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STRUCTURES WITH TWO PARTIAL ORDERS AND THE AMALGAMATION PROPERTY

Abstract

We show that partially ordered sets, lattices, semilattices, Boolean algebras, Heyting algebras with a further coarser or finer partial order, or a linearization, or an auxiliary relation have the strong amalgamation property, Fraïssé limits and, in many cases, an ω-categorical model completion with quantifier elimination.

The same applies to causal spaces, introduced by Kronheimer and Penrose in connection with foundational problems in general relativity. Our main tool is the superamalgamation property, thus we provide arguments suggesting the usefulness of the superamalgamation property also in pure model theory, not only in algebraic logic.

Keywords: superamalgamation property, Fraïssé limit, auxiliary relation, causal space.

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1. Introduction

The amalgamation property has proved a very useful tool in algebra [18, 20, 24], algebraic topology [26], logic [14, 19, 22], particularly, algebraic logic [15, 23, 33, 34], and is deeply connected to category theory, e. g. [4, 24]. It has also been considered in some philosophical arguments [35, 38]. In particular, the amalgamation property is a key ingredient in the proof of the existence of Fraïssé limits, which are countable universal homogeneous structures, relative to some class of models [14, 19]. For example, the Fraïssé limit of the class of finite linearly ordered sets is the ordered set of the rationals, the Fraïssé limit of the class of finite graphs is the random graph. A carefully compiled list of references about the amalgamation property can be found in [9, Section 3].

Ordered structures with a further binary relation are ubiquitous in mathematics [1, 8, 6, 12, 13, 30, 36], with connections to physics [27, 37] and computer science [5, 17], just to state a few examples. In particular, auxiliary relations are an important tool in the theory of continuous lattices and domains, with many applications to the theory of computation, to the semantics of programming languages and to other branches of mathematics [17]. Auxiliary relations in ordered sets with further structure arise also in different contexts, related to topological dualities, algebraic logic and various generalizations of topology. See [6] for a survey. Causal spaces have been introduced in [27] in connection with foundational problems in general relativity. In an equivalent formulation, a *causal space* is a poset with an antireflexive auxiliary relation.

Because of the mentioned abundance of examples, we believe that it is interesting to study the amalgamation property for ordered structures with additional relations. As a further motivation, let us recall that there are many methods which can be used in order to combine theories which are already known to have the amalgamation property, in such a way to get further theories with the property. See e. g. [16, 32] and further references there. The examples we present might be useful in order to evaluate exact boundaries for such combining methods.

In detail, we show that if some theory of partially ordered sets, possibly with further operations and predicates, has the superamalgamation property, then superamalgamation is maintained by adding a further coarser or finer partial order, or a linearization, or an auxiliary relation; see Theorem 2.2. In particular, this applies to the theories mentioned in the abstract; see Theorem 2.1. Recall that the superamalgamation property is a strengthening of the amalgamation property which applies to classes with at least a partial order relation. So far, the main applications of the superamalgamation property have been found in algebraic logic, e. g., [15, 23, 33]. Our results show that the superamalgamation property has also some model-theoretical interest. A similar situation occurred in [32], where the reader might find a more in-depth discussion.

While, as we mentioned, we deal with widely studied theories, the results presented here are new, to the best of our knowledge, namely, the amalgamation property for such theories has not been considered in the literature. On the other hand, amalgamation properties have been discussed for other theories of ordered structures with additional relations. As a result which might be somewhat connected with the present note, Celani [7] proved a version of the superamalgamation property for quasi-modal algebras. Quasi-modal algebras are not first-order structures, in that they are endowed with an operator sending elements of some structure to ideals, not to other elements. However, quasi-modal algebras are in a one-to-one correspondence both with Boolean algebras with a subordination and also with Boolean algebras endowed with a pre-contact [11], [2, Remarks 2.5 and 2.6], [6, Sect. 8.3]. Moreover, the amalgamation property for Contact (Boolean) Algebras has been proved in [12] and [3]. As another example, in [30, Theorem 5.14] we have proved the strong amalgamation property for the class of semilattices with a "specialization".

We leave as an open problem whether the methods here and the just mentioned results share a common generalization. It seems that our results here can be easily generalized to contact relations, if one forgets about additivity (or distributivity), Condition C_4 on [12, p. 346]. Generally the expression *weak contact* is used for such relations. In this connection, a comparison with [32] might be relevant. In [32] we proved results similar to the present ones, but dealing with further operations, not relations. Again, the superamalgamation property turned out to be relevant, but the results in [32] do not apply to additive operations; in particular, we have recently shown that the superamalgamation property alone is not sufficient in order to deal with additive operations [31, Example 7.2]. So the question of the right hypotheses appropriate for additive contact relations is still completely open.

In conclusion, the main motivations for our study are the following.

(A) It is safe to say that contemporary algebraic logic and model theory are full of new sophisticated techniques and astonishing results. However the amalgamation property, though a classical notion, still maintains its importance.

(B) Most of the theories we are dealing with have been widely studied and are relevant to various fields, even outside mathematics, including physics and computer science. The proofs we present provide uniform results which apply to many more theories, by the way, possible candidates for further applications.

(C) As a motivation only just hinted above, B. Jónsson [21] asked whether there are characterizations of amalgamable theories just in terms of the forms of their axioms. It seems that Jónsson problem has solution only in extremely rare cases. On the other hand, there are many methods which can be used in order to *combine* theories which are already known to have the amalgamation property, in such a way to get further theories with the property. See e. g. [16, 32] and further references there. A further motivation for our study is to try to evaluate exact boundaries for such combining methods; this is also the reason why we state some results in their possibly most general form.

(D) As in [32], here we provide model-theoretical applications of the superamalgamation property, while, so far, to the best of our knowledge, the applications of the superamalgamation property have been essentially limited to algebraic logic. As hinted above, in [31] we showed that in certain situations the superamalgamation property is not adequate for getting preservation results. The main point of the present paper, together with [31, 32], seems to be an exact evaluation of the relevance of the superamalgamation property, when dealing with results about preservation of amalgamation in expansions.

2. Statements of the main results

In what follows *theory* is always intended in the sense of a *first-order theory*. We say that a relation $\ll$ is *finer* than another relation $\leq$ on the same domain if $a \ll b$ implies $a \leq b$, for all a, b in the domain; in this situation we also say that $\leq$ is *coarser* than $\ll$. Note that some authors use this terminology the other way round[1]. Recall that if $\mathbf{S}$ is a set with a partial order $\leq$ (henceforth, *poset*, for short), an *auxiliary relation* [17, Definition I-1.11] is a binary antisymmetric and transitive relation $\ll$ which is finer than the order and satisfies the following condition: $w \leq x \ll y \leq z$ implies $w \ll z$.

In order to have the joint embedding property, we assume that *Boolean algebras* and *Heyting algebras* are nontrivial (that is $0 \neq 1$) and that auxiliary relations in Boolean algebras and Heyting algebras satisfy $0 \ll 0$ and $1 \ll 1$. The definition of the superamalgamation property will be recalled in the next section.

Our main results are the following.

Theorem 2.1. *Suppose that T^- is either the theory of partial orders, or lattices, or join semilattices, or meet semilattices, or Boolean algebras, or Heyting algebras. In a language with an added binary relation $\ll$, let T be any extension of T^- obtained by adding one of the following axioms:*

(1) $\ll$ is an order relation coarser than $\leq$, or

(2) $\ll$ is a linearization of $\leq$, or

(3) $\ll$ is a reflexive order relation finer than $\leq$, or

(4) $\ll$ is an auxiliary relation.

[1]The terminology we adopt is particularly common when dealing with equivalence relations. In the present terminology $\ll$ is finer than $\leq$ if (the diagram of) $\ll$ is contained in $\leq$. Apparently, this contrasts the topological notion, where a topology τ is finer than σ when $\tau \supseteq \sigma$. However, note that if τ is finer than σ and K denotes closure, then $p \in K_\tau x$ implies $p \in K_\sigma x$, namely, the adherence relation in τ is finer—in the present sense—than the the adherence relation in σ.

Alternatively, let T be the theory of causal spaces.

Then T has the strong amalgamation property, more generally, the superamalgamation property with respect to $\leq$.

The class of finite models of T has a Fraïssé limit $\mathbf{M}$. Except possibly for lattices and Heyting algebras, the first-order theory of $\mathbf{M}$ is ω-categorical, has quantifier elimination and is the model completion of T.

The next theorem provides a more general result which applies to arbitrary theories with the superamalgamation property. Note that lattices, semilattices, etc. have the superamalgamation property. See Theorem 3.2.

THEOREM 2.2. *Suppose that T^- is a theory of ordered structures. If T^- has the superamalgamation property, then T has the superamalgamation property, where T is any theory obtained from T by adding one of the axioms (1) - (4) in Theorem 2.1.*

Both in the above statement and in Theorem 2.1, we still get superamalgamation if we simultaneously add any family $(\ll^i)_{i\in I}$ of further relations satisfying possibly distinct axioms among (1) - (4).

Parts (1) - (3) of Theorems 2.1 and 2.2 will be proved at the end of Section 3. When T^- is the theory of partial orders, Theorem 2.1(4) is a special case of Theorem 4.2 which will be proved in Section 4. The same applies to the theory of causal spaces. The remaining cases follow from Theorem 5.2 in Section 5. Details are given at the end of Section 5.

The superamalgamation property is necessary in some arguments: the strong amalgamation property alone is not enough. See Remark 6.1 below.

We can merge Theorems 2.1 and 2.2 with [32, Theorem 4.1 and Corollary 4.3], under the corresponding hypotheses. Namely, we still get superamalgamation if we also add families of operations satisfying the conditions (A1e)–(C3) listed in [32], with respect to $\leq$, for example, isotone, idempotent, closure operations. This is because, in both cases, the relevant assumption is the superamalgamation property.

3. Ordered structures with the superamalgamation property

In this section we give a proof of Parts (1)–(3) of Theorems 2.1 and 2.2. The section is a good introduction to the methods which will be used in the subsequent sections.

We first recall the basic notions. *Models* are intended in the classical model-theoretical sense [19]. An *ordered structure* is a model with a partial order $\leq$ and, possibly, further relations, functions and constants. A class $\mathcal{K}$ of models of the same type has the *amalgamation property* (AP) if, whenever $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\iota_1 \colon \mathbf{C} \to \mathbf{A}$ and $\kappa_1 \colon \mathbf{C} \to \mathbf{B}$ are embeddings, there are a model $\mathbf{D} \in \mathcal{K}$ and embeddings $\iota \colon \mathbf{A} \to \mathbf{D}$ and $\kappa \colon \mathbf{B} \to \mathbf{D}$ such that $\iota_1 \circ \iota = \kappa_1 \circ \kappa$. If, in addition, $\mathbf{D}$, ι and κ can be always chosen in such a way that $\iota(A) \cap \kappa(B) = (\iota_1 \circ \iota)(C)$, then $\mathcal{K}$ is said to have the *strong amalgamation property* (SAP).

$$\begin{tikzcd}
 & \mathbf{D} & \\
\mathbf{A} \arrow[ur, dashed, "\iota"] & & \mathbf{B} \arrow[ul, dashed, "\kappa"'] \\
 & \mathbf{C} \arrow[ul, "\iota_1"] \arrow[ur, "\kappa_1"'] &
\end{tikzcd}$$

Suppose further that $\mathcal{K}$ is a class of ordered structures with order $\leq$. Then $\mathcal{K}$ has the *superamalgamation property with respect to* $\leq$, or simply the *superamalgamation property*, when $\leq$ is understood, if, in addition, for every $a \in A \setminus \iota_1(C)$ and $b \in B \setminus \kappa_1(C)$, if $\iota(a) \leq_D \kappa(b)$, then there is $c \in C$ such that $a \leq_A \iota_1(c)$, $\kappa_1(c) \leq_B b$, and also the corresponding conclusion holds when $\iota(b) \leq_D \kappa(a)$. Note that, since $\leq$ is transitive, the superamalgamation property determines $\leq$ uniquely on $\iota_1(\mathbf{A}) \cup \kappa_1(\mathbf{B})$. The superamalgamation property has found many applications in algebraic logic [15, 23, 33]. We sometimes use the definition also for an arbitrary binary relation R in place of the order $\leq$. This generalization has some use, as well [33]. A first-order theory T has the (strong, super) amalgamation property if the class of all models of T has the (strong, super) amalgamation property.

Lemma 3.1. *Suppose that $(D, \leq_D, \ll_D)$ is a set with two partial orders and $\ll_D$ is coarser than $\leq_D$. If $(E, \leq_E)$ is a poset extending $(D, \leq_D)$, then there is a partial order $\ll_E$ on E such that $\ll_E$ is coarser than $\leq_E$ and $(E, \leq_E, \ll_E)$ extends $(D, \leq_D, \ll_D)$.*

Proof: Let $e \ll_E f$ on E if $e \leq_E f$, or there are $h, k \in D$ such that $e \leq_E h \ll_D k \leq_E f$. By construction, $\ll_E$ is coarser than $\leq_E$. Moreover, $(E, \ll_E)$ extends $(D, \ll_D)$. Indeed, if $e, f \in D$ and $e \ll_D f$, then $e \ll_E f$ by the second clause in the definition of $\ll_E$ and since $\leq_E$ is reflexive. In the other direction, if $e, f \in D$ and $e \ll_E f$ is given by $e \leq_E f$, then $e \leq_D f$, since $e, f \in D$ and $(E, \leq_E)$ extends $(D, \leq_D)$, thus $e \ll_D f$, since by assumption $\ll_D$ is coarser than $\leq_D$. On the other hand, if $e, f \in D$ and $e \ll_E f$ is given by $e \leq_E h \ll_D k \leq_E f$, then $e \leq_D h \ll_D k \leq_D f$, since $e, h, k, f \in D$. Thus $e \ll_D h \ll_D k \ll_D f$, since $\ll_D$ is coarser than $\leq_D$, then $e \ll_D f$ by transitivity of $\ll_D$.

We now check that $\ll_E$ is transitive. If $e \ll_E f \ll_E g$ is witnessed by

$$e \leq_E h \ll_D k \leq_E f \leq_E p \ll_D q \leq_E g, \tag{3.1}$$

then $k \leq_E p$, by transitivity of $\leq_E$. Since $k, p \in D$, we have $k \leq_D p$, hence $k \ll_D p$, since $\ll_D$ is coarser than $\leq_D$. Hence $h \ll_D k \ll_D p \ll_D q$, thus $h \ll_D q$, by transitivity of $\ll_D$. Then (3.1) reads $e \leq_E h \ll_D q \leq_E g$, which means $e \ll_E g$ by definition. The other cases are immediate from transitivity of $\leq_E$.

It remains to check that $\ll_E$ is antisymmetric. Suppose that $e \ll_E f \ll_E e$ is witnessed by (3.1) with $g = e$. We have proved above that $h \ll_D k \ll_D p \ll_D q$. Moreover, $q \leq_E g = e \leq_E h$, hence $q \leq_E h$, $q \leq_D h$ and $q \ll_D h$, since $q, h \in D$ and $\ll_D$ is coarser than $\leq_D$. Thus $h \ll_D k \ll_D p \ll_D q \ll_D h$, hence $h = k = p = q$, by transitivity and antisymmetry of $\ll_D$. Finally, from $h = k$, $p = q$ and (3.1) with $g = e$ we get $e \leq_E h \leq_E f \leq_E p \leq_E e$, hence $e = f$, by transitivity and antisymmetry of $\leq_E$. The other cases are much simpler. For example, if

$e \leq_E h \ll_D k \leq_E f \leq_E e$, then $k \leq_E f \leq_E e \leq_E h$, hence $k \leq_E h$, $k \leq_D h$, $k \ll_D h$, thus $k = h$ by antisymmetry of $\ll_D$. Hence $e \leq_E h \leq_E f \leq_E e$, thus $e = f$. $\square$

The next theorem collects some classical results, sometimes not explicitly stated in the literature. The theorem follows from some proofs in [15, 20, 25]. See the proof of [32, Theorem 2.4] for full details. Note also that there are many situations in which amalgamation automatically implies superamalgamation, e. g. [15, 28].

THEOREM 3.2. *Partially ordered sets, meet semilattices, join semilattices, lattices, Boolean algebras and Heyting algebras have the superamalgamation property. The same applies to the classes of finite such structures.*

Remark 3.3. For later use, we recall the arguments from [20] relevant for the case of partially ordered sets. Without loss of generality, assume that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are posets such that $\mathbf{C} \subseteq \mathbf{A}, \mathbf{B}$ (here $\subseteq$ means *substructure*) and $A \cap B = C$. A superamalgamating structure is $\mathbf{D}$ with $D = A \cup B$ and $\leq_D$ equal to $\leq_A \cup \leq_B \cup (\leq_A \circ \leq_B) \cup (\leq_B \circ \leq_A)$, where, say, $(a, b) \in \leq_A \circ \leq_B$ means that there is $c \in A \cap B = C$ such that $a \leq_A c$ and $c \leq_B b$. Superamalgamation holds by construction; the arguments on [20, p. 203] (there $\mathbf{C}$ is used in place of $\mathbf{D}$ here) show that if both $\leq_A$ and $\leq_B$ are transitive, then $\leq_D$ is transitive and the inclusions from $\mathbf{A}$ and $\mathbf{B}$ to $\mathbf{D}$ are embeddings. Note that the argument, so far, works generally for transitive relations, not necessarily order relations.

Then the arguments on [20, p. 204] go on proving that if both $\leq_A$ and $\leq_B$ are transitive and antisymmetric, then $\leq_D$ is transitive and antisymmetric, thus an order relation, since reflexivity is straightforward (note that Jónsson uses the term "asymmetric" for what nowadays is commonly called antisymmetric). Some further elaborations can be found in [29].

We now give the proof of Parts (1)–(3) of Theorems 2.1 and 2.2. Recall that the theorems deal with a theory T^- with the superamalgamation property and state that superamalgamation is preserved by adding a new relation $\ll$ satisfying certain properties. We will recall these properties in the course of the proof.

Proof: *(Theorems 2.1 and 2.2, Superamalgamation, Part (1))* This is the case when $\ll$ is supposed to be an order relation coarser than $\leq$.

Without loss of generality, assume that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are members of T such that $\mathbf{C} \subseteq \mathbf{A}, \mathbf{B}$ and $A \cap B = C$. Let $\mathbf{A}^-$, $\mathbf{B}^-$ and $\mathbf{C}^-$ be the reducts of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ to the language of T^-. By Theorem 3.2, or by the assumption in Theorem 2.2, there is a model $\mathbf{E}^-$ superamalgamating $\mathbf{A}^-$ and $\mathbf{B}^-$ over $\mathbf{C}^-$ in the appropriate class (posets, semilattices, etc.). Because of SAP (a consequence of superamalgamation), we may assume that $\mathbf{E}^-$ extends both $\mathbf{A}^-$ and $\mathbf{B}^-$. Let $D = A \cup B$.

Since partial orders have the superamalgamation property, the $\{\leq, \ll\}$-reducts of $\mathbf{A}$ and $\mathbf{B}$ can be superamalgamated over the $\{\leq, \ll\}$-reduct of $\mathbf{C}$ by an amalgamating structure $(D, \leq, \ll)$ over D. Here we are using superamalgamation for two orders at a time, but the argument recalled in Remark 3.3 applies to this case, as well, and coarseness conditions are preserved. In particular, say, $(A, \ll)$ embeds in $(D, \ll)$.

Since the superamalgamation property determines the order on $A \cup B$ uniquely, $(D, \leq)$ is a substructure of $(E, \leq)$. By Lemma 3.1, there is an order $\ll_E$ on E such that $(D, \leq, \ll)$ embeds in $(E, \leq, \ll_E)$ and $\ll_E$ is coarser than $\leq_E$. If we expand $\mathbf{E}^-$ by adding $\ll_E$, we get a model $\mathbf{E}$ of T. Since $(A, \ll)$ embeds in $(D, \ll)$ and $(D, \ll)$ embeds in $(E, \ll_E)$, we get that $(A, \ll)$ embeds in $(E, \ll_E)$, and the same for $(B, \ll)$. On the other hand, since $\mathbf{E}^-$ extends both $\mathbf{A}^-$ and $\mathbf{B}^-$, this takes care of the language of T^-. Thus $\mathbf{E}$ superamalgamates $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$.

(Part (2)) Recall that this is the case when $\ll$ is supposed to be a linearization of $\leq$.

If $\ll$ is a linearization of $\leq$ in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, then, in particular, $\ll$ is an order coarser than $\leq$. We can thus apply (1) in order to get a model $\mathbf{E}$ amalgamating $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$, where $\ll_E$ in $\mathbf{E}$ is a (not necessarily linear) order coarser than $\leq_E$. Let $\mathbf{F}$ be obtained from $\mathbf{E}$ by replacing $\ll_E$ by a linearization $\ll_F$ of $\ll_E$. The only thing to be checked is that $\mathbf{F}$ extends $\mathbf{A}$ and $\mathbf{B}$. If $a, b \in A$, $a \neq b$ and $a \ll_F b$, then it is not the case that $b \ll_A a$, since $\ll_E$ extends $\ll_A$, and $\ll_F$ is antisymmetric and coarser than $\ll_E$. Since $\ll_A$ is a linear order, $a \ll_A b$. The case $a \neq b \in B$ is treated in the same way. Thus $\mathbf{F}$ amalgamates $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$.

(Part (3)) This is the case when $\ll$ is supposed to be a reflexive order relation finer than $\leq$.

Given $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, argue as in (1) to obtain a model $\mathbf{E}^-$ superamalgamating $\mathbf{A}^-$ and $\mathbf{B}^-$ over $\mathbf{C}^-$ and extending both $\mathbf{A}^-$ and $\mathbf{B}^-$. Define $\ll$ on E by setting $e \ll f$ if either $e = f$, or (e, f) belongs to $\ll_A \cup \ll_B \cup (\ll_A \circ \ll_B) \cup (\ll_B \circ \ll_A)$. The arguments in Remark 3.3 show that $\ll$ is a partial order extending both $\ll_A$ and $\ll_B$. Since, by assumption, $\ll_A$ is finer than $\leq_A$, and similarly for $\mathbf{B}$, we get that $\ll$ is finer than $\leq$ in E, since $\leq$ is transitive on E. Expanding $\mathbf{E}^-$ by adding $\ll$, as defined above, we get a superamalgamating model.

In each of the cases (1)–(3), the last statement in Theorem 2.2 follows from the fact that in the above arguments the definition of $\leq_E$ does not depend on the construction of $\ll_E$, so that we can perform the construction simultaneously for each $\ll^i$, observing that no axiom ties together such relations.

(Theorem 2.1, Fraïssé limits, quantifier elimination and model-completion, Parts (1)–(3)) The remaining statements in Theorem 2.1(1)–(3) follow from the just proved amalgamation property using standard model-theoretical arguments. In the cases of partial orders, semilattices and lattices, the joint embedding property for T follows from the amalgamation property, since we are allowed to consider $\mathbf{C}$ as an empty structure. In the cases of Boolean and Heyting algebras, by the assumptions stated right before Theorem 2.1, there is a unique $\emptyset$-generated structure, which can be taken as $\mathbf{C}$, thus the amalgamation property for T provides the joint embedding property in such cases, as well. Note that if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are finite, then, in view of the last statement in Theorem 3.2, the above proof of superamalgamation provides a finite amalgamating model $\mathbf{E}$ in each case. Hence the class of finite models of T has the amalgamation property and the joint embedding property.

Then use [19, Chapter 7], specifically, Theorems 7.1.2 and 7.4.1 therein. As far as the statement about the model completion is concerned, see [22, Fact 2.1(3)]; the proof here does not apply to lattices and Heyting algebras, since such theories are not locally finite. $\square$

4. Posets with an auxiliary relation; causal spaces

If $\mathbf{S} = (S, \leq)$ is a poset, an *auxiliary relation* [17, Definition I-1.11] on $\mathbf{S}$ is a binary relation $\ll$ finer than the order and satisfying

$$w \leq x \ll y \leq z \text{ implies } w \ll z, \tag{A}$$

for all $w, x, y, z \in S$. Since $\ll$ is finer than $\leq$ and $\leq$ is antisymmetric, $\ll$ is antisymmetric, as well. Note that the definition neither requires $\ll$ to be reflexive, nor to be antireflexive. Since $\leq$ is assumed to be reflexive, (A) is equivalent to the conjunction of the following two conditions.

$$w \leq x \ll y \text{ implies } w \ll y, \tag{A1}$$

$$x \ll y \leq z \text{ implies } x \ll z. \tag{A2}$$

Since $\ll$ is finer than $\leq$, it follows that $\ll$ is transitive.

If one further requires that $\ll$ is antireflexive, then the structure $(S, \leq, \ll)$ is a *causal space* in the terminology from [27]. If this is the case, $\ll$ is a strict partial order. Causal spaces should not to be confused with the different notion of a *causal set* [40]. In [27] another relation $\to$ is considered, but it can be defined in terms of $\leq$ and $\ll$ by a quantifier-free formula, hence $\to$ does not modify the notion of an embedding. The formal similarities between the theory of posets with an auxiliary relation and the theory of causal spaces might have a deeper meaning; see [37] for a discussion.

Condition (A) has been considered in models with further structure, as one of the defining properties for a *subordination* relation, e. g., clause (S4) in either [1, Definition 2.1], [8, Definition 9], or [6, Definition 1]. Partial orders with a further binary relation satisfying (A) have been considered also in [10], in connection with Conway theory of combinatorial games. See the quoted papers for credits to original sources and further references.

We now embark on the proofs of some generalizations of Theorems 2.1 and 2.2. The proofs will occupy this section and the next one. The present section essentially deals with posets with a further binary relation, which might be a comparable order, an auxiliary relation or, possibly, a relation satisfying a set of distinct properties. In the next section we will first prove an extension of Lemma 3.1, according to which an additional binary relation on a poset $\mathbf{P}$ can be lifted to a poset extending $\mathbf{P}$. The superamalgamation property for all the theories considered here then allows us to prove Theorem 2.2 in a slightly more general form in which the additional relation $\ll$ is allowed to satisfy a range of possibilities.

CONVENTIONS 4.1. We will consider the following properties of a binary transitive relation R.

$\mathfrak{r}$. R is reflexive; $\mathfrak{ar}$. R is antireflexive; $\mathfrak{as}$. R is antisymmetric.

We will consider two binary relations $\ll$ and $\leq$. By F and C, respectively, we mean the conditions that $\ll$ is finer, coarser, respectively, than $\leq$. $A1$ and $A2$ will refer to the conditions in equations (A1) and (A2).

In what follows we will consider various combinations of the above properties. Notice that, for certain combinations, we get a trivial conclusion, or a trivial class of structures. For example, if both F and C hold, then necessarily $\ll$ is equal to $\leq$, hence all the results we prove reduce to corresponding results for a single relation. If $\ll$ is reflexive and $A1$ or $A2$ holds, then necessarily $\ll$ is coarser than $\leq$, thus if also F holds, then again $\ll$ is equal to $\leq$. On the other hand, we cannot have both $\mathfrak{r}$ and $\mathfrak{ar}$ for the same relation (unless the domain is empty). Similarly, we cannot have $\leq$ reflexive, $\ll$ antireflexive and C. In order for the statements of the following theorems to be formally true we must allow structures with empty domain.

THEOREM 4.2. *Suppose that $P, Q \subseteq \{\mathfrak{r}, \mathfrak{ar}, \mathfrak{as}\}$ and $N \subseteq \{F, C\}$.*

Let $T = T_{P,Q,N}$ be the theory asserting that $\ll$ and $\leq$ are transitive binary relations, that $\leq$ satisfies the properties in P, $\ll$ satisfies the properties in Q and that the properties in N are satisfied.

Then T has the superamalgamation property with respect to $\leq$. The class of finite models of T has a Fraïssé limit $\mathbf{M}$. The first-order theory of $\mathbf{M}$ is ω-categorical, has quantifier elimination and is the model completion of T.

PROOF: Superamalgamation follows from the arguments in [20] recalled in Remark 3.3, since possible coarseness relations are preserved by the construction. The last two sentences in the theorem are proved as in the final part of the proof of Theorem 2.1(1)–(3) in Section 3. □

THEOREM 4.3. *The conclusions in Theorem 4.2 still hold under the assumptions* $P, Q \subseteq \{\mathfrak{r}, \mathfrak{ar}, \mathfrak{as}\}$ *and* $N \subseteq \{F, C, A1, A2\}$*, provided that*

(b1) $F \in N$ *and*

(b2) $\mathfrak{as} \in P$*, or* $\mathfrak{ar} \in Q$*, or* $\mathfrak{as} \notin Q$*.*

PROOF: The proof is divided into cases, according to whether $A1$, $A2$ belong to N.

(0) $A1 \notin N$ and $A2 \notin N$. This follows from Theorem 4.2.

(I) $A1 \notin N$ and $A2 \in N$.

Define $\leq_D$ over $D = A \cup B$ as in Remark 3.3, thus $(D, \leq_D)$ superamalgamates the $\leq$-reducts of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. Recall that $\leq_D$ has been defined as the union of $\leq_A$ $\leq_B$ $\leq_A \circ \leq_B$ and $\leq_B \circ \leq_A$.

Define $\ll$ over $D = A \cup B$ to be the union of $\ll_A$, $\ll_B$, $\ll_A \circ \leq_B$ and $\ll_B \circ \leq_A$, and consider the model $\mathbf{D} = (D, \leq, \ll)$. In order to lighten the notation, from now on unsubscripted relations are always meant as interpreted in $\mathbf{D}$.

We first show that, say, $\mathbf{A}$ is actually a substructure of $\mathbf{D}$. The $\leq$-part follows from the mentioned Remark 3.3. If $a, a' \in A$ and $a \ll_A a'$, then $a \ll a'$, by the definition of $\ll$. In the other direction, suppose that $a, a' \in A$ and $a \ll a'$ holds in $\mathbf{D}$ because, say, (a, a') belongs to $\ll_A \circ \leq_B$, that is,

$$a \ll_A c \leq_B a', \tag{4.1}$$

for some $c \in D$. Then $c \in A$, because of the $\ll_A$ relation, and $c \in B$, because of the $\leq_B$ relation, thus $c \in C$, since $C = A \cap B$. Also, by the $\leq_B$ relation, $a' \in B$, thus $a' \in C$. Since both c and a' belong to C, we get $c \leq_C a'$, since $\mathbf{B}$ extends $\mathbf{C}$. But also $\mathbf{A}$ extends $\mathbf{C}$, thus $c \leq_A a'$. Then (4.1) reads $a \ll_A c \leq_A a'$. By (A2), which holds in $\mathbf{A}$ by assumption, we have $a \ll_A a'$. The other cases are similar or simpler. Similarly, $\mathbf{B}$ is a substructure of $\mathbf{D}$.

We now show that (A2) holds in $\mathbf{D}$. The proof can be given by analyzing all possible cases, but we will provide a more uniform argument. Suppose that $x \ll y$, $y \leq z$ and, say, $x \in A$. In any case, $x \ll y$ is witnessed by a relation of the form $x \ll_A w$, for some $w \in D$, possibly followed by some $\leq_A$ or some $\leq_B$ relation added on the right. For example, if $x \ll y$ is given by $x \ll_B c \leq_A w$, for $c \in C$, then $x \in A \cap B = C$, hence also $x \ll_A c$, since **C** embeds both in **A** and in **B**. Moreover, $y \leq z$ is witnessed by $\leq_A$ and $\leq_B$ relations only. Thus if $x \ll y \leq z$, then x and z can be connected by a chain of relations starting with $\ll_A$ and with possibly further occurrences of $\leq_A$ and $\leq_B$ on the right. Now we show how to reduce the length of such chains of relations. If $x \ll_A w \leq_A v$, then $x \ll_A v$ by (A2) in **A**, thus we may assume that no $\leq_A$ appears immediately after $\ll_A$. If $x \ll_A w \leq_B v \leq_B u$, then $x \ll_A w \leq_B u$ by transitivity of $\leq_B$ in **B**. If $x \ll_A w \leq_B v \leq_A u$, then $w, v \in A \cap B$, hence, by an argument usual by now, $w \leq_A v$. The above relation then reads $x \ll_A w \leq_A v \leq_A u$, hence $x \ll_A u$ by (A2) in **A**. Iterating the above reductions, we get that if $x \in A$ and $x \ll y \leq z$, then $x \ll_A z$, or $x \ll_A w \leq_B z$. The case $x \in B$ is symmetrical. In each case $x \ll z$, by the definition of $\ll$.

Since, by the assumption $F \in N$, $\ll$ is finer than $\leq$ in both **A** and **B**, and $\leq$ is transitive in **D**, we have that $\ll$ is finer than $\leq$ in **D**. Transitivity of $\ll$ then follows from the already proved condition (A2) in **D**.

If $\ll$ is antireflexive in **A** and **B**, then $\ll$ is antireflexive in **D**, since $D = A \cup B$ and both **A** and **B** embed in **D**, as we have already shown. For the same reason, if $\ll$ is reflexive in **A** and **B**, then $\ll$ is reflexive in **D**.

Assume that $\mathfrak{as} \in Q$, that is, $\ll$ is supposed to be antisymmetric; in particular $\ll$ is antisymmetric in **A** and **B**. By the assumption (b2), $\mathfrak{ar} \in Q$, that is, $\ll$ is antireflexive, or $\mathfrak{as} \in P$, that is, $\leq$ is antisymmetric. In the former case $\ll$ is necessarily antisymmetric, since it is both transitive and antireflexive. In the latter case, if $x \ll y$, $y \ll x$, then $x \leq y$, $y \leq x$, since $\ll$ is finer than $\leq$. Thus $x = y$, by antisymmetry of $\leq$.

(II) $A1 \in N$ and $A2 \notin N$. This case is symmetrical to (I).

(III) $A1 \in N$ and $A2 \in N$. This case follows from the next theorem, whose proof refers to the above arguments, but does not use the statement of Theorem 4.3 itself. $\square$

THEOREM 4.4. *The conclusions in Theorem 4.2 still hold under the assumptions* $P, Q \subseteq \{\mathfrak{r}, \mathfrak{ar}, \mathfrak{as}\}$ *and* $\{A1, A2\} \subseteq N \subseteq \{F, C, A1, A2\}$.

Proof: Recall that we need to prove superamalgamation for the theory asserting that $\ll$ and $\leq$ are transitive, that $\leq$ satisfies the properties in P, $\ll$ satisfies the properties in Q and the properties in N are satisfied.

The proof is similar to the proof of case (I) in Theorem 4.3. However, the definition of $\ll$ in $\mathbf{D}$ must be adapted to this case, and we need to show how to get by without the assumptions (b1) and (b2).

Similar to the proof of Theorem 4.3, define D and $\leq$ as in Remark 3.3. In the present case define $\ll$ over $D = A \cup B$ to be the union of $\ll_A$, $\ll_B$, $\ll_A \circ \ll_B$, $\ll_B \circ \ll_A$, $\ll_A \circ \leq_B$, $\leq_A \circ \ll_B$, $\ll_B \circ \leq_A$ and $\leq_B \circ \ll_A$.

The proof that $\mathbf{D}$ extends both $\mathbf{A}$ and $\mathbf{B}$ is not significantly different from (I) in 4.3. If, say, $a, a' \in A$ and $a \ll a'$ in $\mathbf{D}$, then this is witnessed by $a \ll_A a'$ (and there is nothing to prove), or by $a \ll_B a'$ (and then $a, a' \in C$, hence $a \ll_A a'$, again), or by relations of the form $a \trianglelefteq_A c \blacktriangleleft_B a'$ or $a \trianglelefteq_B c \blacktriangleleft_A a'$, for appropriate $\trianglelefteq, \blacktriangleleft$. In any case, $c \in C$ and, say, in the former case, $a' \in B$, hence $a' \in C$, thus the relations can be evaluated in $\mathbf{A}$, providing again $a \ll_A a'$, since (A1) and (A2) hold in $\mathbf{A}$ and $\ll_A$ is transitive.

The proof that $\mathbf{D}$ satisfies (A1) and (A2), too, presents no significant variation with respect to the proof in case (I) in 4.3 that $\mathbf{D}$ satisfies (A2). If, say, $x \leq y$ and $y \ll z$, then these relations are witnessed by chains of relations of the form $u \ll_X v$ or $u \leq_Y v$, for $\mathbf{X}$ and $\mathbf{Y}$ either equal to $\mathbf{A}$ or to $\mathbf{B}$, and at least one $\ll_X$ appearing among the relations witnessing $y \ll z$. Then a reduction similar to the one used in the proof of (A2) in (I) shows that we can reduce to at most two relations, with $\ll_X$ appearing at least once. Indeed, adjacent $\mathbf{A}$-subscripted relations can be evaluated in $\mathbf{A}$, hence can be reduced to a single relation, using (A1), or (A2), or transitivity of $\leq_A$ or of $\ll_A$. Similarly for adjacent $\mathbf{B}$-subscripted relations. If we have a chain of the form

$$w \trianglelefteq_A v \blacktriangleleft_B u \sqsubseteq_A t, \tag{4.2}$$

then necessarily $v, u \in A \cap B = C$, hence $v \blacktriangleleft u$ can be evaluated in $\mathbf{A}$, hence (4.2) reduces to a single relation in $\mathbf{A}$. Since in the above reductions we use (A1), (A2) and transitivity of $\ll$, at least one $\ll$ appears in the reduced expression, since, as we mentioned, at least one $\ll$ appears in the original relations witnessing $y \ll z$.

The proof in the above paragraph also gives transitivity of $\ll$. We now comment on the difference between the present proof and case (I) in the proof of 4.3. In case (I) we used (b1) in order to prove transitivity of $\ll$; we now explain why the additional assumption (b1) is not necessary when we have both (A1) and (A2) at our disposal. The typical example is given by $x, z \in A$, $y \in B$, with $x \ll y \ll z$ given by $x \ll_A c \leq_B y$ and $y \ll_B d \leq_A z$. Note that if we want (A2) to hold in $\mathbf{D}$, the above relations imply $x \ll y$ and $y \ll z$; in other words, the satisfaction of (A2) necessarily requires that $\ll_A \circ \leq_B$ and $\ll_B \circ \leq_A$ appear in the definition of $\ll$ in $\mathbf{D}$. On the other hand, from the above conditions we get $c \leq_B y \ll_B d$, and we need (A1) (not (A2)!) in $\mathbf{B}$ in order to get $c \ll_B d$, hence, as usual, $c \ll_A d$. Now from $x \ll_A c \ll_A d \leq_A z$ we get $x \ll z$ by transitivity of $\ll$ and (A2) in $\mathbf{A}$. However, we needed (A1) in order to perform the above computations, hence the argument cannot be carried over in case (I) in 4.3.

Now we deal with antisymmetry. Suppose that $\ll$ is antisymmetric in $\mathbf{A}$ and $\mathbf{B}$ and $x \ll y \ll x$. If both x and y are in A, then $x = y$ by antisymmetry of $\ll$ in $\mathbf{A}$, and similarly for $\mathbf{B}$. Hence suppose, say, $x \in A$ and $y \in B$. We first deal with the case when $x \ll y \ll x$ is witnessed by $x \ll_A c \ll_B y$ and $y \ll_B d \ll_A x$, for $c, d \in C$, then we will show that all the other cases can be reduced to the preceding case. From $c \ll_B y \ll_B d$ we get $c \ll_B d$ by transitivity of $\ll_B$ and then $c \ll_C d$. Symmetrically $d \ll_C c$, hence $c = d$, by antisymmetry of $\ll_C$. Thus $x \ll_A c = d \ll_A x$, hence $x = c$, by antisymmetry of $\ll_A$. Symmetrically, $y = c$, thus $x = y$. Now consider the general case in which $x \leq_A c$ might appear among the relations witnessing $x \ll y$. Since we have already proved that $\ll$ is transitive, $x \ll x$, and this relation can be evaluated in $\mathbf{A}$, since we have already shown that $\mathbf{A}$ embeds in $\mathbf{D}$. From $x \ll_A x \leq_A c$ we get $x \ll_A c$ by (A2) in $\mathbf{A}$. Symmetrically, if $d \leq_A x$, then $d \leq_A x \ll_A x$, hence $d \ll_A x$, by (A1). Performing the symmetrical computations with respect to y, we can reduce all the $\leq$ relations to $\ll$ relations. This means that we always can reduce to the case we have already treated.

If $\ll$ is finer (coarser) than $\leq$ in all the models to be amalgamated, then, by the definitions, $\ll$ is finer (coarser) than $\leq$ in $\mathbf{D}$. $\square$

5. Ordered structures with the superamalgamation property (continued)

In this section we show that versions of Theorems 4.2, 4.3, 4.4 apply, more generally, to classes of ordered structures, provided the superamalgamation property holds. The proofs in the previous section are heavily used here. Recall the conditions (A1), (A2) and the Conventions 4.1 from the previous section.

LEMMA 5.1. *Suppose that $Q \subseteq \{\mathfrak{r}, \mathfrak{ar}, \mathfrak{as}\}$ and $N \subseteq \{F, C, A1, A2\}$.*

Let $(D, \leq_D)$ and $(E, \leq_E)$ be posets, and suppose that $(E, \leq_E)$ extends $(D, \leq_D)$. If $\ll_D$ is a binary transitive relation on D satisfying Q, and N is satisfied in $(D, \leq_D, \ll_D)$, then there is a transitive relation $\ll_E$ on E such that Q and N are satisfied in $(E, \leq_E, \ll_E)$ and moreover $(E, \leq_E, \ll_E)$ extends $(D, \leq_D, \ll_D)$.

PROOF: Again, we will consider various cases.

(0) $A1 \notin N$ and $A2 \notin N$. This case is divided into two subcases. (0i) $C \notin N$. The subcase is elementary; it is enough to let $e \ll_E f$ on E if and only if $e, f \in D$ and $e \ll_D f$ in D (if $\mathfrak{r} \in Q$, in addition we let $e \ll_E e$, for all $e \in E$). By construction, $\ll_E$ extends $\ll_D$. Since $\ll_D$ is transitive on D, the new relation $\ll_E$ is transitive. Similarly, if $\ll_D$ is antisymmetric, then $\ll_E$ is antisymmetric. Suppose that $F \in N$, that is, $\ll$ is required to be finer that $\leq$. If this holds in $\mathbf{D}$, then it holds in $\mathbf{E}$, as well.

(0ii) $C \in N$. This is essentially the content of Lemma 3.1. Just note that antisymmetry of $\ll_D$ is used there only in the proof of antisymmetry of $\ll_E$.

(I) $A1 \notin N$ and $A2 \in N$. First, note that if $C \in N$, that is, $\ll$ is assumed to be coarser than $\leq$, then (A2) (and also (A1)) are automatically verified, if $\ll$ is transitive. Hence, in view of (0ii) above, in what follows we may assume that $C \notin N$.

Let $e \ll_E f$ on $\mathbf{E}$ if $e \in D$ and there is $g \in D$ such that $e \ll_D g \leq_E f$ (again, if $\mathfrak{r} \in Q$, we additionally let $e \ll_E e$, for all $e \in E$). Condition (A2) is verified, since $\leq_E$ is transitive. We have that $\mathbf{E}$ actually extends $\mathbf{D}$, since if $e \ll_D f$, then we get $e \ll_E f$ by taking $g = f$. In the other direction, if $f \in D$ and $e \ll_E f$, as witnessed by $e \ll_D g \leq_E f$, then $g \leq_D f$, since $(E, \leq_E)$ extends $(D, \leq_D)$. Thus $e \ll_D g \leq_D f$, hence $e \ll_D f$ by (A2) in $\mathbf{D}$. In particular, if $\ll_D$ is antireflexive, then $\ll_E$ is antireflexive, since then $\mathfrak{r} \notin Q$ and if $e \ll_E e$, then $e \in D$. If $\ll_D$ is finer than $\leq_D$, then $\ll_E$ is finer than $\leq_E$, by transitivity of $\leq_E$.

We now check transitivity of $\ll_E$. If $e \ll_D g \leq_E f \ll_D h \leq_E k$, then $e, g, f, h \in D$, hence we get $e \ll_D h$ by (A2) and transitivity of $\ll$ in $\mathbf{D}$, thus $e \ll_D h \leq_E f$, that is, $e \ll_E f$. This proves transitivity of $\ll_E$.

It remains to deal with antisymmetry when $\mathfrak{as} \in Q$. If $e \ll_E f \ll_E e$ is witnessed by $e \ll_D g \leq_E f \ll_D h \leq_E e$, then $e, g, f, h \in D$, hence $e \ll_D g \leq_D f \ll_D h \leq_D e$, since $\leq_E$ extends $\leq_D$, thus $e \ll_D f \ll_D e$ by (A2) in $\mathbf{D}$. Then $e = f$ by antisymmetry of $\ll_D$.

(II) The case $A1 \in N$ and $A2 \notin N$ is proved symmetrically.

(III) $A1 \in N$ and $A2 \in N$. The case is similar to (I), setting $e \ll_E f$ on $\mathbf{E}$ if there are $h, g \in D$ such that $e \leq_E h \ll_D g \leq_E f$ (as usual, if $\mathfrak{r} \in Q$, we additionally let $e \ll_E e$, for all $e \in E$). We point out the main differences.

Transitivity of $\ll_E$ needs only a small variation. If

$$e \leq_E h \ll_D k \leq_E f \leq_E p \ll_D q \leq_E g \tag{5.1}$$

then $k \leq_E p$, hence $k \leq_D p$, since $k, p \in D$. From $h \ll_D k \leq_D p \ll_D q$ we get $h \ll_D q$ by (A2) and transitivity of $\ll_D$. Hence $e \leq_E h \ll_D q \leq_E g$, which is what we had to show.

A small detail needs to be worked out also in the case $\mathfrak{ar} \in P$. If, by contradiction, $e \ll_E e$, for some $e \in E$, that is, $e \leq_E h \ll_D g \leq_E e$, for some $h, g \in D$, then $g \leq_E e \leq_E h$, hence $g \leq_E h$, $g \leq_D h$, then $h \ll_D g \leq_D h$, thus $h \ll_D h$, by (A2), contradicting antireflexivity of $\ll_D$, that is, contradicting $\mathfrak{ar} \in P$.

The proof that $\ll_E$ is antisymmetric if $\ll_D$ is antisymmetric is more involved. Suppose that $e \ll_E f \ll_E e$ is witnessed by (5.1) with $g = e$. Then $k, p \in D$, $k \leq_E p$, hence $k \leq_D p$, thus we get $h \ll_D p$ from (A2) and $h \ll_D k \leq_D p$. Symmetrically, $p \ll_D h$, hence $h = p$, by antisymmetry of $\ll_D$, hence also $h \ll_D p = h$. Similarly, $k = q$, using (A1). From (5.1), recalling that, at present, $g = e$, we have $k = q \leq_E g = e \leq_E h$, thus $k \leq_D h$. Then from $h \ll_D h$ and (A1) we get $k \ll_D h$. Since also $h \ll_D k$, we get $h = k$ by antisymmetry of $\ll_D$. Then (5.1) reads $e \leq_E h = k \leq_E f \leq_E p = h = k = q \leq_E g = e$, thus we get $e = f$ by transitivity and antisymmetry of $\leq_E$. □

THEOREM 5.2. *Suppose that $\mathcal{K}$ is a class of ordered structures with order $\leq$ and $\mathcal{K}$ has the superamalgamation property. Let $Q \subseteq \{\mathfrak{r}, \mathfrak{ar}, \mathfrak{as}\}$ and $N \subseteq \{F, C, A1, A2\}$. Suppose further that*

(a) $A1 \notin N$, $A2 \notin N$, or

(b) $F \in N$, or

(c) $\{A1, A2\} \subseteq N$.

Let $\mathcal{K}_{Q,N}$ be the class of expansions of models of $\mathcal{K}$ in a language with a further binary relation $\ll$, with the conditions that $\ll$ is transitive, $\ll$ satisfies the properties in Q and that the properties in N are satisfied.

Then $\mathcal{K}_{Q,N}$ has the superamalgamation property with respect to $\leq$.

PROOF: The proof presents no essential difference with respect to the proof of superamalgamation in the proof of Theorem 2.1(1) given at the end of Section 3. Here just consider members of $\mathcal{K}$, resp., $\mathcal{K}_{Q,N}$ in place of models of T^-, resp., T and use the more general Theorems 4.2, 4.3, 4.4 and Lemma 5.1 in place of Remark 3.3 and Lemma 3.1. Note that, in the cases at hand, $P = \{\mathfrak{r}, \mathfrak{as}\}$, hence clause (b2) in Theorem 4.3 is automatically satisfied. □

PROOF: *(Proof of Theorems 2.1 and 2.2 (continued))* Amalgamation for Part (4) is the special case $Q = \emptyset$ and $N = \{F, A1, A2\}$ of Theorem 5.2. For the theory of causal spaces just take $Q = \{\mathfrak{ar}\}$ instead. The proof at the end of Section 3 for the existence of Fraïssé limits, etc., works in the present cases, as well. □

6. Further remarks

Remark 6.1. In this remark we show that the superamalgamation property is necessary in Theorem 2.2(1), as well as in Theorem 5.2, even when $\ll$

is assumed to be a partial order. In other words, we do need Theorem 3.2 in the proof of Theorem 2.1(1). See (iii) below. Some of the arguments in the present remark partially overlap with [39].

(i) We first observe that the theory T_{uc} of partial orders which are expressible as pairwise incomparable unions of chains (an "antichain of chains") has SAP. The theory T_{uc} is a first-order universal theory: just add to the theory of partial orders the sentence

$$\forall xyz(x \geq z \,\&\, y \geq z \Rightarrow x \geq y \text{ or } y \geq x), \tag{6.1}$$

together with its dual. SAP for T_{uc} follows from SAP for the theory of linearly ordered sets. For every chain in the amalgamating base **C**, amalgamate each corresponding triplet of chains as in the case of linearly ordered sets. If **A** and **B** have more chains not connected to elements of **C**, just take their disjoint unions in the amalgamating model **D**.

(ii) On the other hand, if we add another binary relation $\ll$ and add to T_{uc} axioms asserting that $\ll$ is transitive and coarser than $\leq$, AP fails.

Let $C = \{c, d, e\}$, with all the elements pairwise incomparable both with respect to $\leq$ and to $\ll$, and both $\leq$, $\ll$ reflexive. Let $A = C \cup \{a\}$ with $d \leq a$, $d \ll a$, $c \ll a$, $a \ll a$ and no further $\ll$-relation, in particular, not $e \ll a$. Let $B = C \cup \{b\}$ with $d \leq b$, $d \ll b$, $e \ll b$, $b \ll b$ and not $c \ll b$. If **D** is an amalgamating structure which is a $\leq$-union of disjoint chains, a and b should be $\leq$-comparable, say, $a \leq b$. If $\ll$ is coarser than $\leq$ on **D**, then $b \ll a$. If $\ll$ is transitive, then $e \ll a$ in **D**, but this contradicts the requirement that **A** embeds in **D**, since $e \ll a$ fails in **A**.

(iii) More generally, the counterexample in (ii) above shows that the class of posets which are union of at most 3 chains and with a further coarser partial order does not have AP in the class of posets which are union of disjoint chains and with a further coarser transitive binary relation.

In particular, if T^- is T_{uc} and T is obtained by adding an axiom asserting that $\ll$ is a partial order coarser than $\leq$, then T does not have AP. This shows that the assumption that $\mathcal{K}$ has the superamalgamation property in Theorems 2.2(1) and 5.2 (case (a), N= {C}) cannot be weakened to SAP.

Remark 6.2. (a) On the other hand, in the proof of Theorem 2.2(3) only the strong amalgamation property is used (if we want to obtain the weaker conclusion that T has the strong amalgamation property).

(b) The strong amalgamation property is necessary in the above remark, the amalgamation property alone is not enough. Recall that distributive lattices have AP but not SAP [18].

The theory of distributive lattices with a further finer order relation $\ll$ does not have AP. Let $\mathbf{C}$ be a 3-element $\leq$-chain with middle element c and all distinct elements $\ll$-incomparable. Both in $\mathbf{A}$ and in $\mathbf{B}$ add a complement a, resp. b, of c. Let a be $\ll$-incomparable with all the other elements of A, while let $b \ll 1$ in $\mathbf{B}$. Since complements are unique in distributive lattices, we have $\iota(a) = \kappa(b)$ in any amalgamating structure, but then $\iota(a)$ incomparable with $\iota(1)$ contradicts $\iota(a) = \kappa(b) \leq \kappa(1) = \iota(1)$.

(c) Our proofs of Theorem 2.2(2),(4) do use the superamalgamation property, but we do not know whether a proof can be found which uses only the strong amalgamation property, provided, as above, we want to obtain the weaker conclusion that T has the strong amalgamation property.

Problem 6.3. Study the Fraïssé limits whose existence follows from Theorems 2.1, 4.2–4.4.

In particular, may the Fraïssé limit of the theory of causal spaces have some physical significance?

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