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Helena Marciuš
Tin Perkov

HENNESSY–MILNER THEOREMS FOR TOPOLOGICAL SEMANTICS OF MODAL LOGIC

Abstract

Bisimulation is the basic equivalence between models in modal logic. Although bisimulation implies modal equivalence, the converse does not hold in general. Hennessy–Milner-style theorems find various conditions under which the converse holds, e.g. finiteness. In Kripke semantics, the condition of finiteness can be improved to image-finiteness, i.e., a model may be infinite, but without infinite branching. The present paper proposes local finiteness as an analogue of this property for topological semantics, proves appropriate Hennessy–Milner theorem analogues, and shows how locally finite topologies, in fact, directly correspond to image-finite Kripke models.

Keywords: topological semantics, bisimulation, Hennessy–Milner theorem.

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1. Introduction

Bisimulation is the basic equivalence between models in modal logic, since it provides natural back-and-forth conditions which make related worlds modally equivalent, i.e., such that the same modal formulas are satisfied at them.

However, while bisimilar models are modally equivalent, the converse does not hold in general (a counterexample can be found in e.g. [3], p. 68).

In case of basic modal logic and Kripke semantics, Hennessy–Milner theorem [5] establishes a condition that models must satisfy for the converse to hold. A sufficient condition is that the models are image-finite, meaning each world has access to a finite number of worlds.

For various purposes, like generalization or applicability, modal semantics other than Kripke semantics are developed. Topological semantics for modal logic was introduced by McKinsey and Tarski [6]. They proposed two interpretations for $\Diamond$ – as the closure operator (usually called c-semantics) and as the derived set operator (usually called d-semantics). An appropriate notion of bisimulation is defined for both semantics, so that bisimilar worlds are modally equivalent. As expected, the converse does not hold in general, but it does under the assumption that the models are finite.

In a 2021 lecture, Bezhanishvili [2] posed the question: What is the analogue of image-finiteness for topological models? In other words, can we obtain a Hennessy–Milner analogue under less restrictive condition than the finiteness of models? This paper offers an answer: Hennessy–Milner analogues for both c- and d-semantics hold under the assumption that models are locally finite, i.e., each point has a finite open neighborhood (Section 3).

While the basic modal system **K** is the logic of all Kripke frames, its extension **S4** is the logic of all reflexive and transitive frames (cf. e.g. [3] for details). On the other hand, **S4** is the logic of all topological spaces w.r.t. c-semantics (see e.g. [8]). Moreover, any reflexive and transitive Kripke frame can be naturally transformed to an Alexandroff space on which the same modal formulas are valid under c-semantics, and vice versa. The present paper shows that this transformation in particular establishes a

1-1 correspondence between image-finite Kripke frames and locally finite topological spaces.

Esakia [4] (see also [8]) proved that **wK4**, an extension of **K** which is the modal logic of irreflexive and weakly transitive Kripke frames (weak transitivity means that $wRuRv$ always implies $w = v$ or wRv), is sound and complete w.r.t. d-semantics. Moreover, every Alexandroff space can be transformed to an irreflexive and weakly transitive Kripke frame and vice versa. In this case we also prove that there is a 1-1 correspondence between image-finite Kripke frames and locally finite topological spaces (Section 4).

2. Preliminaries

We assume familiarity with the syntax and Kripke semantics of basic modal logic. We will consider $\vee$, $\neg$ and $\Diamond$ to be primitives, while other Boolean connectives and $\Box$ are defined as abbreviations in the usual way. This paper relates Kripke semantics to topological semantics. We will consider two different topological semantics for modal logic, usually called *c-semantics* and d-semantics. This section briefly overviews basic notions and facts in topology and topological semantics for modal logic.

A *topological space* is (X, τ), where X is a set, and τ a family of subsets of X which contains $\emptyset$ and X, and is closed under unions and finite intersections. We say τ is a *topology* on X. The elements of τ are called *open sets*, while their complements are called *closed sets*. By duality, the family of all closed sets is closed under intersections and finite unions. An open set containing $x \in X$ is called an *open neighborhood* of x.

Let $A \subseteq X$. The *closure* $Cl(A)$ of A is the smallest closed set containing A. Clearly, $x \in Cl(A)$ if and only if each open neighborhood of x intersects A. A *limit point* of A is $x \in X$ such that each open neighborhood of x intersects A at some point other than x. The set of all limit points of A is denoted by $d(A)$. An *isolated point* of A is $x \in A$ such that x has an open neighborhood which intersects A only at x.

We say that a topological space is an *Alexandroff space* if it is closed under arbitrary intersections.

We say that a topological space is *locally finite* if each point has a finite open neighborhood. It is easy to see that a locally finite space is an Alexandroff space (see e.g. [7]).

We say that a topological space is *scattered* if every non-empty subset has an isolated point.

A *topological model* for basic modal logic is $M = (X, \tau, \nu)$, where (X, τ) is a topological space with $X \neq \emptyset$, and ν is a function called *valuation*, which maps each propositional variable p to a subset $\nu(p) \subseteq X$.

In the Boolean case, the valuation of a topological model extends to all formulas in a straightforward way. However, for formulas containing a modality, the two semantics diverge.

In c-semantics, $\nu(\Diamond\varphi) = Cl(\nu(\varphi))$. In other words, $x \in \nu(\Diamond\varphi)$ if and only if for all $U \in \tau$ such that $x \in U$ there is $y \in U$ such that $y \in \nu(\varphi)$.

In d-semantics, $\nu(\Diamond\varphi) = d(\nu(\varphi))$. In other words, $x \in \nu(\Diamond\varphi)$ if and only if for all $U \in \tau$ such that $x \in U$ there is $y \neq x$ such that $y \in U$ and $y \in \nu(\varphi)$.

We use the notation $\Vdash$ for satisfaction relation in Kripke semantics, $\Vdash_c$ for c-semantics, and $\Vdash_d$ in d-semantics.

We say that points x and x' from some models M and M', respectively, are *modally equivalent* and we write $x \equiv x'$ if they satisfy the same formulas. This depends on semantics, and although it is usually clear from the context what we mean, when we feel the need, we will emphasize this by using the terminology *c-modally equivalent* or *d-modally equivalent*, as appropriate.

3. Analogues of Hennessy–Milner theorem

Bisimulations are relations between structures such that related points are propositionally equivalent, and appropriate back-and-forth conditions are satisfied. We assume the reader is familiar with the precise definition for Kripke semantics. For topological c-semantics, a *c-bisimulation* between models $M = (X, \tau, \nu)$ and $M' = (X', \tau', \nu')$ is a non-empty relation $Z \subseteq X \times X'$ such that for all $(x, x') \in Z$ we have:

(at) $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any propositional variable p

(forth) for each open neighborhood U of x there is an open neighborhood U' of x' such that for all $y' \in U'$ there is $y \in U$ such that yZy'

(back) for each open neighborhood U' of x' there is an open neighborhood U of x such that for all $y \in U$ there is $y' \in U'$ such that yZy'

We say that points $x \in X$ and $x' \in X'$ are *c-bisimilar* if there is a c-bisimulation Z between X and X' such that xZx'.

This definition was given in [1], along with a proof of the basic property that bisimilarity implies modal equivalence, and also a proof that the converse holds in the case of finite models. More precisely, the relation of modal equivalence between finite topological models is a c-bisimulation. We show that the same holds over locally finite topological models.

THEOREM 3.1. *Let $M = (X, \tau, \nu)$ and $M' = (X', \tau', \nu')$ be locally finite topological models. If $x \in X$ and $x' \in X'$ are c-modally equivalent, then they are c-bisimilar.*

PROOF: We prove that the relation of modal equivalence is a c-bisimulation between M and M'. The condition (at) from the definition of c-bisimulation is trivially fulfilled.

We now prove the (forth) condition. Let $x \in X$ and $x' \in X'$ be modally equivalent and let $U \in \tau$ such that $x \in U$. Suppose the condition (forth) is not satisfied for x, x' and U. Then we have:

for every $U' \in \tau'$ containing x', there exists $y' \in U'$ such that for every $y \in U$ we have $y \not\equiv y'$. $(*)$

Let $V = \bigcap_{x \in U \in \tau} U$ and let $V' = \bigcap_{x' \in U' \in \tau'} U'$. Since (X, τ) and (X', τ') are locally finite topologies, V and V' are finite sets. Moreover, since locally finite spaces are Alexandroff spaces, we have $x \in V \in \tau$ and $x' \in V' \in \tau'$.

Now, by $(*)$ applied to V', there exists $y' \in V'$ such that for every $y \in U$, and so in particular for every $y \in V$, there exists a formula φ_y such that $y \nVdash_c \varphi_y$ and $y' \Vdash_c \varphi_y$. Put $\varphi := \bigwedge_{y \in V} \varphi_y$. Then $V \cap \nu(\varphi) = \emptyset$, and therefore $x \not\Vdash_c \Diamond\varphi$. On the other hand, for every $U' \in \tau'$ such that $x' \in U'$, since $V' \subseteq U'$, we have $y' \in U' \cap \nu'(\varphi)$, and therefore $x' \Vdash_c \Diamond\varphi$. This contradicts the assumption that $x \equiv x'$.

The (back) condition is proved similarly. □

In d-semantics, the definition of bisimulation is adjusted in an obvious way. A *d-bisimulation* between models $M = (X, \tau, \nu)$ and $M' = (X', \tau', \nu')$ is a non-empty relation $Z \subseteq X \times X'$ such that for all $(x, x') \in Z$ we have:

(at) $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any propositional variable p

(forth) for each open neighborhood U of x there is an open neighborhood U' of x' such that for all $y' \in U' \setminus \{x'\}$ there is $y \in U \setminus \{x\}$ such that yZy'

(back) for each open neighborhood U' of x' there is an open neighborhood U of x such that for all $y \in U \setminus \{x\}$ there is $y' \in U' \setminus \{x'\}$ such that yZy'

By induction on the complexity of a formula, it is easy to prove that d-bisimilarity implies modal equivalence. Now we prove the converse in the case of locally finite topological models.

Theorem 3.2. *Let $M = (X, \tau, \nu)$ and $M' = (X', \tau', \nu')$ be locally finite topological models. If $x \in X$ and $x' \in X'$ are d-modally equivalent, then they are d-bisimilar.*

Proof: The proof is completely analogous to the proof of Theorem 3.1. We present only the part of the proof of (forth) in which a care is needed to observe the smallest neighborhoods V and V' of x and x', respectively, without x and x' themselves.

By an analogous reasoning as in the proof of Theorem 3.1, we conclude that there is $y' \in V' \setminus \{x'\}$ such that for every $y \in V \setminus \{x\}$ there is φ_y such that $y \not\Vdash_d \varphi_y$ and $y' \Vdash_d \varphi_y$. Put $\varphi := \bigwedge_{y \in V \setminus \{x\}} \varphi_y$. Then $V \setminus \{x\} \cap \nu(\varphi) = \emptyset$. Hence, $x \not\Vdash_d \Diamond\varphi$. On the other hand, analogously as in the proof of Theorem 3.1, we obtain $x' \Vdash_d \Diamond\varphi$, thus arriving to a similar contradiction.
□

4. Image-finite models correspond to locally finite topologies

Hennessy–Milner analogues proved in the previous section are a good signal that local finiteness corresponds to image-finiteness. In this section we show that this is in fact a 1-1 correspondence, under a natural transformation from reflexive and transitive Kripke frames (for c-semantics), or irreflexive and weakly transitive Kripke frames (for d-semantics), to Alexandroff spaces and vice versa. We will use the same well-known transformation as presented in e.g. [8], with additional assumptions of image-finiteness and local finiteness, respectively.

4.1. c-semantics

First we establish the correspondence between locally finite spaces and image-finite, reflexive and transitive frames.

THEOREM 4.1. *There is a 1-1 correspondence between the class of of locally finite topological spaces and the class of image-finite reflexive and transitive Kripke frames.*

PROOF: Let (W, R) be an image-finite reflexive and transitive Kripke frame. Let τ_R be the set of all R-upward closed subsets of W, i.e. all $U \subseteq W$ such that $w \in U$ and wRu always implies $u \in U$. In other words, τ_R contains $\emptyset$ and domains of all generated subframes of (W, R). Then τ_R is obviously a topology on W.

Let $w \in W$ and denote $R[w] = \{u \in W : wRu\}$. Since R is a transitive relation, $R[w]$ is R-upward closed. Since the frame is image-finite, $R[w]$ is finite. So, $R[w]$ is a finite open neighborhood of w. Hence, (W, τ_R) is a locally finite topological space.

Conversely, let (X, τ) be a locally finite topological space. We define relation $R_\tau \subseteq X \times X$ by putting $xR_\tau y$ if and only if $x \in Cl(\{y\})$.

Let $x \in X$ and let $U = \bigcap_{x \in V \in \tau} V$. Since (X, τ) is locally finite, U is a finite open neighborhood of x. Clearly, $y \in U$ if and only if $y \in V$ for every open neighborhood V of x if and only if $x \in Cl(\{y\})$, i.e., $y \in R_\tau[x]$, thus $R_\tau[x] = U$. Hence, (X, R_τ) is an image-finite frame.

Furthermore, it is easy to see that $R = R_{\tau_R}$, and on the other hand, $\tau = \tau_{R_\tau}$.

Thus a 1-1 correspondence between locally finite spaces and image-finite reflexive and transitive frames is established. □

By induction on the complexity of a formula, it is easy to prove that for each modal formula φ the following holds: $(W, R) \Vdash \varphi$ if and only if $(W, \tau_R) \Vdash_c \varphi$. We present only the modal case. Let $x \in W$ and let ν be a valuation. We have the following:

$$\begin{aligned}
(W, R, \nu), x \Vdash \Diamond\varphi &\iff \exists y \in R[x], (W, R, \nu), y \Vdash \varphi \\
&\iff R[x] \cap \nu(\varphi) \neq \emptyset \\
&\iff \forall V \in \tau_R \text{ such that } x \in V, V \cap \nu(\varphi) \neq \emptyset \\
&\iff x \in Cl(\nu(\varphi)) \\
&\iff (W, \tau_R, \nu), x \Vdash_c \Diamond\varphi
\end{aligned}$$

Since $\tau = \tau_{R_\tau}$, it also holds that $(W, \tau) \Vdash_c \varphi$ if and only if $(W, R_\tau) \Vdash \varphi$.

4.2. *d*-semantics

Using the similar transformation, we can establish 1-1 correspondence between irreflexive, weakly transitive image-finite frames and locally finite spaces.

Theorem 4.2. *There is a 1-1 correspondence between the class of locally finite topological spaces and the class of image-finite irreflexive, weakly transitive Kripke frames.*

Proof: Suppose (W, R) is an image-finite irreflexive and weakly transitive frame. As before, the collection τ_R of all R-upward closed sets is a topology on W. Let $w \in W$. Since R is weakly transitive and irreflexive relation, $R[w] \cup \{w\}$ is an open neighborhood of w. Since $R[w]$ is a finite set, it follows that (W, τ_R) is a locally finite topological space.

Conversely, let (X, τ) be a locally finite topological space. We define relation $R_\tau \subseteq X \times X$ by putting $xR_\tau y$ if and only if $x \in d(\{y\})$.

Since $x \notin d(\{x\})$, R_τ is an irreflexive relation. To see that R_τ is weakly transitive, let $x \neq z$, $x \in d(\{y\})$ and $y \in d(\{z\})$. Let U be an open neigbourhood of x. Then $y \in U \setminus \{x\}$. Since U is also an open neighborhood of y, it follows that $z \in U \setminus \{y\}$. Since $x \neq z$, we have that $z \in U \setminus \{x\}$ and therefore $x \in d(\{z\})$.

Let $x \in X$ and let $U = \bigcap_{x \in V \in \tau} V$. Since (X, τ) is a locally finite topological space, U is a finite open neighborhood of x. Suppose $y \in U \setminus \{x\}$. Then $y \in V \setminus \{x\}$ for every open neighborhood V of x, so it follows that $x \in d(\{y\})$. Conversely, if $y \in R_\tau[x]$, then $y \in U \setminus \{x\}$. Hence, $R_\tau[x] = U \setminus \{x\}$, so (X, R_τ) is an image-finite frame.

Moreover, it is easy to see that $R = R_{\tau_R}$ and $\tau = \tau_{R_\tau}$.

It follows that there is a 1-1 correspondence between locally finite spaces and image-finite irreflexive and weakly transitive frames. $\square$

Also, it is easy to check in a similar way as in the case of c-semantics that the same formulas are satisfied on related Kripke and topological models.

5. Conclusion

We have established that a sensible topological property, local finiteness, corresponds to image-finiteness of relational structures, in a sense that it enables the converse of the basic property of bisimulations, i.e., that modal equivalence between points in locally finite topological modal models implies bisimilarity. Since there is a well-known correspondence between Kripke models and Alexandroff spaces, and locally finite topologies are all Alexandroff, it would be of interest to find a topological property strictly more general than local finiteness which would enable Hennessy–Milner-like result in the future. We believe there is no simple answer to this, since the proof of Hennessy–Milner theorem requires finitely many accessible worlds, so that conjunctions we construct in the proof are finite. In a topological model, a natural analogy is to require the existence of a finite open neighborhood, but this immediately results in an Alexandroff topology. So, it seems that a more general property leading to a Hennessy–Milner-like result would require a different proof technique. We leave these considerations for future work.

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Helena Marciuš
University of Zagreb
Faculty of Science, Department of Mathematics
Bijenička cesta 30
Zagreb, Croatia
e-mail: helena.marcius@math.hr

Tin Perkov
University of Zagreb
Faculty of Teacher Education
Savska cesta 77
Zagreb, Croatia
e-mail: tin.perkov@ufzg.hr

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