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Akbar Rezaei
Arsham Borumand Saeid

PRE-HILBERT ALGEBRAS WITH CONDITION (GE3)

Abstract

GE-algebras are introduced as a generalization of Hilbert algebras in 2020. In this paper, we show that a self distributive GE-algebra is equivalent to the pimpl-RML algebra, and a transitive GE algebra is equivalent to the pi-pre-BBBCC algebra with condition (An). Further, we prove that pimpl-BE algebras are a subclass of pimpl-RML algebras and pre-Hilbert algebras with condition (GE3) are equivalent to the pi-pre-BBBCC algebras with condition (An).

Keywords: RML algebra, pi-RML algebra, pre-BBBCC algebra, pi-pre-BBBCC algebra, BCK-algebra, (self distributive) BE-algebra, (transitive) GE-algebra.

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1. Introduction

The study of logical algebras has been a rich area for expanding and connecting various algebraic structures that model logical systems. Among the most fundamental of these are **BCI** and **BCK-algebras**, introduced by Iséki [5], Iséki and Tanaka [6], which abstract the implicative fragment of propositional logic. Mundici [10] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, while Meng [8] established that implicative commutative semigroups are equivalent to a class of BCK-algebras. These algebras are characterized by a binary operation $\rightarrow$ and a constant 1, satisfying specific axioms related to implication [2, 9]. This foundation has inspired a wave of generalizations and variations, leading to a diverse properties of related algebraic structures. A key development in this effort was the introduction of **BE-algebras** by Kim and Kim [7], which weaken certain axioms of BCK-algebras. These algebras, defined by the axioms (Re), (M), (L), and the exchange property (Ex), provided a broader framework for studying algebraic logic. Rezaei et al. [11] proved that every Hilbert algebra is a self distributive BE-algebra and commutative self distributive BE-algebra is a Hilbert algebra. A comprehensive effort to systematize and further generalize these concepts was undertaken by Iorgulescu [3, 4], who proposed an extensive taxonomy of algebras including **RML**, **pre-BBBCC**, and their "pi" variants by meticulously combining subsets of fundamental properties like antisymmetry (An), transitivity (Tr), exchange (Ex), and self-distributivity. More recently, new structures have emerged. **GE-algebras** were defined by Bandaru et al. [1] via the identity (GE3): $x \rightarrow (y \rightarrow z) = x \rightarrow [y \rightarrow (x \rightarrow z)]$, offering a new perspective on the interaction between implication and self-applicability. Recently, Walendziak [13] defined **pre-Hilbert algebras** by weakening the axioms of Hilbert algebras, a well-known algebraic counterpart to Henkin's Positive Implicative Logic. While these definitions appear distinct, a natural question arises: how are these various classes of algebras BCI, BCK, BE, RML, GE, pre-Hilbert, and their pi and self-distributive variants related? The literature contains scattered results, but a unified hierarchical overview has been lacking. This paper aims to synthesize these connections by:

- Clarifying the precise relationships between these algebras using their defining properties.
- Proving new equivalences, such as between self-distributive GE-algebras and pimpl-RML algebras, and between transitive GE-algebras and pi-pre-BBBCC algebras with condition (An).
- Providing counterexamples to show where these relationships break down, establishing the properness of inclusions.
- Presenting a comprehensive hierarchy that situates these algebraic structures within a single framework, illuminating the logical landscape they form.

2. Preliminaries

Recall the list of properties:

(An) **(Antisymmetry)** $x \to y = 1$ and $y \to x = 1 \quad \Rightarrow \quad x = y$,

(B) $(y \to z) \to [(x \to y) \to (x \to z)] = 1$,

(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,

(*) $y \to z = 1 \quad \Rightarrow \quad (x \to y) \to (x \to z) = 1$,

(**) $y \to z = 1 \quad \Rightarrow \quad (z \to x) \to (y \to x) = 1$,

(C) $[x \to (y \to z)] \to [y \to (x \to z)] = 1$,

(D) $y \to [(y \to x) \to x] = 1$,

(Ex) **(Exchange)** $x \to (y \to z) = y \to (x \to z)$,

(K) $x \to (y \to x) = 1$,

(L) **(Last element)** $x \to 1 = 1$,

(M) $1 \to x = x$,

(N) $1 \to x = 1 \quad \Rightarrow \quad x = 1$,

(Re) **(Reflexivity)** $x \to x = 1$,

(Tr) **(Transitivity)** $x \to y = 1$ and $y \to z = 1 \quad \Rightarrow \quad x \to z = 1$,

(P) $x \to (y \to z) \leq (x \to y) \to (x \to z)$,

(pimpl) $x \to (y \to z) = (x \to y) \to (x \to z)$,

(pi) $y \to (y \to x) = y \to x$.

A proper RML algebra is a RML algebra ([3, P8]) (i.e. verifying (Re), (M), (L)) not verifying (Ex), (An), (B), (*), (BB), (**), (Tr), (pi).

A proper pi-RML algebra is a pi-RML algebra ([3, P8-pi]) (i.e. verifying (Re), (M), (L), (pi)) not verifying (Ex), (An), (B), (*), (BB), (**), (Tr), (pimpl).

A proper BCI algebra is a BCI algebra ([2, Def. 1.1.1] and [3, PO1]) (i.e. verifying (BB), (D), (Re), (N), (An) or, equivalently, (BB), (D), (Re), (An), or, equivalently, (BB), (M), (An), or, equivalently, (B), (C), (Re), (An)) not verifying (L).

A proper BCK algebra is a BCK algebra ([9, Def. 1.1] and [3, PO2]) (i.e. verifying (BB), (D), (Re), (L), (An), or, equivalently, (BB), (M), (L), (An), or, equivalently, (B), (C), (K), (An)) not verifying (pi) (hence (pimpl)).

A proper BE algebra is a BE algebra ([7, Def. 1] and [3, PO6]) (i.e. verifying (Re), (M), (L), (Ex) (hence (D)) not verifying (An), (Tr), (*) (hence (B)), (**) (hence (BB)), (pi).

A proper pi-BE algebra is a pi-BE algebra ([3, PO6-pi]) (i.e. verifying (Re), (M), (L), (Ex) (hence (D), (pi)) not verifying (An), (B), (BB), (*), (**), (Tr), (pimpl).

A proper pre-BBBCC algebra is a pre-BBBCC algebra ([3, P23]) (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D)) not verifying (An), (Ex), (pi).

A proper pi-pre-BBBCC algebra is a pi-pre-BBBCC algebra ([3, P23-pi]) (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D), (pi)) not verifying (An), (Ex), (pimpl).

3. Pre-Hilbert and GE Algebras

In 2021, Bandaru et al. ([1]) defined a new algebraic structure was named GE-algebra as follows:

A GE-algebra ([1, Def. 3.1]) is a non-empty set X with a constant 1 and a binary operation "$\rightarrow$" satisfying axioms:

(GE1) $x \rightarrow x = 1$,

(GE2) $1 \rightarrow x = x$,

(GE3) $x \rightarrow (y \rightarrow z) = x \rightarrow [y \rightarrow (x \rightarrow z)]$, for all $x, y, z \in X$.

(GE1) used instead of the property (Re) and respectively, (GE2) instead of the property (M).

In 2024, Walendziak defined the notation of pre-Hilbert algebras as a generalization of Hilbert algebras. A pre-Hilbert algebra ([13, Def. 3.5]) is an algebra $(X, \rightarrow, 1)$ of type $(2, 0)$ verifying (M), (K) and (P).

Remark 3.1. It is shown that in a given GE-algebra the properties (L) and (pi) hold ([1, Th. 3.3(1)-(2)]). Hence every GE-algebra is a RML algebra. The following example shows that the converse does not hold in general.

Example 3.2. Let $X = \{1, a, b\}$. Consider the Table 1.

Table 1: The $\rightarrow$ operation defining RML algebra $(X; \rightarrow, 1)$.

$\rightarrow$	1	a	b
1	1	a	b
a	1	1	a
b	1	b	1

Then $(X; \rightarrow, 1)$ is a RML algebra, but neither a GE-algebra nor self-distributive. Indeed,

$$\begin{aligned} a \to (b \to a) = a \to b = a &\neq 1 \\ &= a \to 1 \\ &= a \to (b \to 1) \\ &= a \to [b \to (a \to a)], \end{aligned} \tag{3.1}$$

and

$$a \to (b \to a) = a \to b = a \neq 1 = a \to 1 = (a \to b) \to (a \to a).$$

Further, every GE-algebra is a pi-RML algebra ([12, Pro. 3.3]), and every self distributive GE-algebra is an RML algebra with condition (pimpl), (briefly, pimpl-RML algebra). The following example shows that there is a pi-RML algebra which is not a GE-algebras.

Example 3.3. Let $X = \{1, a, b, c\}$. Consider the Table 2.

Table 2: The $\to$ operation defining pi-RML algebra $(X; \to, 1)$.

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	c	1	c
c	1	1	1	1

Then $(X; \to, 1)$ is a pi-RML algebra, but not GE-algebra, since

$$\begin{aligned} a \to (b \to a) = a \to c = c &\neq 1 \\ &= a \to 1 \\ &= a \to (b \to 1) \\ &= a \to [b \to (a \to a)]. \end{aligned} \tag{3.2}$$

Proposition 3.4. Every pimpl-RML algebra is a GE-algebra.

Proof: Let $(X, \to, 1)$ be a pimpl-RML algebra. It is sufficient to prove

(GE3). Given $x, y, z \in X$, using three times (pimpl), (R), (L) and (M), we get

$$\begin{aligned} x \to [y \to (x \to z)] &= x \to [(y \to x) \to (y \to z)] \\ &= [x \to (y \to x)] \to [x \to (y \to z)] \\ &= [(x \to y) \to (x \to x)] \to [x \to (y \to z)] \\ &= [(x \to y) \to 1] \to [x \to (y \to z)] \\ &= 1 \to [x \to (y \to z)] \\ &= x \to (y \to z). \end{aligned} \tag{3.3}$$

Thus, (GE3) is valid. □

The converse of Proposition 3.4 is not generally true as shown in the following example.

Example 3.5. Let $X = \{1, a, b, c\}$. Consider the Table 3.

Table 3: The $\to$ operation defining GE-algebra $(X; \to, 1)$.

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	1
c	1	1	1	1

Then $(X; \to, 1)$ is a GE-algebra, but not a pimpl-RML algebra, because (pimpl) does not holds. Indeed,

$$a \to (b \to c) = a \to 1 = 1 \neq c = 1 \to c = (a \to b) \to (a \to c).$$

The following corollary is a direct consequence of Remark 3.1 and Proposition 3.4.

Corollary 3.6. The notions of a self distributive GE-algebra and a pimpl-RML algebra are equivalent.

Remark 3.7. From Example 3.5 we conclude that the condition self distributivity in Corollary 3.6 is necessary, and so we can not remove it.

Corollary 3.8. Every pimpl-BE algebra (self distributive BE-algebra) is a pimpl-RML algebra.

Proof: By applying [1, Th. 3.13] and Corollary 3.6. □

A GE-algebra $(X, \to, 1)$ ([1, Def. 3.5]) is said to be *transitive* if it satisfies

$$\text{(TGE)} \quad (x \to y) \to [(z \to x) \to (z \to y)] = 1,$$

for all $x, y, z \in X$.

Note that the property of (TGE) is just the well known property (B). From [12, Re. 2.9 and Ex. 2.10], we can see that every transitive GE-algebra do not have to verify (pimpl).

Proposition 3.9. Every pi-pre-BBBCC algebra with condition (An) is a transitive GE-algebra.

Proof: From [3, Th. 2.4] it follows that $(Ex) \Leftrightarrow (BB)$. By [3, Pro. 3.6] we deduce that also satisfies (pimpl). Hence by a similar argument of Proposition 3.4 the proof is obvious. □

Proposition 3.10. Every transitive GE-algebra is a pi-pre-BBBCC algebra.

Proof: Using [1, Th. 3.4] and Remark 3.1, we can see that (BB) and (pi) hold. □

The following corollary is a direct consequence of Propositions 3.9 and 3.10.

Corollary 3.11. The notions of a transitive GE-algebra and a pi-pre-BBBCC algebra with condition (An) are equivalent.

From [12, Th. 3.6], every transitive GE-algebra is a pre-Hilbert algebra with condition (GE3) and vice versa, and so by Corollary 3.11, the notions of a pi-pre-BBBCC algebra with condition (An) and a pre-Hilbert algebra with condition (GE3) are equivalent.

4. Conclusions and Summary of Relations

- GE-algebras are a proper subclass of pi-RML algebras (as shown by Example 3.3).
- pimpl-GE algebras are a proper subclass of GE-algebras (as shown by Example 3.5) and are equivalent to pimpl-RML algebras.
- Transitive GE-algebras are a subclass of GE-algebras and are equivalent to pi-pre-BBBCC algebras with condition (An).
- pimpl-BE algebras are a subclass of pimpl-RML algebras.
- Pre-Hilbert algebras with condition (GE3) are equivalent to pi-pre-BBBCC algebras with condition (An).

Now, we can draw the following diagram:

$$
\begin{tikzcd}
\text{Hilbert algebra} \arrow[dr] & \text{BE-algebra} \arrow[r] & \text{RML algebra} \\
\begin{array}{c}\text{pre-Hilbert algebra}\\ \text{with (GE3)}\end{array} \arrow[u] \arrow[d, shift right] & \text{pimpl-BE algebra} \arrow[u] \arrow[d] & \text{pi-RML algebra} \arrow[u] \\
\text{transitive GE-algebra} \arrow[u, shift right] \arrow[r] \arrow[d, shift right] & \text{GE-algebra} \arrow[ur] & \\
\begin{array}{c}\text{pi-pre BBBCC algebra}\\ \text{with (An)}\end{array} \arrow[u, shift right] & \text{pimpl-GE algebra} \arrow[u] \arrow[r, shift left] & \text{pimpl-RML algebra} \arrow[l, shift left] \arrow[uu]
\end{tikzcd}
$$

The diagram clearly shows that pi-pre-BBBCC algebra with condition (An) leads to pre-Hilbert algebra which then leads to ransitive GE-algebra, and also connects to pre-Hilbert algebra with condition (GE).

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Akbar Rezaei
Payame Noor University
Department of Mathematics
P.O. Box 19395-4697
Tehran, Iran
e-mail: rezaei@pnu.ac.ir

Arsham Borumand Saeid
Shahid Bahonar University of Kerman
Department of Pure Mathematics
Faculty of Mathematics and Computer
Kerman, Iran
e-mail: arsham@uk.ac.ir

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