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A NOTE ON THE NUMBER OF SUBNORMAL MODAL LOGICS

Abstract

In the article we discuss the notion of a subnormal modal logic and, in particular, the notion of an intermediate subnormal modal logic. The goal of the article is to determine the cardinality of the set of all intermediate subnormal modal logics. Thus we define a set of certain formulas, such that each subset of the defined set determines a different intermediate subnormal modal logic. As we demonstrate, since the set in question is countable, the set of all intermediate subnormal modal logics has cardinality of the continuum.

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1. Preliminaries

As it is usually defined [1], a normal modal logic is one which contains all formulas of the form $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ (the K-schema) and is closed under the necessitation rule (RN: $\frac{\alpha}{\Box\alpha}$)[1]. The weakest normal modal logic is usually denoted by **K**. Hence, a logic is *non*-normal when it either rejects at least some instances of the K-schema, or restricts (or rejects) the scope of RN (or a little bit of both). Non-normal logics have been extensively studied in, among others, [3, 4, 5, 7, 8, 9, 11, 12]. Among non-normal modal logics we can distinguish those which are closed under RN but do not contain the K-schema. This is the meaning of the word 'subnormal' adopted in [2], where the weakest (classical) subnormal modal logic is studied, that is the system **N** – first introduced in [6], which is simply a classical propositional logic (in the language with $\Box$) extended with RN.

Such notion of a subnormal modal logic allows for logics which are not necessarily 'under' the system **K**. For instance, a logic determined by PC, RN and the schema $\Box\alpha \to \alpha$, which was studied in [10], is neither weaker, nor stronger than **K**, it is incomparable to **K**. Hence, we can further distinguish certain subnormal modal logics. Having **N** and **K** before our eyes, it seems natural to consider those logics which are stricly stronger than **N** and strictly weaker than **K**, that is – intermediate logics.

In the article, we want to answer the question: how many logics intermediate between **N** and **K** are there? Our strategy is as follows. Given a standard modal language $\mathcal{L}$ and a standard definition of a formula, we know there are no more than continuum-many sets of formulas of any kind, and we will treat logics as special sets of formulas. In order to show that the number of all intermediate subnormal modal logics is continuum, we will distinguish a countable set of certain theorems of **K** which are not theorems of **N**, and show that each subset of the distinguished set determines a different logic. Since there are continuum-many subsets of a countable set, it will follow that there are continuum-many intermediate subnormal

[1] Here we will only speak of classical modal logics, that is logics containing PC – the set of all classical propositional tautologies, and closed under *Modus Ponens*.

modal logics.[2]

We will adopt a standard definition of a formula. However, since our investigations will depend on the syntactic differences between formulas, we will not omit parentheses or simplify the notation in a way that would obscure our intentions.

Definition 1.1. Let $Var := \{p_0, p_1, p_2, \ldots\}$ be the set of denumerably many propositional variables. We take standard connectives $\mathcal{C} = \{\neg, \wedge, \vee, \rightarrow \Box\}$, their syntactic function is as usual. The language $\mathcal{L}$ is defined as $\mathcal{L} = (Var, \mathcal{C})$. By Fm we denote the set of all (well-formed) formulas of $\mathcal{L}$, defined as follows:

1. $Var \subseteq Fm$,
2. if $\alpha, \beta \in Fm$, then $\neg\alpha, \Box\alpha, (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta) \in Fm$,
3. nothing more is in Fm.

We will use $(\alpha \leftrightarrow \beta)$ as an abbreviation of $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$.

Given the requirement that a logic must be closed under RN, it might seem natural to think of subnormal modal logics as not containing the K-schema by definition. However, since we are interested only in those logics which do not contain the K-schema, it is unnecessary to require from logic not to contain the K-schema. Given RN, the difference between normal and subnormal modal logics lies in *not* requiring subnormal modal logics to contain the K-schema, in opposition to such a requirement for normal modal logics. This gives us a more general definition of a subnormal modal logic.

Before we formulate the definition, we need to mention substitutions. As usual, a substitution is a mapping $s \colon Var \longrightarrow Fm$ extendable to a function $\bar{s} \colon Fm \longrightarrow Fm$ in a standard way, so that $\bar{s}((\alpha \circ \beta)) = (\bar{s}(\alpha) \circ \bar{s}(\beta))$ for all $\circ \in \{\wedge, \vee, \rightarrow\}$, and $\bar{s}(\triangle\alpha) = \triangle\bar{s}(\alpha)$ for $\triangle \in \{\neg, \Box\}$. Since each substitution s uniquely determines $\bar{s}$, we will write s instead of $\bar{s}$. The set

[2] This result has also been proved by the author of this article in [7] using a different kind of semantics.

of all substitutions will be denoted by 'Sub'. Moreover, for any $X \subseteq Fm$, we define $Sub(X) := \{s(\alpha)\colon \alpha \in X \text{ and } s \in Sub\}$.

Definition 1.2. Let $X \subseteq Fm$. A subnormal modal logic determined by X, denoted by $L(X)$, is the smallest set of formulas satisfying the following conditions:

1. $PC \subseteq L(X)$,
2. $Sub(X) \subseteq L(X)$,
3. if $\alpha \in L(X)$ and $(\alpha \to \beta) \in L(X)$, then $\beta \in L(X)$,
4. if $\alpha \in L(X)$, then $\Box\alpha \in L(X)$.

Formulas from $Sub(X)$ are called *axioms of* $L(X)$. A special case of subnormal modal logics are **K** and **N**. We have:

$$\mathbf{K} = L(\{(\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1))\}) \text{ and } \mathbf{N} = L(\varnothing).$$

Definition 1.3. A subnormal modal logic L is called *intermediate* iff L is a proper subset of **K**[3].

It follows that an intermediate subnormal modal logic cannot contain the schema $(\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))$, for then it would be identical with **K**.

2. Semantics

We are going to use semantics introduced by Fitting, Marek and Truszczyński in [2]. The key idea behind this kind of semantics is to be able to distinguish logically equivalent formulas under the scope of $\Box$ without invalidating the necessitation rule, since **N** is not closed under the extensionality rule $(\frac{\alpha \leftrightarrow \beta}{\Box\alpha \leftrightarrow \Box\beta})$ (see [7], p. 76). It is achieved by associating a binary relation on possible worlds with *each* formula. So now $\Box\gamma$ is true at a possible world

[3] Although it seems natural to exclude **N** from intermediate logics, we let it slide, since we do not need to exclude the empty set of axioms in further proofs for them to work.

x when α is true at all worlds α-accessible from x. And even though, for instance, $p \vee q$ is logically equivalent with $\neg(\neg p \wedge \neg q)$, $\Box(p \vee q)$ and $\Box\neg(\neg p \wedge \neg q)$ are *not* logically equivalent, since $p \vee q$ may be true at all $p \vee q$-accessible worlds from x, while $\neg(\neg p \wedge \neg q)$ may fail at some $\neg(\neg p \wedge \neg q)$-accessible world from x. Even logically equivalent formulas display different semantic behaviour, since the semantics takes into account the very *shape* of a formula as a factor in defining truth-conditions for $\Box$. For more philosophy behind the semantics for subnormal modal logic see [7], Part I.

Definition 2.1. A model is a structure $\mathbb{M} = (M, \{R_\alpha\}_{\alpha \in Fm}, e)$, where M is a non-empty set (of possible worlds), each R_α is a binary relation on M, and e is a function (valuation) from $M \times Var$ to $\{0, 1\}$. Truth (satisfaction) of a formula at a given possible world – symbolically: $(\mathbb{M}, x) \Vdash \alpha$ – is defined as follows.

(a) $(\mathbb{M}, x) \Vdash p_i \iff e(x, p_i) = 1$, for $p_i \in Var$,

(b) $(\mathbb{M}, x) \Vdash \neg\alpha \iff (\mathbb{M}, x) \nVdash \alpha$,

(c) $(\mathbb{M}, x) \Vdash (\alpha \wedge \beta) \iff (\mathbb{M}, x) \Vdash \alpha$ and $(\mathbb{M}, x) \Vdash \beta$,

(d) $(\mathbb{M}, x) \Vdash (\alpha \vee \beta) \iff (\mathbb{M}, x) \Vdash \alpha$ or $(\mathbb{M}, x) \Vdash \beta$,

(e) $(\mathbb{M}, x) \Vdash (\alpha \to \beta) \iff (\mathbb{M}, x) \nVdash \alpha$ or $(\mathbb{M}, x) \Vdash \beta$,

(f) $(\mathbb{M}, x) \Vdash \Box\alpha \iff (\forall y \in M)(xR_\alpha y \Rightarrow (\mathbb{M}, y) \Vdash \alpha)$.

For $Y \subseteq Fm$, we write $(\mathbb{M}, x) \Vdash Y$ to mean that $(\mathbb{M}, x) \Vdash \delta$ for all $\delta \in Y$. We also write $\mathbb{M} \Vdash \alpha$ when $(\mathbb{M}, x) \Vdash \alpha$ for all $x \in M$.

It follows that the necessitation rule preserves satisfaction in a model.

Lemma 2.2. *If* $\mathbb{M} \Vdash \alpha$*, then* $\mathbb{M} \Vdash \Box\alpha$*.*

Proof: Suppose $\mathbb{M} \nVdash \Box\alpha$. Then $(\mathbb{M}, b) \nVdash \Box\alpha$ for some $b \in M$. Hence there is $c \in M$ such that $bR_\alpha c$ and $(\mathbb{M}, c) \nVdash \alpha$. So there is $c \in M$ such that $(\mathbb{M}, c) \nVdash \alpha$. Therefore $\mathbb{M} \nVdash \alpha$. $\Box$

We also state the following lemma as a formality.

Lemma 2.3. *If* $\mathbb{M} \Vdash \alpha$ *and* $\mathbb{M} \Vdash (\alpha \to \beta)$*, then* $\mathbb{M} \Vdash \beta$*.*

In [2] we find the proof of the completeness theorem.

THEOREM 2.4. $\alpha \in \mathbf{N} \iff$ *for every model* $\mathbb{M}, \mathbb{M} \Vdash \alpha$.

In the light of Lemmas 2.2, 2.3 and Theorem 2.4, in order to prove that a given model $\mathbb{M}$ satisfies all theorems of a logic $L(X)$ for $X \subseteq Fm$, it suffices to show that $\mathbb{M}$ satisfies specific axioms of $L(X)$, that is $Sub(X)$.

3. Cardinality

In this section we will take advantage of the fact that it is not generally the case in subnormal modal logics that $\Box\alpha$ and $\Box\beta$ are equivalent when α and β are equivalent. Take a model $\mathbb{M} = (\{a\}, \{R_\gamma\}_{\gamma \in Fm}, e)$ such that $e(a, p_i) = 0$ for all $p_i \in Var$, and

$$R_\gamma = \begin{cases} \{(a,a)\} & \text{if } \gamma = ((p_0 \wedge p_1) \wedge p_2), \\ \varnothing & \text{if } \gamma \neq ((p_0 \wedge p_1) \wedge p_2), \end{cases}$$

for every $\gamma \in Fm$. It follows that $(\mathbb{M}, a) \nVdash \Box((p_0 \wedge p_1) \wedge p_2)$ and $(\mathbb{M}, a) \Vdash \Box(p_0 \wedge (p_1 \wedge p_2))$, since $((p_0 \wedge p_1) \wedge p_2) \neq (p_0 \wedge (p_1 \wedge p_2))$, even though

$$(((p_0 \wedge p_1) \wedge p_2) \leftrightarrow (p_0 \wedge (p_1 \wedge p_2))) \in PC.$$

Hence $(\Box((p_0 \wedge p_1) \wedge p_2) \leftrightarrow \Box(p_0 \wedge (p_1 \wedge p_2))) \notin \mathbf{N}$.

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of natural numbers, and let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$.

DEFINITION 3.1. For any $\alpha \in Fm$ and any $n \in \mathbb{N}^+$ we define:

1. $\alpha^1 = \alpha$,
2. $\alpha^{n+1} = (\alpha^n \wedge \alpha)$.

DEFINITION 3.2. Let $\gamma \in Fm$ and $p_i \in Var$. We will say that γ is a p_i-implication iff $\gamma = (\Box p_i \to \Box p_i^n)$ for some $n \in \mathbb{N}^+ \setminus \{1\}$. We further define $I(p_i) := \{\delta \in Fm \colon \delta \text{ is a } p_i\text{-implication}\}$. A subnormal modal logic determined by a subset of $I(p_i)$ will be called a p_i-logic.

Definition 3.3. Let $X \subseteq I(p_i)$. We define

$$nat(X) := \{n \in \mathbb{N} \colon (\exists \beta \in X)(\beta = (\Box p_i \to \Box p_i^n))\}.$$

From these definitions it follows immediately that:

Lemma 3.4. *Let $X \subseteq I(p_i)$.*

- *If $n \in nat(X)$, then $n \geqslant 2$,*
- $(\Box p_i \to \Box p_i^n) \in X \iff n \in nat(X)$,
- $|I(p_i)| = \aleph_0$.

Moreover, we can show that each p_i-logic is in fact intermediate. To do that, we need to prove that every substitution of every p_i-implication is a theorem of $\mathbf{K}$, and that no p_i-logic proves the schema $(\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))$.

Lemma 3.5. $s(\delta) \in \mathbf{K}$, *for any $s \in Sub, \delta \in I(p_i)$.*

Proof: Let $\delta = (\Box p_i \to \Box p_i^k)$, for $k \geqslant 2$. We have $(p_i \to p_i^k) \in PC$, hence $(p_i \to p_i^k) \in \mathbf{K}$. By necessitation, it follows that $\Box(p_i \to p_i^k) \in \mathbf{K}$, therefore, by K-schema and *Modus Ponens*, $(\Box p_i \to \Box p_i^k) \in \mathbf{K}$. And finally, since $\mathbf{K}$ is closed under substitutions, $s((\Box p_i \to \Box p_i^k)) \in \mathbf{K}$. $\Box$

To show that no p_i-logic proves the K-schema, it suffices to show that the strongest of them, that is $L(I(p_i))$, does not prove the K-schema.

Lemma 3.6. $(\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)) \notin L(I(p_i))$ *for some $\alpha, \beta \in Fm$.*

Proof: Consider the model $\mathbb{M} = (\{a\}, \{R_\gamma\}_{\gamma \in Fm}, e)$ such that $e(a, p_0) = 1$ and $e(p_j, a) = 0$ for $j \neq 0$, and moreover:

$$R_\gamma = \begin{cases} \varnothing & \text{if } \gamma = (p_0 \to p_1) \text{ or } (\exists k \geqslant 2)(\exists s \in Sub)(\gamma = s(p_i^k)), \\ \{(a, a)\} & \text{otherwise}, \end{cases}$$

for all $\gamma \in Fm$. Take any $s \in Sub$. It follows that $R_{s(p_i^k)} = \varnothing$ for all $k \geqslant 2$. Thus $(\mathbb{M}, a) \Vdash \Box s(p_i^k)$ for all $k \geqslant 2$, hence $(\mathbb{M}, a) \Vdash (\Box s(p_i) \to \Box s(p_i^k))$. Finally, $(\mathbb{M}, a) \Vdash s((\Box p_i \to \Box p_i^k))$ for all $k \geqslant 2$. It means that $\mathbb{M}$ satisfies all substitutions of all formulas in $I(p_i)$, hence $\mathbb{M}$ satisfies all theorems of

$L(I(p_i))$. Now consider the formula $(\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1))$. We have $R_{(p_0 \to p_1)} = \varnothing$, hence $(\mathbb{M}, a) \Vdash \Box(p_0 \to p_1)$. We also have $R_{p_0} = \{(a,a)\}$ and $e(p_0, a) = 1$, hence $(\mathbb{M}, a) \Vdash \Box p_0$. But $R_{p_1} = \{(a,a)\}$ and $e(p_1, a) = 0$, so $(\mathbb{M}, a) \nVdash \Box p_1$. Thus $(\mathbb{M}, a) \nVdash (\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1))$. □

Corollary 3.7. For all $X \subseteq I(p_i)$, $(\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)) \notin L(X)$.

Corollary 3.7 implies that **N** does not prove the K-schema.

Corollary 3.8. $L(X)$ is intermediate, for all $X \subseteq I(p_i)$.

Let us now prove the most crucial lemma which says that powers of p_i are substitutionally independent – you cannot get p_i^m from p_i^k by any subtitution if $m \neq k$ and m, k are greater than 2. Even though $(p_i^m \leftrightarrow p_i^k) \in PC$ for all $m, k \in \mathbb{N}$, what matters for us is that p_i^m and p_i^k as formulas are different objects if $m \neq k$.

Lemma 3.9. *For all $s \in Sub, n, k \in \mathbb{N}$ such that $n \geqslant 2$ and $k \geqslant 2$,*

$$\text{if } n \neq k, \text{ then } p_i^n \neq s(p_i^k).$$

Proof: Suppose $n \neq k$. Since $n \geqslant 2$ and $k \geqslant 2$, we have $p_i^n = (p_i^{n-1} \wedge p_i)$ and $s(p_i^k) = s((p_i^{k-1} \wedge p_i)) = (s(p_i^{k-1}) \wedge s(p_i))$. Consider $s(p_i)$. Either $s(p_i) \in Var$, or $s(p_i) \notin Var$.

(Case 1) $s(p_i) \in Var$. Now we have two more cases: either $s(p_i) = p_i$, or $s(p_i) \neq p_i$.

(Case 1.1) $s(p_i) = p_i$. Then $s(p_i^k) = p_i^k$. And since $n \neq k$, we get $p_i^n \neq s(p_i^k)$.

(Case 1.2) $s(p_i) \neq p_i$. Then $s(p_i^k)$ contains a variable different from p_i and p_i^n contains only a variable p_i. Hence, $p_i^n \neq s(p_i^k)$.

(Case 2) $s(p_i) \notin Var$. We know that either $s(p_i)$ contains connectives other than $\wedge$, or $s(p_i)$ contains only $\wedge$ and no other connectives.

(Case 2.1) $s(p_i)$ contains connectives other than $\wedge$. Then $s(p_i^k)$ contains connectives other than $\wedge$, and since p_i^n contains only $\wedge$ as a connective, we get $p_i^n \neq s(p_i^k)$.

(Case 2.1) $s(p_i)$ contains only $\wedge$ and no other connectives. Then $s(p_i) = (\alpha \wedge \beta)$ for some $\alpha, \beta \in Fm$. Hence we have $s(p_i^k) = (s(p_i^{k-1}) \wedge\ (\alpha \wedge\ \beta))$. We also have $p_i^n = (p_i^{n-1} \wedge p_i)$. Now, the last two elements of p_i^n – being a sequence of symbols – are p_i and). The last two elements of $s(p_i^k)$ are) and) – the same symbol repeated. Therefore, $p_i^n \neq s(p_i^k)$. $\square$

Finally, we can prove the following theorem.

THEOREM 3.10. *For all $X \cup \{\alpha\} \subseteq I(p_i)$: if $\alpha \notin X$, then $\alpha \notin L(X)$.*

PROOF: Suppose $\alpha \notin X$. Consider a model $\mathbb{M} = (\{a\}, \{R_\gamma\}_{\gamma \in Fm}, e)$ such that $e(a, q) = 0$ for all $q \in Var$ and

$$R_\gamma = \begin{cases} \varnothing & \text{if } \gamma = p_i \text{ or } (\exists n \in nat(X))(\exists s \in Sub)(\gamma = s(p_i^n)), \\ \{(a,a)\} & \text{otherwise,} \end{cases}$$

for all $\gamma \in Fm$. We will show that $(\mathbb{M}, a) \Vdash L(X)$ and $(\mathbb{M}, a) \nVdash \alpha$.

To show that $\mathbb{M}$ satisfies all theorems of $L(X)$, it is enough to prove that $\mathbb{M}$ satisfies all axioms of $L(X)$, that is all formulas in $Sub(X)$. So take any $\delta \in X, s \in Sub$. We have $\delta = (\Box p_i \to \Box p_i^k)$ for some $k \geqslant 2$. Hence, $k \in nat(X)$, and so $R_{s(p_i^k)} = \varnothing$. Therefore, $(\mathbb{M}, a) \Vdash \Box s(p_i^k)$, and then $(\mathbb{M}, a) \Vdash (\Box s(p_i) \to\ \Box s(p_i^k))$. Finally, $(\mathbb{M}, a) \Vdash s((\Box p_i \to \Box p_i^k))$.

We know that $\alpha = (\Box p_i \to \Box p_i^m)$ for some $m \geqslant 2$. Since $\alpha \notin X$, we have $m \notin nat(X)$. Let us show that

$$(\forall n \in nat(X))(\forall s \in Sub)(p_i^m \neq s(p_i^n)). \qquad (\star)$$

Take any $n \in nat(X), s \in Sub$. We have $m \neq n$, $m \geqslant 2$ and $n \geqslant 2$. By Lemma 3.9 we infer $p_i^m \neq s(p_i^n)$.

By $(\star)$ and the fact that $p_i^m \neq p_i$ we get $R_{p_i^m} \neq \varnothing$, hence $R_{p_i^m} = \{(a,a)\}$. We also have $(\mathbb{M}, a) \nVdash p_i^m$, so $(\mathbb{M}, a) \nVdash \Box p_i^m$. Since $R_{p_i} = \varnothing$, we get $(\mathbb{M}, a) \Vdash \Box p_i$. Thus, $(\mathbb{M}, a) \nVdash (\Box p_i \to \Box p_i^m)$. $\square$

Corollary 3.11. $(\forall X, Y \subseteq I(p_i))(X \neq Y \Rightarrow L(X) \neq L(Y))$.

Proof: Without loss of generality assume that $\beta \in X$ and $\beta \notin Y$. Then $\beta \in L(X)$ and, by Theorem 3.10, $\beta \notin L(Y)$. □

Corollary 3.12. Cardinality of the set of all p_i-logics is continuum, for any $p_i \in Var$.

Corollary 3.13. Cardinality of the set of all intermediate subnormal modal logics is continuum.

Corollary 3.14. Cardinality of the set of all subnormal modal logics is continuum.

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