(\mathcal{L})$, we can conclude that $\xi = \tau\xi_1 + \omega\xi_2 \in \mathcal{V}(\mathcal{L})$, where $\tau, \omega \in \mathbb{R}$ with $\tau, \omega > 0$. Moreover, the kernel of ξ is given by $\ker \xi = \ker \xi_1 \cap \ker \xi_2$. It is evident that $\xi_1, \xi_2 \in \mathcal{V}(\mathcal{L})$ and $\tau, \omega \in \mathbb{R}$ with $\tau, \omega > 0$. Therefore, for any $q, v \in \mathcal{L}$, we have the following:

$$\xi(1) = (\tau\xi_1 + \omega\xi_2)(1) = \tau\xi_1(1) + \omega\xi_2(1) = \tau(\xi_1(1)) + \omega(\xi_2(1)) = 0.$$

Thus, (VL_1) holds. Furthermore, we have the following:

$$\begin{aligned} \xi(q * v) &= (\tau\xi_1 + \omega\xi_2)(q * v) \\ &= (\tau\xi_1)(q * v) + (\omega\xi_2)(q * v) \\ &\geq \tau(\xi_1(v) - \xi_1(q)) + \omega(\xi_2(v) - \xi_2(q)) \quad \text{since } \tau, \omega > 0 \\ &= \tau\xi_1(v) - \tau\xi_1(q) + \omega\xi_2(v) - \omega\xi_2(q) \\ &= (\tau\xi_1 + \omega\xi_2)(v) - (\tau\xi_1 + \omega\xi_2)(q) \\ &= \xi(v) - \xi(q). \end{aligned}$$

Therefore, we can conclude that (VL_2) holds. As a result, $\xi \in \mathcal{V}(\mathcal{L})$. Furthermore,

$$\begin{aligned} q \in \ker \xi &\Leftrightarrow \xi(q) = 0 \Leftrightarrow (\tau\xi_1 + \omega\xi_2)(q) = 0 \Leftrightarrow \tau\xi_1(q) + \omega\xi_2(q) = 0 \\ &\stackrel{\tau, \omega > 0}{\Leftrightarrow} \xi_1(q) = \xi_2(q) = 0 \Leftrightarrow q \in \ker \xi_1 \cap \ker \xi_2. \end{aligned}$$

Hence, $\ker \xi = \ker \xi_1 \cap \ker \xi_2$.

(iii) In general, if $\xi_i \in \mathcal{V}(\mathcal{L})$, $\tau_i \in \mathbb{R}$, $\tau_i > 0$ and $1 \leq i \leq n$, then $\xi = \sum_{i=1}^n \tau_i \xi_i \in \mathcal{V}(\mathcal{L})$ and $\ker \xi = \bigcap_{i=1}^n \ker \xi_i$.

PROPOSITION 3.6. Assume $\mathcal{L}$ is a CKL -algebra. If $\xi \in \mathcal{V}(\mathcal{L})$, then $\ker \xi \in \mathcal{Id}(\mathcal{L})$.

PROOF: It is evident that $1 \in \ker \xi$. Let us consider $q, q * v \in \ker \xi$. This implies $\xi(q), \xi(q * v) = 0$, and so by (LV_2) , we obtain the following inequality:

$$0 = \xi(q * v) \geq \xi(v) - \xi(q) \geq \xi(v).$$

Hence, $\xi(v) \leq 0$. By Proposition 3.4(ii), $\xi \geq 0$, it follows that $\xi(b) \geq 0$, for any $b \in \mathcal{L}$. Consequently, we have $\xi(v) \geq 0$. As a result, $\xi(v) = 0$, which implies that $v \in \ker \xi$. We have to mention that (I3) and (I4) do not need to be checked in the case of a CKL-algebra. Therefore, $\ker \xi \in \mathcal{Id}(\mathcal{L})$. $\square$

PROPOSITION 3.7. Let us assume that $\mathbb{I} \in \mathcal{Id}(\mathcal{L})$. We will now define the function $\xi_{\mathbb{I}} : \mathcal{L} \rightarrow \mathbb{R}$ as follows:

$$\xi_{\mathbb{I}}(q) = \begin{cases} 0 & \text{if } q \in \mathbb{I} \\ f & \text{if } q \notin \mathbb{I} \end{cases} \quad (3.3)$$

where $f \in \mathbb{R}$, $f > 0$. Then the following statements hold:

- (i) $\xi_{\mathbb{I}} \in \mathcal{V}(\mathcal{L})$,
- (ii) $\ker \xi_{\mathbb{I}} = \mathbb{I}$.

PROOF: (i) It is evident that $1 \in \mathbb{I}$ since $\mathbb{I} \in \mathcal{Id}(\mathcal{L})$. As a result, we have $\xi_{\mathbb{I}}(1) = 0$, which satisfies property (VL_1) . To prove property (VL_2) , if there are $q, v \in \mathcal{L}$ with $\xi(q * v) = \xi(q) = 0$, and $\xi(v) = f$. But then $q, q * v \in \mathbb{I}$, whereas $v \notin \mathbb{I}$, which is a contradiction.

(ii) The proof is straightforward and clear. $\square$

Remark 3.8. According to Propositions 3.6 and 3.7, there exists a relationship between $\mathcal{V}(\mathcal{L})$ and $\mathcal{Id}(\mathcal{L})$. However, it is important to note that this relationship is not a one-to-one correspondence.

PROPOSITION 3.9. If ξ is a valuation on the positive L -algebra $\mathcal{L}$, and $f \in \mathcal{L}$, then the function ξ_f defined as $\xi_f(q) = \xi(f * q)$ for any $q \in \mathcal{L}$ is a member of $\mathcal{V}(\mathcal{L})$. Furthermore, if ξ is a 1-valuation, then ξ_f is a 1-valuation if $f = 1$ and in this case $\xi_f = \xi$.

PROOF: It is clear that $\xi_f(1) = 0$. Now, let us consider $q, v, f \in \mathcal{L}$. Since $\mathcal{L}$ is a positive L -algebra, we can deduce the following:

$$\xi_f(q) - \xi_f(v) = \xi(f*q) - \xi(f*v) \leq \xi((f*v)*(f*q)) = \xi(f*(v*q)) = \xi_f(v*q).$$

Hence $\xi_f \in \mathcal{V}(\mathcal{L})$. Now, we have

$$\xi_f(q) = 0 \Leftrightarrow \xi(f*q) = 0 \Leftrightarrow f*q = 1 \Leftrightarrow f \leq q.$$

So if $f = 1$, then $q = 1$ and so ξ_f is 1-valuation and clearly, $\xi_f = \xi$. $\square$

DEFINITION 3.10. [11] Let $\mathcal{L}$ and $\mathcal{K}$ be two L -algebras. Then a map $\Theta : \mathcal{L} \rightarrow \mathcal{K}$ is called an L -homomorphism if for any $q, v \in \mathcal{L}$ we have $\Theta(q*v) = \Theta(q) *_H \Theta(v)$.

If Θ is an injective, then Θ is called a *monomorphism*, and if Θ is onto, then Θ is called an *epimorphism*. In addition, if Θ is a bijective function, then Θ is called an *isomorphism*.

Remark 3.11. Let $(\mathcal{L}_1, *_1, 1_{\mathcal{L}_1})$ and $(\mathcal{L}_2, *_2, 1_{\mathcal{L}_2})$ be two L -algebras. Then $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ with the following operations is an L -algebra.

$$(q, v) * (f, y) = (q *_1 f, v *_2 y), \text{ and } \mathbf{1} = (1_{\mathcal{L}_1}, 1_{\mathcal{L}_2}),$$

for any $(q, v), (f, y) \in \mathcal{L}$.

PROPOSITION 3.12. Let $(\mathcal{L}_1, *_1, 1_{\mathcal{L}_1})$ and $(\mathcal{L}_2, *_2, 1_{\mathcal{L}_2})$ be two L -algebras, and $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We can say that $\mathcal{L}$ has a valuation ξ if each $\mathcal{L}_i$ has a valuation ξ_i for $i = 1, 2$, and $\xi = \xi_1 + \xi_2$.

PROOF: Let ξ_1 and ξ_2 be valuations on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Define $\xi : \mathcal{L} \rightarrow \mathbb{R}$ by $\xi(q) = \xi_1(q_1) + \xi_2(q_2)$, for any $q = (q_1, q_2)$. Then $\xi(1) = \xi((1_{\mathcal{L}_1}, 1_{\mathcal{L}_2})) = \xi_1(1_{\mathcal{L}_1}) + \xi_2(1_{\mathcal{L}_2}) = 0$. Now, let $q = (q_1, q_2), v = (v_1, v_2) \in \mathcal{L}$. Then, by Remark 3.11, we have

$$\begin{aligned} \xi(q*v) &= \xi(q_1 *_1 v_1, q_2 *_2 v_2) \\ &= \xi_1(q_1 *_1 v_1) + \xi_2(q_2 *_2 v_2) \\ &\geq \xi_1(v_1) - \xi_1(q_1) + \xi_2(v_2) - \xi_2(q_2) \\ &= \xi_1(v_1) + \xi_2(v_2) - (\xi_2(q_1) + \xi_2(q_2)) \\ &= \xi(v) - \xi(q). \end{aligned}$$

Hence, $\xi \in \mathcal{V}(\mathcal{L})$. $\square$

4. Quotient L -algebras induced by valuation maps

In this section, we will define a map that will allow us to introduce a congruence relation on $\mathcal{L}$. We will then explore various properties of the quotient L -algebras that arise from this congruence relation.

Note. Let's clarify that we will consider $\mathcal{L}$ as a CKL -algebra from now on unless mentioned otherwise.

According to Remark 3.11, we can define a mapping $d_\xi : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{R}$ for $\xi \in \mathcal{V}(\mathcal{L})$. For any $q, v \in \mathcal{L}$, the mapping is defined as follows:

$$d_\xi(q, v) = \xi(q * v) + \xi(v * q). \quad (4.1)$$

Remark 4.1. Clearly, $d_\xi(1, q) = \xi(q)$.

PROPOSITION 4.2. The following inequalities are satisfied by every d_ξ :

- (i) $d_\xi(q, q) = 0$.
- (ii) $d_\xi(q, v) \geq 0$.
- (iii) $d_\xi(q, v) = d_\xi(v, q)$.
- (iv) $d_\xi(q, b) \leq d_\xi(q, v) + d_\xi(v, b)$.
- (v) $d_\xi(q, v) \geq d_\xi(q * f, v * f)$.
- (vi) $d_\xi(q, v) \geq d_\xi(f * q, f * v)$.
- (vii) $d_\xi(q * v, f * y) \geq d_\xi(q * v, f * v) + d_\xi(f * v, f * y)$.
- (viii) $|\xi(q) - \xi(v)| \leq d_\xi(q, v)$.

PROOF: It is evident that (i), (ii), and (iii) are true.

(iv) Assume $q, v, b \in \mathcal{L}$. Then

$$\begin{aligned} d_\xi(q, v) + d_\xi(v, b) &= \xi(q * v) + \xi(v * q) + \xi(v * b) + \xi(b * v) \\ &\geq \xi(q * b) + \xi(b * q) \quad \text{by Proposition 3.4(v)} \\ &= d_\xi(q, b). \end{aligned}$$

Hence, $d_\xi(q, b) \leq d_\xi(q, v) + d_\xi(v, b)$.

(v) Let $q, v, f \in \mathcal{L}$. Then by Proposition 2.5(vi), $v * q \leq (q * f) * (v * f)$ and by Proposition 3.4(i), $\xi((q * f) * (v * f)) \leq \xi(v * q)$. By the similar way, we can see that $\xi((v * f) * (q * f)) \leq \xi(q * v)$, and so $d_\xi(q, v) \geq d_\xi(q * f, v * f)$.

(vi) By Propositions 2.5(vii) and 3.4(i), the proof is straightforward.

(vii) This is just a special case of (iv).

(viii) Let $q, v \in \mathcal{L}$. Then:

$$\xi(v) - \xi(q) \leq \xi(q * v) \leq \xi(v * q) + \xi(q * v) = d_\xi(q, v).$$

and

$$\xi(q) - \xi(v) \leq \xi(v * q) \leq \xi(q * v) + \xi(v * q) = d_\xi(q, v).$$

Hence $-d_\xi(q, v) \leq \xi(q) - \xi(v) \leq d_\xi(q, v)$. Thus $|\xi(q) - \xi(v)| \leq d_\xi(q, v)$. $\square$

Let $\xi \in \mathcal{V}(\mathcal{L})$. Define $\equiv_\xi$ for any $q, v \in \mathcal{L}$ by

$$(q, v) \equiv_\xi \Leftrightarrow d_\xi(q, v) = 0. \quad (4.2)$$

PROPOSITION 4.3. The binary relation $\equiv_\xi$ is a congruence relation on $\mathcal{L}$, called the congruence relation induced by ξ .

PROOF: Proposition 4.2(i) and (iii) demonstrate that $d_\xi(q, v)$ possesses the properties of reflexivity and symmetry. Let $q, v, b \in \mathcal{L}$ such that $(q, v) \equiv_\xi$ and $(v, b) \equiv_\xi$. By Proposition 4.2(iv), we have $d_\xi(q, b) \leq d_\xi(q, v) + d_\xi(v, b) = 0$. Therefore, by Proposition 4.2(ii), we have $d_\xi(q, b) = 0$, and so $(q, b) \equiv_\xi$. Now, suppose that $d_\xi(q, v) = d_\xi(b, r) = 0$. By applying Proposition 4.2(v) and (vi) we get,

$$0 = d_\xi(q, v) \geq d_\xi(q * b, v * b), \quad 0 = d_\xi(b, r) \geq d_\xi(v * b, v * r),$$

Moreover, by utilizing Proposition 4.2(ii) and (vii), we can infer that:

$$d_\xi(q * b, v * b) = d_\xi(v * b, v * r) = 0,$$

Applying the principle of transitivity, we can thus conclude that $(q * b, v * r) \equiv_\xi$. Consequently, we can infer that $\equiv_\xi$ is a congruence relation. $\square$

Remark 4.4. Let q, v be elements of $\mathcal{L}$. Since ξ belongs to $\mathcal{V}(\mathcal{L})$, we obtain

$$\begin{aligned}\xi(q * v) = 0 &\Leftrightarrow \xi(1) + \xi(q * v) = 0 \Leftrightarrow \xi((q * v) * 1) + \xi(1 * (q * v)) = 0 \\ &\Leftrightarrow d_\xi(q * v, 1) = 0.\end{aligned}$$

By the similar discussion, we have $\xi(v * q) = 0 \Leftrightarrow d_\xi(v * q, 1) = 0$. Hence,

$$\begin{aligned}(q, v) \in \equiv_\xi &\Leftrightarrow d_\xi(q, v) = 0 \\ &\Leftrightarrow \xi(q * v) = \xi(v * q) = 0 \\ &\Leftrightarrow d_\xi(q * v, 1) = 0, \text{ and } d_\xi(v * q, 1) = 0.\end{aligned}$$

Remark 4.5. Let ξ be an element of $\mathcal{V}(\mathcal{L})$. Define $[q]_\xi$ as the set $\{v \in \mathcal{L} \mid (q, v) \in \equiv_\xi\}$, where $q \in \mathcal{L}$. The collection of all such $[q]_\xi$ is denoted by $\frac{\mathcal{L}}{\equiv_\xi}$. By Proposition 4.3, $\equiv_\xi$ is a congruence relation, the operation $*$ can be defined on $\frac{\mathcal{L}}{\equiv_\xi}$ as $[q]_\xi * [v]_\xi = [q * v]_\xi$ for all $q, v \in \mathcal{L}$. Hence, by Remark 4.4, $\langle \frac{\mathcal{L}}{\equiv_\xi}, *, [1]_\xi \rangle$ forms an L -algebra.

PROPOSITION 4.6. Assuming $\xi \in \mathcal{V}(\mathcal{L})$. Then $\sim_{\ker(\xi)}$ is equivalent to $\equiv_\xi$.

PROOF: By Proposition 3.6, $\ker(\xi) \in \mathcal{Id}(\mathcal{L})$. Also, if $\equiv_\xi$ is a congruence relation providing an L -algebra, then it is already defined by the ideal $[1]_\xi$ as ideals are in one-to-one correspondence to congruences whose quotient is an L -algebra. In addition, $(q, v) \in \sim_{\ker(\xi)}$. Then, according to (2.2), we have $q * v, v * q \in \ker(\xi)$. Consequently, $\xi(q * v), \xi(v * q) = 0$. Therefore, $d_\xi(q, v) = 0$. Thus, using (4.2), we can conclude that $(q, v) \in \equiv_\xi$. Hence, $\sim_{\ker(\xi)} \subseteq \equiv_\xi$.

Conversely, if $(q, v) \in \equiv_\xi$, then $d_\xi(q, v) = 0$. Consequently, $\xi(q * v) + \xi(v * q) = 0$. According to Proposition 3.4(ii), this implies that $\xi(q * v) = \xi(v * q) = 0$, and therefore, $q * v, v * q \in \ker(\xi)$. Thus, by (2.2), we can deduce that $(q, v) \in \sim_{\ker(\xi)}$, which in turn implies that $\equiv_\xi \subseteq \sim_{\ker(\xi)}$. Therefore, we conclude that $\sim_{\ker(\xi)} = \equiv_\xi$. $\square$

Remark 4.7. Suppose $q \in [1]_\xi$. This implies that $d_\xi(q, 1) = 0$, and consequently, $(q, 1) \equiv_\xi$. According to Proposition 4.6, it follows that $(q, 1) \in \sim_{\ker(\xi)}$. Thus, $q \in \ker(\xi)$, leading to the conclusion that $\xi(q) = 0$.

THEOREM 4.8. Assume $\xi_1, \xi_2 \in \mathcal{V}(\mathcal{L})$, where $[1]_{\xi_1} = [1]_{\xi_2}$. Then $\frac{\mathcal{L}}{\equiv_{\xi_1}} = \frac{\mathcal{L}}{\equiv_{\xi_2}}$.

PROOF: Since $[1]_{\xi_1}, [1]_{\xi_2}$ are defining ideals for the congruences $\equiv_{\xi_1}, \equiv_{\xi_2}$, respectively, we get that if they are equal, the congruences are either. $\square$

COROLLARY 4.9. Let $\xi \in \mathcal{V}(\mathcal{L})$. Then $\bar{\xi} \in \mathcal{V}\left(\frac{\mathcal{L}}{\equiv_\xi}\right)$, where $\bar{\xi} : \frac{\mathcal{L}}{\equiv_\xi} \rightarrow \mathbb{R}$ and $\bar{\xi}([q]_\xi) = \xi(q)$, for any $q \in \mathcal{L}$.

PROOF: Straightforward. $\square$

In the following, we have two general facts in universal algebra: congruences of $\mathcal{L}/\equiv$ are in one-to-one correspondence with congruences of $\mathcal{L}$ that contain $\equiv$. Thus, we have:

PROPOSITION 4.10. Assuming that $\xi \in \mathcal{V}(\mathcal{L})$ and $\mathbb{I} \in \mathcal{Id}(\mathcal{L})$ such that $[1]_\xi \subseteq \mathbb{I}$, then we have:

$$(i) \quad q \in \mathbb{I} \text{ if and only if } [q]_\xi \in \frac{\mathbb{I}}{\equiv_\xi}.$$

$$(ii) \quad \frac{\mathbb{I}}{\equiv_\xi} \in \mathcal{Id}\left(\frac{\mathcal{L}}{\equiv_\xi}\right).$$

PROOF: (i) Let $[q]_\xi \in \frac{\mathbb{I}}{\equiv_\xi}$. Then there exists $v \in \mathbb{I}$ such that $[q]_\xi = [v]_\xi$, which implies that $d_\xi(q, v) = 0$. By Remark 4.4, we have $d_\xi(q * v, 1) = d_\xi(v * q, 1) = 0$. Thus $q * v$ and $v * q$ belong to $[1]_\xi \subseteq \mathbb{I}$. So, $v * q \in \mathbb{I}$. Since $v \in \mathbb{I}$ and $\mathbb{I} \in \mathcal{Id}(\mathcal{L})$ we conclude that $q \in \mathbb{I}$. Conversely, if $q \in \mathbb{I}$, then $d_\xi(q, 1) = 0$ and so $q \in [1]_\xi$. Hence, $[q]_\xi = [1]_\xi \in \frac{\mathbb{I}}{\equiv_\xi}$. Therefore, $[q]_\xi \in \frac{\mathbb{I}}{\equiv_\xi}$.

(ii) By (i), the proof is clear. $\square$

THEOREM 4.11. *There exists a one-to-one corresponding between ideals of $\mathcal{L}$ and ideals of $\frac{\mathcal{L}}{\equiv_{\xi}}$ that contain $[1]_{\xi}$.*

PROOF: By referring to Proposition 4.10, the proof becomes evident. $\square$

THEOREM 4.12. *If $\mathcal{K}$ is an L -algebra, $\Theta : \mathcal{L} \rightarrow \mathcal{K}$ is an L -homomorphism and $\xi \in \mathcal{V}(\mathcal{K})$, then $\xi \circ \Theta \in \mathcal{V}(\mathcal{L})$.*

PROOF: The proof is simple and direct. $\square$

THEOREM 4.13. *Suppose $\mathcal{K}$ is an L -algebra, $\Theta : \mathcal{L} \rightarrow \mathcal{K}$ is an L -epimorphism, and $\xi \in \mathcal{V}(\mathcal{K})$. It follows that $\frac{\mathcal{L}}{\equiv_{\xi \circ \Theta}}$ is isomorphic to $\frac{\mathcal{K}}{\equiv_{\xi}}$.*

PROOF: According to Theorem 4.12, we know that $\xi \circ \Theta \in \mathcal{V}(\mathcal{L})$. Additionally, based on Proposition 4.3 and Remark 4.5, we can establish that $\frac{\mathcal{L}}{\equiv_{\xi \circ \Theta}}$ and $\frac{\mathcal{K}}{\equiv_{\xi}}$ are both L -algebras. We can define a mapping $\psi : \frac{\mathcal{L}}{\equiv_{\xi \circ \Theta}} \rightarrow \frac{\mathcal{K}}{\equiv_{\xi}}$ such that $\psi([q]_{\xi \circ \Theta}) = [\Theta(q)]_{\xi}$, for any $q \in \mathcal{L}$. Consequently,

$$\begin{aligned}
 [q]_{\xi \circ \Theta} = [v]_{\xi \circ \Theta} &\Leftrightarrow (q, v) \in \equiv_{\xi \circ \Theta} \\
 &\Leftrightarrow (\xi \circ \Theta)(q * v) = (\xi \circ \Theta)(v * q) = 0 \\
 &\Leftrightarrow \xi(\Theta(q * v)) = \xi(\Theta(v * q)) = 0 \\
 &\Leftrightarrow \xi(\Theta(q) * \Theta(v)) = \xi(\Theta(v) * \Theta(q)) = 0 \\
 &\Leftrightarrow (\Theta(q), \Theta(v)) \in \equiv_{\xi} \\
 &\Leftrightarrow [\Theta(q)]_{\xi} = [\Theta(v)]_{\xi} \\
 &\Leftrightarrow \psi([q]_{\xi \circ \Theta}) = \psi([v]_{\xi \circ \Theta}).
 \end{aligned}$$

Thus, we can conclude that ψ is both one-to-one and well-defined. Additionally, let us assume that $[q]_{\xi \circ \Theta}, [v]_{\xi \circ \Theta} \in \frac{\mathcal{L}}{\equiv_{\xi \circ \Theta}}$. Consequently, we have:

$$\begin{aligned}
 \psi([q]_{\xi \circ \Theta} * [v]_{\xi \circ \Theta}) &= \psi([q * v]_{\xi \circ \Theta}) = [\Theta(q * v)]_{\xi} = [\Theta(q)]_{\xi} * [\Theta(v)]_{\xi} \\
 &= \psi([q]_{\xi \circ \Theta}) * \psi([v]_{\xi \circ \Theta}).
 \end{aligned}$$

Furthermore, if $[v]_\xi \in \frac{\mathcal{K}}{\equiv_\xi}$, then due to the fact that Θ is an L -epimorphism, there exists an $q \in \mathcal{L}$ such that $\Theta(q) = v$. Consequently, we can deduce that $\psi([q]_{\xi \circ \Theta}) = [\Theta(q)]_\xi = [v]_\xi$. Thus, ψ can be considered an L -epimorphism. As a result, we can conclude that $\frac{\mathcal{L}}{\equiv_{\xi \circ f \Theta}} \cong \frac{\mathcal{K}}{\equiv_\xi}$.

Second proof: By using the composition $\mathcal{L} \rightarrow \mathcal{K} \rightarrow \frac{\mathcal{K}}{\equiv_\xi}$, then it is easy too see that the defining congruence of the epimorphism $\mathcal{L} \rightarrow \frac{\mathcal{K}}{\equiv_\xi}$ is given by $\equiv_{\xi \circ \Theta}$. $\square$

5. Topology on L -algebra induced by a valuation map

In the following, we utilize the concept of valuation and equation (4.1) to establish a pseudo metric and a topological L -algebra. Furthermore, we explore the properties of this structure.

We consider a list of elements of $\mathcal{L}$ such as $\{q_n\}$ is called a sequence on $\mathcal{L}$ where for any $\{q_n\}$ and $\{v_n\}$, we have $\{q_n\} * \{v_n\} = \{q_n * v_n\}$.

Remark 5.1. Let $\xi : \mathcal{L} \rightarrow \mathbb{R}$ be a continuous function, and let $\{q_n\}$ be a sequence in $\mathcal{L}$. We say that q_n converges to q , denoted as $q_n \rightarrow q \in \mathcal{L}$, if $\lim_{n \rightarrow \infty} d_\xi(q_n, q) = 0$.

THEOREM 5.2. *If $\xi \in \mathcal{V}(\mathcal{L})$, then d_ξ is a pseudo-metric¹ on $\mathcal{L}$ and so $(\mathcal{L}, d_\xi)$ is a pseudo-metric space.*

PROOF: Based on Proposition 4.2(i), (ii), (iii), and (iv), the proof is straightforward. $\square$

¹By a pseudo-metric on $\mathcal{L}$, we mean a real-valued function $d : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{R}$ satisfying the following properties:

$$d(q, v) \geq 0, \quad d(q, q) = 0, \quad d(q, v) = d(v, q) \quad \text{and} \quad d(q, b) \leq d(q, v) + d(v, b),$$

for every $q, v, b \in \mathcal{L}$. If $d(q, v) = 0$ implies $q = v$, then d is called a metric. The space $(\mathcal{L}, d)$ is called a pseudo-metric space and metric space, respectively.

THEOREM 5.3. *If ξ is a 1-valuation map on $\mathcal{L}$, then the pair $(\mathcal{L}, d_\xi)$ forms a metric space.*

PROOF: By Theorem 5.2, we know that $(\mathcal{L}, \xi)$ is a pseudo-metric space. Consider $q, v \in \mathcal{L}$ such that $d_\xi(q, v) = 0$. This implies that $\xi(q * v) + \xi(v * q) = 0$. According to Proposition 3.4(ii), we have $\xi(q * v) = \xi(v * q) = 0$. Since ξ is a 1-valuation map, it follows that $q * v = v * q = 1$. By property (L_3) , we conclude that $q = v$. Therefore, $(\mathcal{L}, d_\xi)$ is a metric space. $\square$

Note. If $(\mathcal{L}, d)$ is a pseudo-metric space, then for each $q \in \mathcal{L}$ and $\varepsilon > 0$, the set

$$N_\varepsilon(q) = \{v \in \mathcal{L} \mid d(q, v) < \varepsilon\},$$

is called a *ball of radius ε with center at q* .

PROPOSITION 5.4. Consider two L -algebras, $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\xi_1 \in \mathcal{V}(\mathcal{L}_1)$ and $\xi_2 \in \mathcal{V}(\mathcal{L}_2)$. Let $(q, v), (f, y) \in \mathcal{L}_1 \times \mathcal{L}_2$. We define the function $d((q, v), (f, y)) = d_{\xi_1}(q, f) + d_{\xi_2}(v, y)$, and we claim that d is a pseudo metric on $\mathcal{L}_1 \times \mathcal{L}_2$.

PROOF: Clearly, d is just the metric associated with the valuation $\xi_1 + \xi_2$ which is a valuation by Proposition 3.12. Thus, by Propositions 3.12 and 4.2 and Theorem 5.2, the proof is clear. $\square$

COROLLARY 5.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two L -algebras. If ξ_1 and ξ_2 are two 1-valuations on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, then the pair $(\mathcal{L}_1 \times \mathcal{L}_2, d)$ forms a metric space.

THEOREM 5.6. *Suppose we have a pseudo-metric space $(\mathcal{L}, d_\xi)$. We can then define a new pseudo-metric space $(\mathcal{L}^2, d_\xi^*)$, where $\mathcal{L}^2$ is the Cartesian product $\mathcal{L} \times \mathcal{L}$. For any $(q, v), (f, y) \in \mathcal{L}^2$, the pseudo-metric d_ξ^* is given by:*

$$d_\xi^*((q, v), (f, y)) = \max\{d_\xi(q, f), d_\xi(v, y)\}.$$

PROOF: By Proposition 4.2(i), (ii) and (iii), clearly $d_\xi^*((q, v), (f, y)) \geq 0$, $d_\xi^*((q, v), (q, v)) = 0$ and $d_\xi^*((q, v), (f, y)) = d_\xi^*((f, y), (q, v))$. Suppose $(q, v), (b, r), (w, z) \in \mathcal{L}^2$. Then

$$\begin{aligned}
 & d_{\xi}^*((q, v), (b, r)) + d_{\xi}^*((b, r), (w, z)) \\
 &= \max \{d_{\xi}(q, b), d_{\xi}(v, r)\} + \max \{d_{\xi}(b, w), d_{\xi}(r, z)\} \\
 &= \max \{d_{\xi}(q, b) + d_{\xi}(b, w), d_{\xi}(v, r) + d_{\xi}(r, z)\} \\
 &\geq \max \{d_{\xi}(q, w), d_{\xi}(v, z)\} \quad \text{by Proposition 4.2(iv)} \\
 &= d_{\xi}^*((q, v), (w, z)).
 \end{aligned}$$

Hence, $(\mathcal{L}^2, d_{\xi}^*)$ is a pseudo-metric space. $\square$

THEOREM 5.7. *If ξ is a 1-valuation map on $\mathcal{L}$, then $(\mathcal{L}^2, d_{\xi}^*)$ forms a metric space.*

PROOF: The proof of this statement follows a similar approach to the proof of Theorem 5.3. $\square$

THEOREM 5.8. *By referring to Remark 3.11, it can be observed that the operation $*$ is continuous.*

PROOF: Consider $(q, v), (f, y) \in \mathcal{L}^2$. For any $\varepsilon > 0$, if $d_{\xi}^*((q, v), (f, y)) < \frac{\varepsilon}{2}$, then it follows that $\max \{d_{\xi}(q, f), d_{\xi}(v, y)\} < \frac{\varepsilon}{2}$. Consequently, we have $d_{\xi}(q, f), d_{\xi}(v, y) < \frac{\varepsilon}{2}$. By utilizing Proposition 4.2(v), (vi), and (vii), we can deduce that

$$d_{\xi}(q*v, f*y) \leq d_{\xi}(q*v, f*v) + d_{\xi}(f*v, f*y) \leq d_{\xi}(q, f) + d_{\xi}(v, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As a result, we can conclude that the operation $*$ is continuous. $\square$

COROLLARY 5.9. *If $\tau_{d_{\xi}}$ is the topology induced by d_{ξ} , then $(\mathcal{L}, *, \tau_{d_{\xi}})$ is a topological L -algebra.*

PROOF: By Theorem 5.2, $(\mathcal{L}, d_{\xi})$ is a pseudo-metric space. Let $q * v \in N_{\varepsilon}(q * v)$. We claim that $N_{\frac{\varepsilon}{2}}(q) * N_{\frac{\varepsilon}{2}}(v) \subseteq N_{\varepsilon}(q * v)$. Assume $b \in N_{\frac{\varepsilon}{2}}(q) * N_{\frac{\varepsilon}{2}}(v)$. Then there exist $w \in N_{\frac{\varepsilon}{2}}(q)$ and $z \in N_{\frac{\varepsilon}{2}}(v)$ such that $b = w * z$. Hence, $d_{\xi}(q, w) \leq \frac{\varepsilon}{2}$ and $d_{\xi}(v, z) \leq \frac{\varepsilon}{2}$. By Proposition 4.2(v) and (vi) we have $d_{\xi}(q*v, w*v) \leq d_{\xi}(q, w)$ and $d_{\xi}(w*v, w*z) \leq d_{\xi}(v, z)$. By Proposition 4.2(vii),

$$d_\xi(q * v, w * z) \leq d_\xi(q * v, w * v) + d_\xi(w * v, w * z) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $b = w * z \in N_\varepsilon(q * v)$. Therefore, $(\mathcal{L}, *, \tau_{d_\xi})$ is a topological L -algebra.

□

THEOREM 5.10. *We can define a function $d^* : \frac{\mathcal{L}}{\equiv_\xi} \times \frac{\mathcal{L}}{\equiv_\xi} \rightarrow \mathbb{R}$, where for any $[q]_\xi, [v]_\xi \in \frac{\mathcal{L}}{\equiv_\xi}$, we have $d^*([q]_\xi, [v]_\xi) = d_\xi(q, v)$. Then, if ξ is a 1-valuation map, the space $(\frac{\mathcal{L}}{\equiv_\xi}, d^*)$ forms a metric space.*

PROOF: By Theorem 5.7, the proof becomes clear. □

PROPOSITION 5.11. If $(\mathcal{L}, d_\xi)$ is a metric space, then ξ is a 1-valuation map.

PROOF: By Remark 4.1, $\xi(q) = d_\xi(1, q)$, so if d_ξ is a metric, we have $\xi(q) = 0$ if and only if $d_\xi(1, q) = 0$ if and only if $q = 1$. Hence, ξ is a 1-valuation map. □

Note. If $(\mathcal{L}, d)$ is a pseudo-metric space, then

- (i) The set $U \subseteq \mathcal{L}$ is open in $(\mathcal{L}, d_\xi)$ if for each $q \in U$, there exists $\varepsilon > 0$ such that $N_\varepsilon(q) \subseteq U$,
- (ii) The topology τ_{d_ξ} induced by d_ξ is the collection of all open sets in $(\mathcal{L}, d_\xi)$.

PROPOSITION 5.12. A valuation ξ on the topological L -algebra $(\mathcal{L}, \tau_{d_\xi})$ is continuous if and only if for each $\varepsilon > 0$, there exists a neighborhood U of 1 such that $\xi(b) < \varepsilon$, for each $b \in U$.

PROOF: Let $q \in \mathcal{L}$ and ε be a positive number. Consider a neighborhood U of 1 such that $\xi(b) < \varepsilon$ for every $b \in U$. By property (L1), we have $q * q = 1$. Therefore, there exist open neighborhoods V and W of q such that $V * W \subseteq U$. Put $F = V \cap W$. For every $v \in F$, we have $q * v, v * q \in F * F \subseteq U$, which implies $\xi(q * v), \xi(v * q) < \varepsilon$. Thus, by Proposition 4.2(viii), we have $|\xi(v) - \xi(q)| < \varepsilon$. This shows that ξ is continuous.

Conversely, if ξ is continuous on $\mathcal{L}$, then it is continuous in 1. Let ε be a positive number. Since $\xi(1) = 0$, $(-\varepsilon, \varepsilon)$ is an open neighborhood of $\xi(1)$ in $\mathbb{R}$. There exists an open neighborhood U of 1 in $\mathcal{L}$ such that $\xi(U) \subseteq (-\varepsilon, \varepsilon)$. Therefore, $\xi(b) < \varepsilon$ for every $b \in U$.

Short proof: If $q \in N_\varepsilon(q)$, then $\xi(q) < \varepsilon$, so ξ should always be continuous w.r.t. d_ξ . $\square$

A function between two metric spaces is called an *isometry* if it preserves distances. Let $\xi_{\mathcal{L}} \in \mathcal{V}(\mathcal{L})$ and $\xi_{\mathcal{K}} \in \mathcal{V}(\mathcal{K})$, where $\mathcal{K}$ is an L -algebra. Then, a function $f : \mathcal{L} \rightarrow \mathcal{K}$ is called valuation preserving if $\xi_{\mathcal{K}} \circ f = \xi_{\mathcal{L}}$.

PROPOSITION 5.13. Consider $\xi_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{R}$ and $\xi_{\mathcal{K}} : \mathcal{K} \rightarrow \mathbb{R}$ as valuations, where $\mathcal{K}$ is an L -algebra. If $f : \mathcal{L} \rightarrow \mathcal{K}$ is an L -homomorphism, then the following statements are equivalent:

- (i) f is valuation preserving,
- (ii) f is an isometry.

PROOF: Assuming that f is valuation preserving, we have $\xi_{\mathcal{K}}(f(q)) = \xi_{\mathcal{L}}(q)$ for any $q \in \mathcal{L}$. Furthermore, for any $q, v \in \mathcal{L}$, we have the following:

$$\begin{aligned} d_{\xi_{\mathcal{K}}}(f(q), f(v)) &= \xi_{\mathcal{K}}(f(q) * f(v)) + \xi_{\mathcal{K}}(f(v) * f(q)) \\ &= \xi_{\mathcal{K}}(f(q * v)) + \xi_{\mathcal{K}}(f(v * q)) \\ &= \xi_{\mathcal{K}} \circ f(q * v) + \xi_{\mathcal{K}} \circ f(v * q) \\ &= \xi_{\mathcal{L}}(q * v) + \xi_{\mathcal{L}}(v * q) \\ &= d_{\xi_{\mathcal{L}}}(q, v). \end{aligned}$$

Hence, it follows that f is an isometry. Conversely, if f is an isometry, then for any $q \in \mathcal{L}$, we have

$$\xi_{\mathcal{L}}(q) = d_{\xi_{\mathcal{L}}}(q, 1) = d_{\xi_{\mathcal{K}}}(f(q), f(1)) = \xi_{\mathcal{K}}(f(q)) + \xi_{\mathcal{K}}(f(1)) = \xi_{\mathcal{K}}(f(q)).$$

Therefore, f is valuation preserving. $\square$

PROPOSITION 5.14. Consider $\xi \in \mathcal{V}(\mathcal{L})$, $X \subseteq \mathcal{L}$ is closed, and $q \in \mathcal{L}$. If there exists $v \in X$ such that $q \equiv_{\ker(\xi)} v$, then it follows that $q \in X$. Conversely, if $\ker(\xi)$ is a neighborhood of 0, the converse is also true.

PROOF: Assume that there exists $v \in X$ such that $q \equiv_{\ker(\xi)} v$. This implies that $\xi(v * q) = \xi(q * v) = 0$. Consequently, for any $\varepsilon > 0$, we have $\xi(q * v) + \xi(v * q) < \varepsilon$. Therefore, for each $\varepsilon > 0$, the set $N_\varepsilon(q) \cap X$ is non-empty, which implies $q \in X$.

Conversely, let $\ker(\xi)$ be a neighborhood of 1 and $q \in X$. There exists a sequence $\{q_n\}$ in X such that $q_n \mapsto q$. Since $*$ is continuous, it follows that $\xi(q * q_n) \mapsto 0$ and $\xi(q_n * q) \mapsto 0$. Hence, there exists a positive integer n_0 such that $q_{n_0} * q \in \ker(\xi)$ and $q * q_{n_0} \in \ker(\xi)$. Therefore, $q_{n_0} \equiv_{\ker(\xi)} q$. $\square$

THEOREM 5.15. Let $\xi \in \mathcal{V}(\mathcal{L})$. We define the relation $\approx$ as follows:

$$\{q_n\} \approx \{v_n\} \Leftrightarrow d_\xi(q_n, v_n) \mapsto 0,$$

for all sequences $\{q_n\}$ and $\{v_n\}$ in $\mathcal{L}$. We will show that $\approx$ is a congruence relation on the set of all sequences in $\mathcal{L}$.

PROOF: Obviously, by Proposition 4.2(i), (iii) and (iv), $\approx$ is an equivalence relation on $\mathcal{L}$. Let $\{q_n\} \approx \{v_n\}$ and $\{f_n\} \approx \{y_n\}$. Then $d_\xi(q_n, v_n) \mapsto 0$ and $d_\xi(f_n, y_n) \mapsto 0$. By Proposition 4.2(v) and (vi), we have $d_\xi(q_n * f_n, v_n * f_n) \mapsto 0$ and $d_\xi(v_n * f_n, v_n * y_n) \mapsto 0$. By Proposition 4.2(iv), we have $d_\xi(q_n * v_n, f_n * y_n) \mapsto 0$ and so $\{q_n\} * \{v_n\} \approx \{f_n\} * \{y_n\}$. Therefore, $\approx$ is a congruence relation on the set of all sequences in $\mathcal{L}$. $\square$

Note. Let ξ be an element of $\mathcal{V}(\mathcal{L})$. The set of all equivalence classes, denoted by $\widetilde{\{q_n\}}$, is defined as $\{\{v_n\} \mid \{v_n\} \approx \{q_n\}\}$. Set $\widetilde{\mathcal{L}} = \{\widetilde{\{q_n\}} \mid \{q_n\} \text{ is a sequences in } \mathcal{L}\}$. By Theorem 5.15, it is clear that $(\widetilde{\mathcal{L}}, *, \widetilde{\{1\}})$ forms an L -algebra.

PROPOSITION 5.16. Let ξ be a 1-valuation on $\mathcal{L}$. Then, the set $(\widetilde{\mathcal{L}}, *, \widetilde{\{1\}})$ forms an L -algebra, and the set $(\widetilde{\mathcal{L}}, \widetilde{d}_\xi)$ forms a metric space. The pseudo-metric d_ξ induces a metric $\widetilde{d}_\xi$ on $\mathcal{L}$ in the following way:

$$\widetilde{d}_\xi(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) = \lim_n d_\xi(q_n, v_n),$$

for all $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$.

PROOF: It is evident that $\widetilde{d}_\xi$ is a pseudo-metric on $\widetilde{\mathcal{L}}$. For any $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$, if $d_\xi(\{q_n\}, \{v_n\}) = 0$, then it implies that $d_\xi(q_n, v_n) \rightarrow 0$. Consequently, $\{q_n\} \approx \{v_n\}$, which implies $\widetilde{\{q_n\}} = \widetilde{\{v_n\}}$. Hence, we can conclude that $(\widetilde{\mathcal{L}}, \widetilde{d}_\xi)$ forms a metric space. $\square$

PROPOSITION 5.17. Let $\xi \in \mathcal{V}(\mathcal{L})$. Then the mapping $\pi_\xi : \mathcal{L} \rightarrow \widetilde{\mathcal{L}}$ defined by $\pi_\xi(q) = \widetilde{q}$, where $\widetilde{q}$ is the equivalence class of the constant sequence with any element equal to q , is a homomorphism.

PROPOSITION 5.18. Consider $\xi \in \mathcal{V}(\mathcal{L})$. Then $\widetilde{\xi} \in \mathcal{V}(\widetilde{\mathcal{L}})$ where $\widetilde{\xi} : \widetilde{\mathcal{L}} \rightarrow \mathbb{R}$ by $\widetilde{\xi}(\widetilde{\{q_n\}}) = \lim_n \xi(q_n)$ for each sequence in $\mathcal{L}$.

PROOF: It is evident that $\widetilde{\xi}(\widetilde{\{1\}}) = 0$. Now, let $\{q_n\}$ and $\{v_n\}$ be sequences in $\mathcal{L}$. Since both ξ and $*$ are continuous, we can conclude that,

$$\begin{aligned} \widetilde{\xi}(\widetilde{\{v_n\}}) - \widetilde{\xi}(\widetilde{\{q_n\}}) &= \lim_n \xi(v_n) - \lim_n \xi(q_n) = \lim_n \xi(v_n) - \xi(q_n) \\ &\leq \lim_n \xi(q_n * v_n) = \widetilde{\xi}(\widetilde{\{q_n\} * \{v_n\}}). \end{aligned}$$

Hence $\widetilde{\xi}$ is a valuation on $\widetilde{\mathcal{L}}$. $\square$

Note. A sequence $\{q_n\} \subseteq \mathcal{L}$ is said to be d_ξ -Cauchy if it is a Cauchy sequence with respect to the pseudo-metric d_ξ on $(\mathcal{L}, d_\xi)$. The space $(\mathcal{L}, d_\xi)$ is said to be d_ξ -complete if every d_ξ -Cauchy sequence converges to an element in $\mathcal{L}$. Now, let $\{q_n\}$ and $\{v_n\}$ be two d_ξ -Cauchy sequences. It follows that the sequence $\{d_\xi(q_n, v_n)\}$ is convergent, as it is a Cauchy sequence in the real numbers $\mathbb{R}$.

COROLLARY 5.19. The metric space $(\widetilde{\mathcal{L}}, \widetilde{d}_\xi)$ is $\widetilde{d}_\xi$ -complete.

PROPOSITION 5.20. If $\widetilde{\mathcal{L}}, \widetilde{\xi}, \pi_\xi$ and d are defined as described above, then the following properties hold:

- (i) $\widetilde{\xi} \circ \pi_\xi = \xi$, which implies that π_ξ is 1-valuation preserving.
- (ii) ξ is a 1-valuation if and only if $\pi_\xi(q) = \widetilde{\{1\}}$ implies that $q = 1$.
- (iii) $\widetilde{d}_\xi = d_{\widetilde{\xi}}$.
- (iv) π_ξ is continuous.

PROOF: (i) For any $q \in \mathcal{L}$, $\widetilde{\xi} \circ \pi_\xi(q) = \widetilde{\xi}(\pi_\xi(q)) = \lim_n \xi(q) = \xi(q)$.

(ii) Let ξ be a 1-valuation and $\pi_\xi(q) = \widetilde{\{1\}}$. Then $\widetilde{\{q\}} = \widetilde{\{1\}}$ and so $\{q\} \approx \{1\}$. Hence, $\xi(q) = d_\xi(q, 1) = 0$. Since ξ is a 1-valuation, $q = 1$. Conversely, if $\xi(q) = 0$ for any $q \in \mathcal{L}$, then $d_\xi(q, 1) = \xi(q) = 0$ and so $\pi_\xi(q) = \widetilde{\{q\}} = \widetilde{\{1\}}$. Hence $q = 1$. Thus ξ is a 1-valuation.

(iii) For any $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$ since ξ and $*$ are continuous, we have

$$\begin{aligned} d_{\widetilde{\xi}}(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) &= \widetilde{\xi}(\widetilde{\{q_n * v_n\}}) + \widetilde{\xi}(\widetilde{\{v_n * q_n\}}) = \lim_n \xi(q_n * v_n) + \lim_n \xi(v_n * q_n) \\ &= \lim_n (\xi(q_n * v_n) + \xi(v_n * q_n)) = \lim_n d_\xi(q_n, v_n) \\ &= d_\xi(\lim_n q_n, \lim_n v_n) = \widetilde{d}_\xi(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}). \end{aligned}$$

(iv) This follows from Proposition 5.13(i) as π_ξ is valuation-preserving and the fact that isometries are always continuous. $\square$

THEOREM 5.21. *Let $\psi \in \mathcal{V}(\mathcal{K})$ such that $(\mathcal{K}, d_\psi)$ is a d_ψ -complete space, where $\mathcal{K}$ is an L -algebra. If $\xi \in \mathcal{V}(\mathcal{L})$ and $\Theta : \mathcal{L} \rightarrow \mathcal{K}$ is a valuation preserving homomorphism, then there exists a unique valuation preserving homomorphism $\widetilde{\Theta} : \widetilde{\mathcal{L}} \rightarrow \mathcal{K}$ such that $\widetilde{\Theta} \circ \pi_\xi = \Theta$.*

PROOF: Suppose $\Theta : \mathcal{L} \rightarrow \mathcal{K}$ is a valuation-preserving homomorphism. By Proposition 5.13, Θ is an isometry. If $\{q_n\}$ is a sequence in $\mathcal{L}$, then $\{\Theta(q_n)\}$ is a sequence in $\mathcal{K}$. Since $\mathcal{K}$ is d_ψ -complete, $\Theta(q_n)$ converges to v for some $v \in \mathcal{K}$. We define $\widetilde{\Theta}(\widetilde{\{q_n\}}) = v$. We will now show that $\widetilde{\Theta}$ is the unique isometry such that $\widetilde{\Theta} \circ \pi_\xi = \Theta$. Let $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$, $\Theta(q_n) \mapsto q$ and $\Theta(v_n) \mapsto v$. Then

$$\begin{aligned} d_{\widetilde{\xi}}(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) &= \widetilde{d}_\xi(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) \\ &= \lim_n \xi(q_n * v_n) + \lim_n \xi(v_n * q_n) \\ &= \lim_n \psi \circ \Theta(q_n * v_n) + \lim_n \psi \circ \Theta(v_n * q_n) \\ &= \lim_n \psi(\Theta(q_n * v_n)) + \lim_n \psi(\Theta(v_n * q_n)) \end{aligned}$$

$$\begin{aligned}
 &= \lim_n \psi(\Theta(q_n) * \Theta(v_n)) + \lim_n \psi(\Theta(v_n) * \Theta(q_n)) \\
 &= \lim_n \psi(q * v) + \lim_n \psi(v * q) \\
 &= \psi(q * v) + \psi(v * q) \\
 &= \psi(\widetilde{\Theta(\{q_n\})} * \widetilde{\Theta(\{v_n\})}) + \psi(\widetilde{\Theta(\{y_n\})} * \widetilde{\Theta(\{q_n\})}) \\
 &= d_\psi(\widetilde{\Theta(\{q_n\})}, \widetilde{\Theta(\{v_n\})}).
 \end{aligned}$$

The uniqueness of $\widetilde{\Theta}$ is evident. Since the operation in $\mathcal{K}$ is continuous with respect to d_ψ , we can conclude that $\widetilde{\Theta}$ is a homomorphism. Since let $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$, such that $\Theta(q_n) \mapsto q$ and $\Theta(v_n) \mapsto v$. From Θ is a homomorphism, we get $\Theta(q_n * v_n) = \Theta(q_n) * \Theta(v_n) \mapsto q * v$. Thus,

$$\begin{aligned}
 \widetilde{\Theta(\{q_n\} * \{v_n\})} &= \widetilde{\Theta(\{q_n * v_n\})} = \widetilde{\Theta} \circ \pi_\xi(q_n * v_n) = \Theta(q_n * v_n) \\
 &= \Theta(q_n) * \Theta(v_n) = \widetilde{\Theta} \circ \pi_\xi(\{q_n\}) * \widetilde{\Theta} \circ \pi_\xi(\{v_n\}) \\
 &= \widetilde{\Theta(\{q_n\})} * \widetilde{\Theta(\{v_n\})}.
 \end{aligned}$$

Lastly, for each $q \in \mathcal{L}$, we have $\widetilde{\Theta} \circ \pi_\xi(q) = \widetilde{\Theta}(\widetilde{\{q\}}) = \Theta(q)$. Therefore, $\widetilde{\Theta} \circ \pi_\xi = \Theta$. $\square$

Note. A general remark, it is noteworthy that $\frac{\mathcal{L}}{\equiv_\xi}$ is the smallest quotient where $\bar{\xi}([q]) = \xi(q)$ is well-defined and defines a 1-valuation. Therefore, factoring out $\equiv_\xi$ provides a canonical L -algebra with a 1-valuation associated with ξ .

6. Conclusion

In this paper, the notion of valuations on L -algebras is defined, and related properties are investigated. Using a valuation map, a congruence relation is defined and proved that this congruence relation is equivalent with a congruence relation made by ideal. Also, it was shown that image and converse image of any valuation is a valuation. In addition, by using valuation,

a metric space is introduced, and it was shown that the operation “ $*$ ” in an L -algebra is uniformly continuous.

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PRE-HILBERT ALGEBRAS WITH CONDITION (GE3)

Abstract

GE-algebras are introduced as a generalization of Hilbert algebras in 2020. In this paper, we show that a self distributive GE-algebra is equivalent to the pimpl-RML algebra, and a transitive GE algebra is equivalent to the pi-pre-BBBCC algebra with condition (An). Further, we prove that pimpl-BE algebras are a subclass of pimpl-RML algebras and pre-Hilbert algebras with condition (GE3) are equivalent to the pi-pre-BBBCC algebras with condition (An).

Keywords: RML algebra, pi-RML algebra, pre-BBBCC algebra, pi-pre-BBBCC algebra, BCK-algebra, (self distributive) BE-algebra, (transitive) GE-algebra.

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1. Introduction

The study of logical algebras has been a rich area for expanding and connecting various algebraic structures that model logical systems. Among the most fundamental of these are **BCI** and **BCK-algebras**, introduced by Iséki [5], Iséki and Tanaka [6], which abstract the implicative fragment of propositional logic. Mundici [10] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, while Meng [8] established that implicative commutative semigroups are equivalent to a class of BCK-algebras. These algebras are characterized by a binary operation $\rightarrow$ and a constant 1, satisfying specific axioms related to implication [2, 9]. This foundation has inspired a wave of generalizations and variations, leading to a diverse properties of related algebraic structures. A key development in this effort was the introduction of **BE-algebras** by Kim and Kim [7], which weaken certain axioms of BCK-algebras. These algebras, defined by the axioms (Re), (M), (L), and the exchange property (Ex), provided a broader framework for studying algebraic logic. Rezaei et al. [11] proved that every Hilbert algebra is a self distributive BE-algebra and commutative self distributive BE-algebra is a Hilbert algebra. A comprehensive effort to systematize and further generalize these concepts was undertaken by Iorgulescu [3, 4], who proposed an extensive taxonomy of algebras including **RML**, **pre-BBBCC**, and their “pi” variants by meticulously combining subsets of fundamental properties like antisymmetry (An), transitivity (Tr), exchange (Ex), and self-distributivity. More recently, new structures have emerged. **GE-algebras** were defined by Bandaru et al. [1] via the identity (GE3): $x \rightarrow (y \rightarrow z) = x \rightarrow [y \rightarrow (x \rightarrow z)]$, offering a new perspective on the interaction between implication and self-applicability. Recently, Walendziak [13] defined **pre-Hilbert algebras** by weakening the axioms of Hilbert algebras, a well-known algebraic counterpart to Henkin’s Positive Implicative Logic. While these definitions appear distinct, a natural question arises: how are these various classes of algebras BCI, BCK, BE, RML, GE, pre-Hilbert, and their pi and self-distributive variants related? The literature contains scattered results, but a unified hierarchical overview has been lacking. This paper aims to synthesize these

connections by:

- Clarifying the precise relationships between these algebras using their defining properties.
- Proving new equivalences, such as between self-distributive GE-algebras and pimpl-RML algebras, and between transitive GE-algebras and pi-pre-BBCC algebras with condition (An).
- Providing counterexamples to show where these relationships break down, establishing the properness of inclusions.
- Presenting a comprehensive hierarchy that situates these algebraic structures within a single framework, illuminating the logical landscape they form.

2. Preliminaries

Recall the list of properties:

$$(An) \text{ (**Antisymmetry**) } x \rightarrow y = 1 \text{ and } y \rightarrow x = 1 \Rightarrow x = y,$$

$$(B) (y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1,$$

$$(BB) (y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1,$$

$$(*) y \rightarrow z = 1 \Rightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) = 1,$$

$$(**) y \rightarrow z = 1 \Rightarrow (z \rightarrow x) \rightarrow (y \rightarrow x) = 1,$$

$$(C) [x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow z)] = 1,$$

$$(D) y \rightarrow [(y \rightarrow x) \rightarrow x] = 1,$$

$$(Ex) \text{ (**Exchange**) } x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$$

$$(K) x \rightarrow (y \rightarrow x) = 1,$$

$$(L) \text{ (**Last element**) } x \rightarrow 1 = 1,$$

$$(M) \quad 1 \rightarrow x = x,$$

$$(N) \quad 1 \rightarrow x = 1 \quad \Rightarrow \quad x = 1,$$

$$(Re) \quad (\mathbf{Reflexivity}) \quad x \rightarrow x = 1,$$

$$(Tr) \quad (\mathbf{Transitivity}) \quad x \rightarrow y = 1 \quad \text{and} \quad y \rightarrow z = 1 \quad \Rightarrow \quad x \rightarrow z = 1,$$

$$(P) \quad x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z),$$

$$(pimpl) \quad x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z),$$

$$(pi) \quad y \rightarrow (y \rightarrow x) = y \rightarrow x.$$

A proper RML algebra is a RML algebra ([3, P8]) (i.e. verifying (Re), (M), (L)) not verifying (Ex), (An), (B), (*), (BB), (**), (Tr), (pi).

A proper pi-RML algebra is a pi-RML algebra ([3, P8-pi]) (i.e. verifying (Re), (M), (L), (pi)) not verifying (Ex), (An), (B), (*), (BB), (**), (Tr), (pimpl).

A proper BCI algebra is a BCI algebra ([2, Def. 1.1.1] and [3, PO1]) (i.e. verifying (BB), (D), (Re), (N), (An) or, equivalently, (BB), (D), (Re), (An), or, equivalently, (BB), (M), (An), or, equivalently, (B), (C), (Re), (An)) not verifying (L).

A proper BCK algebra is a BCK algebra ([9, Def. 1.1] and [3, PO2]) (i.e. verifying (BB), (D), (Re), (L), (An), or, equivalently, (BB), (M), (L), (An), or, equivalently, (B), (C), (K), (An)) not verifying (pi) (hence (pimpl)).

A proper BE algebra is a BE algebra ([7, Def. 1] and [3, PO6]) (i.e. verifying (Re), (M), (L), (Ex) (hence (D)) not verifying (An), (Tr), (*) (hence (B)), (**), (hence (BB)), (pi).

A proper pi-BE algebra is a pi-BE algebra ([3, PO6-pi]) (i.e. verifying (Re), (M), (L), (Ex) (hence (D), (pi)) not verifying (An), (B), (BB), (*), (**), (Tr), (pimpl).

A proper pre-BBBCC algebra is a pre-BBBCC algebra ([3, P23]) (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D)) not verifying (An), (Ex), (pi).

A proper pi-pre-BBBCC algebra is a pi-pre-BBBCC algebra ([3, P23-pi]) (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D), (pi)) not verifying (An), (Ex), (pimpl).

3. Pre-Hilbert and GE Algebras

In 2021, Bandaru et al. ([1]) defined a new algebraic structure was named GE-algebra as follows:

A GE-algebra ([1, Def. 3.1]) is a non-empty set X with a constant 1 and a binary operation “ $\rightarrow$ ” satisfying axioms:

$$(GE1) \quad x \rightarrow x = 1,$$

$$(GE2) \quad 1 \rightarrow x = x,$$

$$(GE3) \quad x \rightarrow (y \rightarrow z) = x \rightarrow [y \rightarrow (x \rightarrow z)], \text{ for all } x, y, z \in X.$$

(GE1) used instead of the property (Re) and respectively, (GE2) instead of the property (M).

In 2024, Walendziak defined the notation of pre-Hilbert algebras as a generalization of Hilbert algebras. A pre-Hilbert algebra ([13, Def. 3.5]) is an algebra $(X, \rightarrow, 1)$ of type $(2, 0)$ verifying (M), (K) and (P).

Remark 3.1. It is shown that in a given GE-algebra the properties (L) and (pi) hold ([1, Th. 3.3(1)-(2)]). Hence every GE-algebra is a RML algebra. The following example shows that the converse does not hold in general.

Example 3.2. Let $X = \{1, a, b\}$. Consider the Table 1.

Table 1: The $\rightarrow$ operation defining RML algebra $(X; \rightarrow, 1)$.

$\rightarrow$	1	a	b
1	1	a	b
a	1	1	a
b	1	b	1

Then $(X; \rightarrow, 1)$ is a RML algebra, but neither a GE-algebra nor self-distributive. Indeed,

$$\begin{aligned}
a \rightarrow (b \rightarrow a) &= a \rightarrow b = a \neq 1 \\
&= a \rightarrow 1 \\
&= a \rightarrow (b \rightarrow 1) \\
&= a \rightarrow [b \rightarrow (a \rightarrow a)], \quad (3.1)
\end{aligned}$$

and

$$a \rightarrow (b \rightarrow a) = a \rightarrow b = a \neq 1 = a \rightarrow 1 = (a \rightarrow b) \rightarrow (a \rightarrow a).$$

Further, every GE-algebra is a pi-RML algebra ([12, Pro. 3.3]), and every self distributive GE-algebra is an RML algebra with condition (pimpl), (briefly, pimpl-RML algebra). The following example shows that there is a pi-RML algebra which is not a GE-algebras.

Example 3.3. Let $X = \{1, a, b, c\}$. Consider the Table 2.

Table 2: The $\rightarrow$ operation defining pi-RML algebra $(X; \rightarrow, 1)$.

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	c	1	c
c	1	1	1	1

Then $(X; \rightarrow, 1)$ is a pi-RML algebra, but not GE-algebra, since

$$\begin{aligned}
a \rightarrow (b \rightarrow a) &= a \rightarrow c = c \neq 1 \\
&= a \rightarrow 1 \\
&= a \rightarrow (b \rightarrow 1) \\
&= a \rightarrow [b \rightarrow (a \rightarrow a)]. \quad (3.2)
\end{aligned}$$

PROPOSITION 3.4. Every pimpl-RML algebra is a GE-algebra.

PROOF: Let $(X, \rightarrow, 1)$ be a pimpl-RML algebra. It is sufficient to prove

(GE3). Given $x, y, z \in X$, using three times (pimpl), (R), (L) and (M), we get

$$\begin{aligned}
 x \rightarrow [y \rightarrow (x \rightarrow z)] &= x \rightarrow [(y \rightarrow x) \rightarrow (y \rightarrow z)] \\
 &= [x \rightarrow (y \rightarrow x)] \rightarrow [x \rightarrow (y \rightarrow z)] \\
 &= [(x \rightarrow y) \rightarrow (x \rightarrow x)] \rightarrow [x \rightarrow (y \rightarrow z)] \\
 &= [(x \rightarrow y) \rightarrow 1] \rightarrow [x \rightarrow (y \rightarrow z)] \\
 &= 1 \rightarrow [x \rightarrow (y \rightarrow z)] \\
 &= x \rightarrow (y \rightarrow z).
 \end{aligned} \tag{3.3}$$

Thus, (GE3) is valid. $\square$

The converse of Proposition 3.4 is not generally true as shown in the following example.

Example 3.5. Let $X = \{1, a, b, c\}$. Consider the Table 3.

Table 3: The $\rightarrow$ operation defining GE-algebra $(X; \rightarrow, 1)$.

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	1
c	1	1	1	1

Then $(X; \rightarrow, 1)$ is a GE-algebra, but not a pimpl-RML algebra, because (pimpl) does not holds. Indeed,

$$a \rightarrow (b \rightarrow c) = a \rightarrow 1 = 1 \neq c = 1 \rightarrow c = (a \rightarrow b) \rightarrow (a \rightarrow c).$$

The following corollary is a direct consequence of Remark 3.1 and Proposition 3.4.

COROLLARY 3.6. The notions of a self distributive GE-algebra and a pimpl-RML algebra are equivalent.

Remark 3.7. From Example 3.5 we conclude that the condition self distributivity in Corollary 3.6 is necessary, and so we can not remove it.

COROLLARY 3.8. Every pimpl-BE algebra (self distributive BE-algebra) is a pimpl-RML algebra.

PROOF: By applying [1, Th. 3.13] and Corollary 3.6. $\square$

A GE-algebra $(X, \rightarrow, 1)$ ([1, Def. 3.5]) is said to be *transitive* if it satisfies

$$(TGE) \quad (x \rightarrow y) \rightarrow [(z \rightarrow x) \rightarrow (z \rightarrow y)] = 1,$$

for all $x, y, z \in X$.

Note that the property of (TGE) is just the well known property (B). From [12, Re. 2.9 and Ex. 2.10], we can see that every transitive GE-algebra do not have to verify (pimpl).

PROPOSITION 3.9. Every pi-pre-BBBCC algebra with condition (An) is a transitive GE-algebra.

PROOF: From [3, Th. 2.4] it follows that $(Ex) \Leftrightarrow (BB)$. By [3, Pro. 3.6] we deduce that also satisfies (pimpl). Hence by a similar argument of Proposition 3.4 the proof is obvious. $\square$

PROPOSITION 3.10. Every transitive GE-algebra is a pi-pre-BBBCC algebra.

PROOF: Using [1, Th. 3.4] and Remark 3.1, we can see that (BB) and (pi) hold. $\square$

The following corollary is a direct consequence of Propositions 3.9 and 3.10.

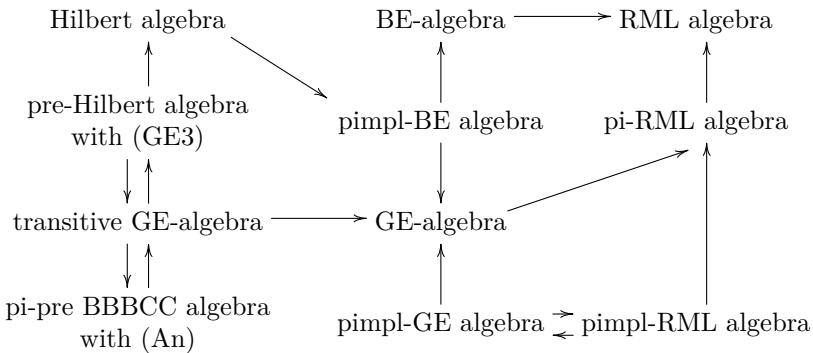
COROLLARY 3.11. The notions of a transitive GE-algebra and a pi-pre-BBBCC algebra with condition (An) are equivalent.

From [12, Th. 3.6], every transitive GE-algebra is a pre-Hilbert algebra with condition (GE3) and vice versa, and so by Corollary 3.11, the notions of a pi-pre-BBBCC algebra with condition (An) and a pre-Hilbert algebra with condition (GE3) are equivalent.

4. Conclusions and Summary of Relations

- GE-algebras are a proper subclass of pi-RML algebras (as shown by Example 3.3).
- pimpl-GE algebras are a proper subclass of GE-algebras (as shown by Example 3.5) and are equivalent to pimpl-RML algebras.
- Transitive GE-algebras are a subclass of GE-algebras and are equivalent to pi-pre-BBBCC algebras with condition (An).
- pimpl-BE algebras are a subclass of pimpl-RML algebras.
- Pre-Hilbert algebras with condition (GE3) are equivalent to pi-pre-BBBCC algebras with condition (An).

Now, we can draw the following diagram:



The diagram clearly shows that pi-pre-BBBCC algebra with condition (An) leads to pre-Hilbert algebra which then leads to transitive GE-algebra, and also connects to pre-Hilbert algebra with condition (GE).

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Patryk Michalczenia 

A NOTE ON THE NUMBER OF SUBNORMAL MODAL LOGICS

Abstract

In the article we discuss the notion of a subnormal modal logic and, in particular, the notion of an intermediate subnormal modal logic. The goal of the article is to determine the cardinality of the set of all intermediate subnormal modal logics. Thus we define a set of certain formulas, such that each subset of the defined set determines a different intermediate subnormal modal logic. As we demonstrate, since the set in question is countable, the set of all intermediate subnormal modal logics has cardinality of the continuum.

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1. Preliminaries

As it is usually defined [1], a normal modal logic is one which contains all formulas of the form $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ (the *K*-schema) and is closed under the necessitation rule (*RN*: $\frac{\alpha}{\Box\alpha}$).¹ The weakest normal modal logic is usually denoted by **K**. Hence, a logic is *non-normal* when it either rejects at least some instances of the *K*-schema, or restricts (or rejects) the scope of *RN* (or a little bit of both). Non-normal logics have been extensively studied in, among others, [3, 4, 5, 7, 8, 9, 11, 12]. Among non-normal modal logics we can distinguish those which are closed under *RN* but do not contain the *K*-schema. This is the meaning of the word ‘subnormal’ adopted in [2], where the weakest (classical) subnormal modal logic is studied, that is the system **N** – first introduced in [6], which is simply a classical propositional logic (in the language with $\Box$) extended with *RN*.

Such notion of a subnormal modal logic allows for logics which are not necessarily ‘under’ the system **K**. For instance, a logic determined by *PC*, *RN* and the schema $\Box\alpha \rightarrow \alpha$, which was studied in [10], is neither weaker, nor stronger than **K**, it is incomparable to **K**. Hence, we can further distinguish certain subnormal modal logics. Having **N** and **K** before our eyes, it seems natural to consider those logics which are strictly stronger than **N** and strictly weaker than **K**, that is – intermediate logics.

In the article, we want to answer the question: how many logics intermediate between **N** and **K** are there? Our strategy is as follows. Given a standard modal language $\mathcal{L}$ and a standard definition of a formula, we know there are no more than continuum-many sets of formulas of any kind, and we will treat logics as special sets of formulas. In order to show that the number of all intermediate subnormal modal logics is continuum, we will distinguish a countable set of certain theorems of **K** which are not theorems of **N**, and show that each subset of the distinguished set determines a different logic. Since there are continuum-many subsets of a countable

¹Here we will only speak of classical modal logics, that is logics containing *PC* – the set of all classical propositional tautologies, and closed under *Modus Ponens*.

set, it will follow that there are continuum-many intermediate subnormal modal logics.²

We will adopt a standard definition of a formula. However, since our investigations will depend on the syntactic differences between formulas, we will not omit parentheses or simplify the notation in a way that would obscure our intentions.

DEFINITION 1.1. Let $Var := \{p_0, p_1, p_2, \dots\}$ be the set of denumerably many propositional variables. We take standard connectives $\mathcal{C} = \{\neg, \wedge, \vee, \rightarrow, \Box\}$, their syntactic function is as usual. The language $\mathcal{L}$ is defined as $\mathcal{L} = (Var, \mathcal{C})$. By Fm we denote the set of all (well-formed) formulas of $\mathcal{L}$, defined as follows:

1. $Var \subseteq Fm$,
2. if $\alpha, \beta \in Fm$, then $\neg\alpha, \Box\alpha, (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta) \in Fm$,
3. nothing more is in Fm .

We will use $(\alpha \leftrightarrow \beta)$ as an abbreviation of $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$.

Given the requirement that a logic must be closed under RN , it might seem natural to think of subnormal modal logics as not containing the K -schema by definition. However, since we are interested only in those logics which do not contain the K -schema, it is unnecessary to require from logic not to contain the K -schema. Given RN , the difference between normal and subnormal modal logics lies in *not* requiring subnormal modal logics to contain the K -schema, in opposition to such a requirement for normal modal logics. This gives us a more general definition of a subnormal modal logic.

Before we formulate the definition, we need to mention substitutions. As usual, a substitution is a mapping $s: Var \rightarrow Fm$ extendable to a function $\bar{s}: Fm \rightarrow Fm$ in a standard way, so that $\bar{s}((\alpha \circ \beta)) = (\bar{s}(\alpha) \circ \bar{s}(\beta))$ for all $\circ \in \{\wedge, \vee, \rightarrow\}$, and $\bar{s}(\Box\alpha) = \Box\bar{s}(\alpha)$ for $\Box \in \{\neg, \Box\}$. Since each substitution s uniquely determines $\bar{s}$, we will write s instead of $\bar{s}$. The set

² This result has also been proved by the author of this article in [7] using a different kind of semantics.

of all substitutions will be denoted by ‘*Sub*’. Moreover, for any $X \subseteq Fm$, we define $Sub(X) := \{s(\alpha) : \alpha \in X \text{ and } s \in Sub\}$.

DEFINITION 1.2. Let $X \subseteq Fm$. A subnormal modal logic determined by X , denoted by $L(X)$, is the smallest set of formulas satisfying the following conditions:

1. $PC \subseteq L(X)$,
2. $Sub(X) \subseteq L(X)$,
3. if $\alpha \in L(X)$ and $(\alpha \rightarrow \beta) \in L(X)$, then $\beta \in L(X)$,
4. if $\alpha \in L(X)$, then $\Box\alpha \in L(X)$.

Formulas from $Sub(X)$ are called *axioms of $L(X)$* . A special case of subnormal modal logics are **K** and **N**. We have:

$$\mathbf{K} = L(\{(\Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1))\}) \text{ and } \mathbf{N} = L(\emptyset).$$

DEFINITION 1.3. A subnormal modal logic L is called *intermediate* iff L is a proper subset of **K**.³

It follows that an intermediate subnormal modal logic cannot contain the schema $(\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta))$, for then it would be identical with **K**.

2. Semantics

We are going to use semantics introduced by Fitting, Marek and Truszczyński in [2]. The key idea behind this kind of semantics is to be able to distinguish logically equivalent formulas under the scope of $\Box$ without invalidating the necessitation rule, since **N** is not closed under the extensionality rule $\left(\frac{\alpha \leftrightarrow \beta}{\Box\alpha \leftrightarrow \Box\beta} \right)$ (see [7], p. 76). It is achieved by associating a

³Although it seems natural to exclude **N** from intermediate logics, we let it slide, since we do not need to exclude the empty set of axioms in further proofs for them to work.

binary relation on possible worlds with *each* formula. So now $\Box\gamma$ is true at a possible world x when α is true at all worlds α -accessible from x . And even though, for instance, $p \vee q$ is logically equivalent with $\neg(\neg p \wedge \neg q)$, $\Box(p \vee q)$ and $\Box\neg(\neg p \wedge \neg q)$ are *not* logically equivalent, since $p \vee q$ may be true at all $p \vee q$ -accessible worlds from x , while $\neg(\neg p \wedge \neg q)$ may fail at some $\neg(\neg p \wedge \neg q)$ -accessible world from x . Even logically equivalent formulas display different semantic behaviour, since the semantics takes into account the very *shape* of a formula as a factor in defining truth-conditions for $\Box$. For more philosophy behind the semantics for subnormal modal logic see [7], Part I.

DEFINITION 2.1. A model is a structure $\mathbb{M} = (M, \{R_\alpha\}_{\alpha \in Fm}, e)$, where M is a non-empty set (of possible worlds), each R_α is a binary relation on M , and e is a function (valuation) from $M \times Var$ to $\{0, 1\}$. Truth (satisfaction) of a formula at a given possible world – symbolically: $(\mathbb{M}, x) \Vdash \alpha$ – is defined as follows.

- (a) $(\mathbb{M}, x) \Vdash p_i \iff e(x, p_i) = 1$, for $p_i \in Var$,
- (b) $(\mathbb{M}, x) \Vdash \neg\alpha \iff (\mathbb{M}, x) \nVdash \alpha$,
- (c) $(\mathbb{M}, x) \Vdash (\alpha \wedge \beta) \iff (\mathbb{M}, x) \Vdash \alpha$ and $(\mathbb{M}, x) \Vdash \beta$,
- (d) $(\mathbb{M}, x) \Vdash (\alpha \vee \beta) \iff (\mathbb{M}, x) \Vdash \alpha$ or $(\mathbb{M}, x) \Vdash \beta$,
- (e) $(\mathbb{M}, x) \Vdash (\alpha \rightarrow \beta) \iff (\mathbb{M}, x) \nVdash \alpha$ or $(\mathbb{M}, x) \Vdash \beta$,
- (f) $(\mathbb{M}, x) \Vdash \Box\alpha \iff (\forall y \in M)(xR_\alpha y \Rightarrow (\mathbb{M}, y) \Vdash \alpha)$.

For $Y \subseteq Fm$, we write $(\mathbb{M}, x) \Vdash Y$ to mean that $(\mathbb{M}, x) \Vdash \delta$ for all $\delta \in Y$. We also write $\mathbb{M} \Vdash \alpha$ when $(\mathbb{M}, x) \Vdash \alpha$ for all $x \in M$.

It follows that the necessitation rule preserves satisfaction in a model.

LEMMA 2.2. *If $\mathbb{M} \Vdash \alpha$, then $\mathbb{M} \Vdash \Box\alpha$.*

PROOF: Suppose $\mathbb{M} \nVdash \Box\alpha$. Then $(\mathbb{M}, b) \nVdash \Box\alpha$ for some $b \in M$. Hence there is $c \in M$ such that $bR_\alpha c$ and $(\mathbb{M}, c) \nVdash \alpha$. So there is $c \in M$ such that $(\mathbb{M}, c) \nVdash \alpha$. Therefore $\mathbb{M} \nVdash \alpha$. $\square$

We also state the following lemma as a formality.

LEMMA 2.3. *If $\mathbb{M} \Vdash \alpha$ and $\mathbb{M} \Vdash (\alpha \rightarrow \beta)$, then $\mathbb{M} \Vdash \beta$.*

In [2] we find the proof of the completeness theorem.

THEOREM 2.4. $\alpha \in \mathbf{N} \iff \text{for every model } \mathbb{M}, \mathbb{M} \Vdash \alpha$.

In the light of Lemmas 2.2, 2.3 and Theorem 2.4, in order to prove that a given model $\mathbb{M}$ satisfies all theorems of a logic $L(X)$ for $X \subseteq Fm$, it suffices to show that $\mathbb{M}$ satisfies specific axioms of $L(X)$, that is $Sub(X)$.

3. Cardinality

In this section we will take advantage of the fact that it is not generally the case in subnormal modal logics that $\Box\alpha$ and $\Box\beta$ are equivalent when α and β are equivalent. Take a model $\mathbb{M} = (\{a\}, \{R_\gamma\}_{\gamma \in Fm}, e)$ such that $e(a, p_i) = 0$ for all $p_i \in Var$, and

$$R_\gamma = \begin{cases} \{(a, a)\} & \text{if } \gamma = ((p_0 \wedge p_1) \wedge p_2), \\ \emptyset & \text{if } \gamma \neq ((p_0 \wedge p_1) \wedge p_2), \end{cases}$$

for every $\gamma \in Fm$. It follows that $(\mathbb{M}, a) \not\models \Box((p_0 \wedge p_1) \wedge p_2)$ and $(\mathbb{M}, a) \models \Box(p_0 \wedge (p_1 \wedge p_2))$, since $((p_0 \wedge p_1) \wedge p_2) \neq (p_0 \wedge (p_1 \wedge p_2))$, even though

$$(((p_0 \wedge p_1) \wedge p_2) \leftrightarrow (p_0 \wedge (p_1 \wedge p_2))) \in PC.$$

Hence $(\Box((p_0 \wedge p_1) \wedge p_2) \leftrightarrow \Box(p_0 \wedge (p_1 \wedge p_2))) \notin \mathbf{N}$.

Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the set of natural numbers, and let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$.

DEFINITION 3.1. For any $\alpha \in Fm$ and any $n \in \mathbb{N}^+$ we define:

1. $\alpha^1 = \alpha$,
2. $\alpha^{n+1} = (\alpha^n \wedge \alpha)$.

DEFINITION 3.2. Let $\gamma \in Fm$ and $p_i \in Var$. We will say that γ is a p_i -implication iff $\gamma = (\Box p_i \rightarrow \Box p_i^n)$ for some $n \in \mathbb{N}^+ \setminus \{1\}$. We further

define $I(p_i) := \{\delta \in Fm : \delta \text{ is a } p_i\text{-implication}\}$. A subnormal modal logic determined by a subset of $I(p_i)$ will be called a p_i -logic.

DEFINITION 3.3. Let $X \subseteq I(p_i)$. We define

$$nat(X) := \{n \in \mathbb{N} : (\exists \beta \in X)(\beta = (\Box p_i \rightarrow \Box p_i^n))\}.$$

From these definitions it follows immediately that:

LEMMA 3.4. Let $X \subseteq I(p_i)$.

- If $n \in nat(X)$, then $n \geq 2$,
- $(\Box p_i \rightarrow \Box p_i^n) \in X \iff n \in nat(X)$,
- $|I(p_i)| = \aleph_0$.

Moreover, we can show that each p_i -logic is in fact intermediate. To do that, we need to prove that every substitution of every p_i -implication is a theorem of $\mathbf{K}$, and that no p_i -logic proves the schema $(\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta))$.

LEMMA 3.5. $s(\delta) \in \mathbf{K}$, for any $s \in Sub, \delta \in I(p_i)$.

PROOF: Let $\delta = (\Box p_i \rightarrow \Box p_i^k)$, for $k \geq 2$. We have $(p_i \rightarrow p_i^k) \in PC$, hence $(p_i \rightarrow p_i^k) \in \mathbf{K}$. By necessitation, it follows that $\Box(p_i \rightarrow p_i^k) \in \mathbf{K}$, therefore, by K -schema and *Modus Ponens*, $(\Box p_i \rightarrow \Box p_i^k) \in \mathbf{K}$. And finally, since $\mathbf{K}$ is closed under substitutions, $s((\Box p_i \rightarrow \Box p_i^k)) \in \mathbf{K}$. $\square$

To show that no p_i -logic proves the K -schema, it suffices to show that the strongest of them, that is $L(I(p_i))$, does not prove the K -schema.

LEMMA 3.6. $(\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)) \notin L(I(p_i))$ for some $\alpha, \beta \in Fm$.

PROOF: Consider the model $\mathbb{M} = (\{a\}, \{R_\gamma\}_{\gamma \in Fm}, e)$ such that $e(a, p_0) = 1$ and $e(p_j, a) = 0$ for $j \neq 0$, and moreover:

$$R_\gamma = \begin{cases} \emptyset & \text{if } \gamma = (p_0 \rightarrow p_1) \text{ or } (\exists k \geq 2)(\exists s \in Sub)(\gamma = s(p_i^k)), \\ \{(a, a)\} & \text{otherwise,} \end{cases}$$

for all $\gamma \in Fm$. Take any $s \in Sub$. It follows that $R_{s(p_i^k)} = \emptyset$ for all $k \geq 2$. Thus $(\mathbb{M}, a) \Vdash \Box s(p_i^k)$ for all $k \geq 2$, hence $(\mathbb{M}, a) \Vdash (\Box s(p_i) \rightarrow \Box s(p_i^k))$.

Finally, $(\mathbb{M}, a) \models s((\Box p_i \rightarrow \Box p_i^k))$ for all $k \geq 2$. It means that $\mathbb{M}$ satisfies all substitutions of all formulas in $I(p_i)$, hence $\mathbb{M}$ satisfies all theorems of $L(I(p_i))$. Now consider the formula $(\Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1))$. We have $R_{(p_0 \rightarrow p_1)} = \emptyset$, hence $(\mathbb{M}, a) \models \Box(p_0 \rightarrow p_1)$. We also have $R_{p_0} = \{(a, a)\}$ and $e(p_0, a) = 1$, hence $(\mathbb{M}, a) \models \Box p_0$. But $R_{p_1} = \{(a, a)\}$ and $e(p_1, a) = 0$, so $(\mathbb{M}, a) \not\models \Box p_1$. Thus $(\mathbb{M}, a) \not\models (\Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1))$. $\square$

COROLLARY 3.7. For all $X \subseteq I(p_i)$, $(\Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1)) \notin L(X)$.

Corollary 3.7 implies that $\mathbf{N}$ does not prove the K -schema.

COROLLARY 3.8. $L(X)$ is intermediate, for all $X \subseteq I(p_i)$.

Let us now prove the most crucial lemma which says that powers of p_i are substitutionally independent – you cannot get p_i^m from p_i^k by any substitution if $m \neq k$ and m, k are greater than 2. Even though $(p_i^m \leftrightarrow p_i^k) \in PC$ for all $m, k \in \mathbb{N}$, what matters for us is that p_i^m and p_i^k as formulas are different objects if $m \neq k$.

LEMMA 3.9. For all $s \in \text{Sub}$, $n, k \in \mathbb{N}$ such that $n \geq 2$ and $k \geq 2$,

$$\text{if } n \neq k, \text{ then } p_i^n \neq s(p_i^k).$$

PROOF: Suppose $n \neq k$. Since $n \geq 2$ and $k \geq 2$, we have $p_i^n = (p_i^{n-1} \wedge p_i)$ and $s(p_i^k) = s((p_i^{k-1} \wedge p_i)) = (s(p_i^{k-1}) \wedge s(p_i))$. Consider $s(p_i)$. Either $s(p_i) \in \text{Var}$, or $s(p_i) \notin \text{Var}$.

(Case 1) $s(p_i) \in \text{Var}$. Now we have two more cases: either $s(p_i) = p_i$, or $s(p_i) \neq p_i$.

(Case 1.1) $s(p_i) = p_i$. Then $s(p_i^k) = p_i^k$. And since $n \neq k$, we get $p_i^n \neq s(p_i^k)$.

(Case 1.2) $s(p_i) \neq p_i$. Then $s(p_i^k)$ contains a variable different from p_i and p_i^n contains only a variable p_i . Hence, $p_i^n \neq s(p_i^k)$.

(Case 2) $s(p_i) \notin \text{Var}$. We know that either $s(p_i)$ contains connectives other than $\wedge$, or $s(p_i)$ contains only $\wedge$ and no other connectives.

(Case 2.1) $s(p_i)$ contains connectives other than $\wedge$. Then $s(p_i^k)$ contains connectives other than $\wedge$, and since p_i^n contains only $\wedge$ as a connective, we get $p_i^n \neq s(p_i^k)$.

(Case 2.1) $s(p_i)$ contains only $\wedge$ and no other connectives. Then $s(p_i) = (\alpha \wedge \beta)$ for some $\alpha, \beta \in Fm$. Hence we have $s(p_i^k) = (s(p_i^{k-1}) \wedge (\alpha \wedge \beta))$. We also have $p_i^n = (p_i^{n-1} \wedge p_i)$. Now, the last two elements of p_i^n – being a sequence of symbols – are p_i and $)$. The last two elements of $s(p_i^k)$ are $)$ and $)$ – the same symbol repeated. Therefore, $p_i^n \neq s(p_i^k)$. $\square$

Finally, we can prove the following theorem.

THEOREM 3.10. *For all $X \cup \{\alpha\} \subseteq I(p_i)$: if $\alpha \notin X$, then $\alpha \notin L(X)$.*

PROOF: Suppose $\alpha \notin X$. Consider a model $\mathbb{M} = (\{a\}, \{R_\gamma\}_{\gamma \in Fm}, e)$ such that $e(a, q) = 0$ for all $q \in Var$ and

$$R_\gamma = \begin{cases} \emptyset & \text{if } \gamma = p_i \text{ or } (\exists n \in nat(X))(\exists s \in Sub)(\gamma = s(p_i^n)), \\ \{(a, a)\} & \text{otherwise,} \end{cases}$$

for all $\gamma \in Fm$. We will show that $(\mathbb{M}, a) \Vdash L(X)$ and $(\mathbb{M}, a) \nVdash \alpha$.

To show that $\mathbb{M}$ satisfies all theorems of $L(X)$, it is enough to prove that $\mathbb{M}$ satisfies all axioms of $L(X)$, that is all formulas in $Sub(X)$. So take any $\delta \in X, s \in Sub$. We have $\delta = (\Box p_i \rightarrow \Box p_i^k)$ for some $k \geq 2$. Hence, $k \in nat(X)$, and so $R_{s(p_i^k)} = \emptyset$. Therefore, $(\mathbb{M}, a) \Vdash \Box s(p_i^k)$, and then $(\mathbb{M}, a) \Vdash (\Box s(p_i) \rightarrow \Box s(p_i^k))$. Finally, $(\mathbb{M}, a) \Vdash s((\Box p_i \rightarrow \Box p_i^k))$.

We know that $\alpha = (\Box p_i \rightarrow \Box p_i^m)$ for some $m \geq 2$. Since $\alpha \notin X$, we have $m \notin nat(X)$. Let us show that

$$(\forall n \in nat(X))(\forall s \in Sub)(p_i^m \neq s(p_i^n)). \quad (\star)$$

Take any $n \in nat(X), s \in Sub$. We have $m \neq n$, $m \geq 2$ and $n \geq 2$. By Lemma 3.9 we infer $p_i^m \neq s(p_i^n)$.

By $(\star)$ and the fact that $p_i^m \neq p_i$ we get $R_{p_i^m} \neq \emptyset$, hence $R_{p_i^m} = \{(a, a)\}$. We also have $(\mathbb{M}, a) \nVdash p_i^m$, so $(\mathbb{M}, a) \nVdash \Box p_i^m$. Since $R_{p_i} = \emptyset$, we get $(\mathbb{M}, a) \Vdash \Box p_i$. Thus, $(\mathbb{M}, a) \nVdash (\Box p_i \rightarrow \Box p_i^m)$. $\square$

COROLLARY 3.11. $(\forall X, Y \subseteq I(p_i))(X \neq Y \Rightarrow L(X) \neq L(Y))$.

PROOF: Without loss of generality assume that $\beta \in X$ and $\beta \notin Y$. Then $\beta \in L(X)$ and, by Theorem 3.10, $\beta \notin L(Y)$. $\square$

COROLLARY 3.12. Cardinality of the set of all p_i -logics is continuum, for any $p_i \in Var$.

COROLLARY 3.13. Cardinality of the set of all intermediate subnormal modal logics is continuum.

COROLLARY 3.14. Cardinality of the set of all subnormal modal logics is continuum.

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