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VALUATIONS ON L-ALGEBRAS

Abstract

The purpose of this paper is to define the notion of valuation maps on L-algebras and introduce its properties. For this, the concepts of valuation and 1-valuation maps are introduced and some characterizations and related properties are investigated. Using a valuation map, a congruence relation is obtained and proved that this congruence relation is equivalent to the congruence relation made by an ideal. Also, showed that the image and inverse image of any valuation is a valuation. Finally, by using a 1-valuation map ξ on L-algebra $\mathcal{L}$, an L-algebra $(\widetilde{\mathcal{L}}, *, \widetilde{\{1\}})$ and a metric space $(\widetilde{\mathcal{L}}, \widetilde{d_\xi})$ has been obtained.

Keywords: L-algebra, valuation map, 1-valuation map, topology, metric space.

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1. Introduction

L-algebras, which are connected to algebraic logic and quantum structures, were first introduced by Rump [11]. Numerous examples have demonstrated the usefulness of L-algebras. Yang and Rump [13] characterized pseudo-MV-algebras and Bosbach's noncommutative bricks as L-algebras. In their work, Wu and Yang [18] provided several approaches to show that orthomodular lattices belong to a particular class of L-algebras. Wu et al. [17] further established that every lattice-ordered effect algebra possesses a natural L-algebra structure.

Several other mathematicians have also investigated the relationship between basic algebras and L-algebras [3]. They showed that a basic algebra satisfying

$$(b \oplus \neg q) \oplus \neg(v \oplus \neg q) = (b \oplus \neg v) \oplus \neg(q \oplus \neg v)$$

can be transformed into an L-algebra. Conversely, if an L-algebra with 0 satisfying certain conditions can be structured as an involutive bounded lattice, then it is necessarily a lattice-ordered effect algebra (see [17]).

Recently, researchers have studied L-algebras from different perspectives. For further information, see [2, 1, 4, 6, 7, 8, 10, 14, 15, 19, 20].

Song, Roh, and Jun [16] introduced the concepts of quasi-valuation maps in BCK/BCI-algebras based on subalgebras and ideals. They also investigated various properties of these maps and established relationships between quasi-valuation maps based on subalgebras and those based on ideals. In another study [9], the continuity of operations in a BCK-algebra was examined using a topology induced by a pseudo-valuation. Moreover, the authors showed that the product of a finite number of a pseudo metric space associated with a pseudo-valuation. They further proved that if a BCK-algebra X possesses a pseudo-valuation, then every quotient space of X admits a pseudometric. The completion of these spaces was also investigated.

In addition, in [5], the concepts of S-quasi-valuation maps and F-quasi-valuation maps were introduced based on subalgebras and filters in hoops, and their associated properties were studied. Relationships between S-quasi-valuation maps and F-quasi-valuation maps were established. Using the F-quasi-valuation map, the authors introduced a (pseudo)metric space and demonstrated that all operations in a hoop are uniformly continuous.

The aim of this paper is to define valuations on L-algebras and investigate their properties. We provide characterizations of valuations and 1-valuation maps and study their associated properties. Using valuation maps, we define a congruence relation and prove that it is equivalent to a congruence relation induced by an ideal. Furthermore, we show that the image and inverse image of a valuation under a homomorphism are also valuations. Finally, we introduce a metric space associated with a valuation map and establish that the operation "$*$" in an L-algebra is uniformly continuous.

2. Preliminaries

This section lists the known definitions and results that will be utilized later.

Definition 2.1 ([7]). An L-algebra is an algebraic structure $(\mathcal{L}; *, 1)$ of type $(2, 0)$ that satisfies the following conditions:

$(L1)$ $q * q = q * 1 = 1$ and $1 * q = q$,

$(L2)$ $(q * v) * (q * b) = (v * q) * (v * b)$,

$(L3)$ if $q * v = v * q = 1$, then $q = v$,

for any $q, v, b \in \mathcal{L}$. Condition $(L1)$ asserts that 1 is a logical unit, while $(L2)$ is linked to the quantum Yang-Baxter equation, it is the so-called cycloid equation. It is important to note that a logical unit is always unique. We have to notice that all axioms are needed to ensure that the following relation is a partial order relation:

$$q \leq v \text{ if and only if } q * v = 1, \tag{2.1}$$

$(L1)$ ensures reflexivity, $(L2)$ implies transitivity and $(L3)$ is used to prove that the relation is antisymmetric. So $(\mathcal{L}, \leq)$ is a poset. If $\mathcal{L}$ has a smallest element 0, then it is called a *bounded L-algebra.*

Proposition 2.2 ([13]). Consider an L-algebra $(\mathcal{L};*,1)$. If $q \leq v$, then for any $q,v,b \in \mathcal{L}$, it follows that $b*q \leq b*v$.

Proposition 2.3 ([13]). The statements below are equivalent for an L-algebra $\mathcal{L}$:

(i) $q \leq v*q$,

(ii) if $q \leq b$, then $b*v \leq q*v$,

(iii) $((q*v)*b)*b \leq ((q*v)*b)*((v*q)*b)$,

for any $q,v,b \in \mathcal{L}$.

Definition 2.4 ([12]). An L-algebra $\mathcal{L}$, that satisfying:

$$q*(v*q)=1, \qquad (K)$$

for any $q,v \in \mathcal{L}$, is called a *KL-algebra*. A *CKL-algebra*, is an L-algebra that satisfies the equation:

$$q*(v*b)=v*(q*b), \qquad (C)$$

for any $q,v,b \in \mathcal{L}$ (see [12]).

Clearly, every CKL-algebra is a KL-algebra, since for any $q,v \in \mathcal{L}$, we have

$$q*(v*q)=v*(q*q)=v*1=1.$$

Note. We have to notice that the notion of KL-algebras concerns the properties in Proposition 2.3.

Proposition 2.5. [4] Assume $(\mathcal{L},*,1)$ is a CKL-algebra. For any $q,v,b \in \mathcal{L}$, the following properties hold:

(i) $q*(v*q)=1$, i.e., $q \leq v*q$,

(ii) $q \leq (q*v)*v$,

(iii) $q \leq v*b$ if and only if $v \leq q*b$,

(iv) if $q \leq v$, then $v*b \leq q*b$,

(v) $((q * v) * b) * b \leq ((q * v) * b) * ((v * q) * b)$,

(vi) $b * v \leq (v * q) * (b * q)$,

(vii) $b * v \leq (q * b) * (q * v)$.

Definition 2.6 ([1]). An L-algebra $(\mathcal{L}, *, 1)$ is a positive L-algebra, if for any $q, v, b \in \mathcal{L}$, the following condition is satisfied:

$$q * (v * b) = (q * v) * (q * b). \qquad (P)$$

Theorem 2.7 ([1]). *Every positive L-algebra is a CKL-algebra.*

Definition 2.8 ([11]). A subset $\mathbb{I}$ of an L-algebra $(\mathcal{L}, *, 1)$ is termed an *ideal of* $\mathcal{L}$, if it meets the following conditions, for all $q, v \in \mathcal{L}$.

(I_1) $1 \in \mathbb{I}$,

(I_2) if $q \in \mathbb{I}$ and $q * v \in \mathbb{I}$, then $v \in \mathbb{I}$,

(I_3) if $q \in \mathbb{I}$, then $(q * v) * v \in \mathbb{I}$,

(I_4) if $q \in \mathbb{I}$, then $v * q \in \mathbb{I}$ and $v * (q * v) \in \mathbb{I}$.

The collection of all ideals of $\mathcal{L}$, is represented by $\mathcal{I}d(\mathcal{L})$.

Proposition 2.9 ([4]). Every ideal of $\mathcal{L}$ is an upset.

If $(\mathcal{L}, *, 1)$ is a CKL-algebra, then for proving that $\mathbb{I} \in \mathcal{I}d(\mathcal{L})$, the conditions (I_3) and (I_4) can be omitted. In fact, for any $q \in \mathbb{I}$, using (C) and (I_1), we have:

$$q * ((q * v) * v) = (q * v) * (q * v) = 1 \in \mathbb{I},$$

for any $v \in \mathcal{L}$. It follows from (I_2) that $(q * v) * v \in \mathbb{I}$. Therefore, (I_3) holds. Furthermore, if $q \in \mathbb{I}$, then for any $v \in \mathcal{L}$, using (K) we have $q * (v * q) = 1 \in \mathbb{I}$ and by (I_2), $v * q \in \mathbb{I}$.

Theorem 2.10 ([11]). *Consider the L-algebra $(\mathcal{L}, *, 1)$. In this context, every ideal $\mathbb{I}$ of $\mathcal{L}$ establishes a congruence relation on $\mathcal{L}$, for any q and v within $\mathcal{L}$, where*

$$q \sim_{\mathbb{I}} v \quad \Leftrightarrow \quad q * v,\ v * q \in \mathbb{I}. \tag{2.2}$$

Conversely, every congruence relation $\sim$ defines an ideal $\mathbb{I} = \{q \in \mathcal{L} \mid q \sim 1\}$.

Note. According to Theorem 2.10, the correspondence between ideals and congruences is one-to-one.

Note. From now on, we let $(\mathcal{L}, *, 1)$ or $\mathcal{L}$, for short, be an L-algebra.

3. Valuations on L-algebras

In this section, our objective is to define valuations on L-algebras and introduce their properties. Additionally, we will provide characterizations of valuations.

Definition 3.1. A real-valued function $\xi : \mathcal{L} \to \mathbb{R}$ is referred to as a *valuation map on* $\mathcal{L}$, if it fulfills the following conditions:

(VL_1) $\xi(1) = 0$,

(VL_2) $\xi(q * v) \geq \xi(v) - \xi(q)$, for any $q, v \in \mathcal{L}$.

The valuation map ξ is classified as a 1-*valuation map*, if it satisfies the following condition, for any $q \in \mathcal{L}$:

$$\xi(q) = 0 \text{ implies } q = 1.$$

The collection of all valuation maps on $\mathcal{L}$ is denoted by $\mathcal{V}(\mathcal{L})$. When considering $\xi \in \mathcal{V}(\mathcal{L})$, the combination $(\mathcal{L}, \xi)$ is referred to as a *valued L-algebra.*

Example 3.2. Consider $\mathcal{L} = \{0, f, y, 1\}$ as a poset, where $0 < f, y < 1$. In this case, $(\mathcal{L}, *, 0, 1)$ forms a bounded L-algebra, where:

$*$	0	f	y	1
0	1	1	1	1
f	y	1	y	1
y	f	f	1	1
1	0	f	y	1

Let us define the map $\xi : \mathcal{L} \to \mathbb{R}$ as follows: $\xi(1) = 0$, $\xi(0) = \xi(f) = \alpha$, and $\xi(y) = \beta$, where $\alpha \geq \beta$. It can be observed that $\xi \in \mathcal{V}(\mathcal{L})$.

Remark 3.3. Consider $\xi \in \mathcal{V}(\mathcal{L})$. It follows that $\xi(q * v) + \xi(v * q) \geq 0$, for any $q, v \in \mathcal{L}$. This is due to the fact that, for all $q, v \in \mathcal{L}$, we have the following relationship:

$$\xi(q * v) + \xi(v * q) \geq \xi(v) - \xi(q) + \xi(q) - \xi(v) = 0.$$

Hence, $\xi(q * v) + \xi(v * q) \geq 0$.

Proposition 3.4. Consider $\xi \in \mathcal{V}(\mathcal{L})$. For any $q, v, b \in \mathcal{L}$, the following relations hold:

(i) ξ is order reversing.

(ii) For any $q \in \mathcal{L}$, $\xi(q) \geq 0$.

(iii) If $q \leq v * b$, then $\xi(b) \leq \xi(q) + \xi(v)$.

(iv) If $\mathcal{L}$ is a KL-algebra, then $\xi(q * v) \leq \xi(q) + \xi(v)$.

If $\mathcal{L}$ is a CKL-algebra, then

(v) $\xi(q * b) \leq \xi(q * v) + \xi(v * b)$.

(vi) $\xi((q * (v * b)) * b) \leq \xi(q) + \xi(v)$.

(vii) $\xi(q * (v * b)) \leq \xi((q * v) * b)$.

Proof: (i) Assume $q, v \in \mathcal{L}$ such that $q \leq v$. According to the property (2.1), it follows that $q * v = 1$. Therefore, by applying the property (VL_2), we obtain:

$$0 = \xi(1) = \xi(q * v) \geq \xi(v) - \xi(q).$$

Hence, $\xi(v) \leq \xi(q)$.

(ii) This follows directly from (VL_1) and (i).

(iii) Assume $q, v, b \in \mathcal{L}$. If $q \leq v * b$, then by (i), we have $\xi(v * b) \leq \xi(q)$. Additionally, by (VL_2), we find that $\xi(v * b) \geq \xi(b) - \xi(v)$. Hence, $\xi(b) - \xi(v) \leq \xi(q)$. Therefore, $\xi(b) \leq \xi(q) + \xi(v)$.

(iv) As $\mathcal{L}$ is a KL-algebra, it follows that for any $q, v \in \mathcal{L}$, we have $v \leq q*v$. By (i), we have $\xi(q * v) \leq \xi(v)$. Also, by (VL_2), $\xi(v) - \xi(q) \leq \xi(q * v)$, and so $\xi(v) \leq \xi(q) + \xi(q * v)$. Hence,

$$\xi(q * v) \leq \xi(v) \leq \xi(q) + \xi(q * v) \leq \xi(q) + \xi(v).$$

Therefore, $\xi(q * v) \leq \xi(q) + \xi(v)$.

(v) Let $q, v, b \in \mathcal{L}$. By Proposition 2.5(vi), we have $q * v \leq (v * b) * (q * b)$. Utilizing property (i), we get $\xi((v * b) * (q * b)) \leq \xi(q * v)$. Furthermore, by (VL_2), we find that $\xi(q * b) - \xi(v * b) \leq \xi(q * v)$. Therefore, we obtain $\xi(q * b) \leq \xi(q * v) + \xi(v * b)$.

(vi) As $\mathcal{L}$ is a CKL-algebra, according to Proposition 2.5(ii), we have $q \leq (q * (v * b)) * (v * b)$. Consequently, using property (i), we can continue as follows:

$$\xi\left((q * (v * b)) * (v * b)\right) \leq \xi(q). \tag{3.1}$$

Additionally, based on Proposition 2.5(ii), we have $v \leq (v * b) * b$. By (i), we obtain:

$$\xi((v * b) * b) \leq \xi(v). \tag{3.2}$$

Hence,

$$\begin{aligned} \xi\left((q * (v * b)) * b\right) &\leq \xi\left((q * (v * b)) * (v * b)\right) + \xi((v * b) * b) \quad \text{by (v)} \\ &\leq \xi(q) + \xi((v * b) * b) \qquad \text{by (3.1)} \\ &\leq \xi(q) + \xi(v). \qquad \text{by (3.2)} \end{aligned}$$

Hence, $\xi((q * (v * b)) * b) \leq \xi(q) + \xi(v)$.

(vii) Consider $q, v, b \in \mathcal{L}$. Proposition 2.5(i) implies that $v \leq q * v$. Furthermore, by Proposition 2.5(iv), we find that $(q * v) * b \leq v * b$. Therefore, using property (i), we can conclude that $\xi(v * b) \leq \xi((q * v) * b)$. Additionally, based on Proposition 2.5(i), we have $v * b \leq q * (v * b)$. Applying property (i) once again, we find that $\xi(q * (v * b)) \leq \xi(v * b)$. Hence, $\xi(q * (v * b)) \leq \xi((q * v) * b)$. □

Set $\ker \xi = \{q \in \mathcal{L} \mid \xi(q) = 0\}$.

Remark 3.5. (i) Consider $\xi \in \mathcal{V}(\mathcal{L})$. It follows that $\tau\xi \in \mathcal{V}(\mathcal{L})$ and $\ker(\tau\xi) = \ker \xi$, for any $\tau \in \mathbb{R}$ with $\tau > 0$. Here, $\tau\xi : \mathcal{L} \to \mathbb{R}$ is defined as $(\tau\xi)(q) = \tau\xi(q)$, for all $q \in \mathcal{L}$.

(ii) Assuming $\xi_1, \xi_2 \in \mathcal{V}(\mathcal{L})$, we can conclude that $\xi = \tau\xi_1 + \omega\xi_2 \in \mathcal{V}(\mathcal{L})$, where $\tau, \omega \in \mathbb{R}$ with $\tau, \omega > 0$. Moreover, the kernel of ξ is given by $\ker \xi = \ker \xi_1 \cap \ker \xi_2$. It is evident that $\xi_1, \xi_2 \in \mathcal{V}(\mathcal{L})$ and $\tau, \omega \in \mathbb{R}$ with $\tau, \omega > 0$. Therefore, for any $q, v \in \mathcal{L}$, we have the following:

$$\xi(1) = (\tau\xi_1 + \omega\xi_2)(1) = \tau\xi_1(1) + \omega\xi_2(1) = \tau(\xi_1(1)) + \omega(\xi_2(1)) = 0.$$

Thus, (VL_1) holds. Furthermore, we have the following:

$$\begin{aligned}
\xi(q * v) &= (\tau\xi_1 + \omega\xi_2)(q * v) \\
&= (\tau\xi_1)(q * v) + (\omega\xi_2)(q * v) \\
&\geq \tau(\xi_1(v) - \xi_1(q)) + \omega(\xi_2(v) - \xi_2(q)) \qquad \text{since } \tau, \omega > 0 \\
&= \tau\xi_1(v) - \tau\xi_1(q) + \omega\xi_2(v) - \omega\xi_2(q) \\
&= (\tau\xi_1 + \omega\xi_2)(v) - (\tau\xi_1 + \omega\xi_2)(q) \\
&= \xi(v) - \xi(q).
\end{aligned}$$

Therefore, we can conclude that (VL_2) holds. As a result, $\xi \in \mathcal{V}(\mathcal{L})$. Furthermore,

$$\begin{aligned}
q \in \ker \xi \;\Leftrightarrow\; \xi(q) = 0 \;\Leftrightarrow\; (\tau\xi_1 + \omega\xi_2)(q) = 0 \;\Leftrightarrow\; \tau\xi_1(q) + \omega\xi_2(q) = 0 \\
\overset{\tau,\omega>0}{\Leftrightarrow} \; \xi_1(q) = \xi_2(q) = 0 \Leftrightarrow \; q \in \ker \xi_1 \cap \ker \xi_2.
\end{aligned}$$

Hence, $\ker \xi = \ker \xi_1 \cap \ker \xi_2$.

(iii) In general, if $\xi_i \in \mathcal{V}(\mathcal{L})$, $\tau_i \in \mathbb{R}$, $\tau_i > 0$ and $1 \leq i \leq n$, then $\xi =$

$\sum_{i=1}^{n} \tau_i \xi_i \in \mathcal{V}(\mathcal{L})$ and $\ker \xi = \bigcap_{i=1}^{n} \ker \xi_i$.

PROPOSITION 3.6. Assume $\mathcal{L}$ is a CKL-algebra. If $\xi \in \mathcal{V}(\mathcal{L})$, then $\ker \xi \in \mathcal{I}d(\mathcal{L})$.

PROOF: It is evident that $1 \in \ker \xi$. Let us consider $q, q * v \in \ker \xi$. This implies $\xi(q), \xi(q * v) = 0$, and so by (LV_2), we obtain the following inequality:

$$0 = \xi(q * v) \geq \xi(v) - \xi(q) \geq \xi(v).$$

Hence, $\xi(v) \leq 0$. By Proposition 3.4(ii), $\xi \geq 0$, it follows that $\xi(b) \geq 0$, for any $b \in \mathcal{L}$. Consequently, we have $\xi(v) \geq 0$. As a result, $\xi(v) = 0$, which implies that $v \in \ker \xi$. We have to mention that (I3) and (I4) do not need to be checked in the case of a CKL-algebra. Therefore, $\ker \xi \in \mathcal{I}d(\mathcal{L})$. □

PROPOSITION 3.7. Let us assume that $\mathbb{I} \in \mathcal{I}d(\mathcal{L})$. We will now define the function $\xi_{\mathbb{I}} : \mathcal{L} \to \mathbb{R}$ as follows:

$$\xi_{\mathbb{I}}(q) = \begin{cases} 0 & \text{if } q \in \mathbb{I} \\ f & \text{if } q \notin \mathbb{I} \end{cases} \tag{3.3}$$

where $f \in \mathbb{R}$, $f > 0$. Then the following statements hold:

(i) $\xi_{\mathbb{I}} \in \mathcal{V}(\mathcal{L})$,

(ii) $\ker \xi_{\mathbb{I}} = \mathbb{I}$.

PROOF: (i) It is evident that $1 \in \mathbb{I}$ since $\mathbb{I} \in \mathcal{I}d(\mathcal{L})$. As a result, we have $\xi_{\mathbb{I}}(1) = 0$, which satisfies property (VL_1). To prove property (VL_2), if there are $q, v \in \mathcal{L}$ with $\xi(q * v) = \xi(q) = 0$, and $\xi(v) = f$. But then $q, q * v \in \mathbb{I}$, whereas $v \notin \mathbb{I}$, which is a contradiction.

(ii) The proof is straightforward and clear. □

Remark 3.8. According to Propositions 3.6 and 3.7, there exists a relationship between $\mathcal{V}(\mathcal{L})$ and $\mathcal{I}d(\mathcal{L})$. However, it is important to note that this relationship is not a one-to-one correspondence.

PROPOSITION 3.9. If ξ is a valuation on the positive L-algebra $\mathcal{L}$, and $f \in \mathcal{L}$, then the function ξ_f defined as $\xi_f(q) = \xi(f * q)$ for any $q \in \mathcal{L}$ is a member of $\mathcal{V}(\mathcal{L})$. Furthermore, if ξ is a 1-valuation, then ξ_f is a 1-valuation if $f=1$ and in this case $\xi_f=\xi$.

PROOF: It is clear that $\xi_f(1)=0$. Now, let us consider $q,v,f\in\mathcal{L}$. Since $\mathcal{L}$ is a positive L-algebra, we can deduce the following:

$$\xi_f(q)-\xi_f(v)=\xi(f*q)-\xi(f*v)\leq\xi((f*v)*(f*q))=\xi(f*(v*q))=\xi_f(v*q).$$

Hence $\xi_f\in\mathcal{V}(\mathcal{L})$. Now, we have

$$\xi_f(q)=0 \;\Leftrightarrow\; \xi(f*q)=0\Leftrightarrow\; f*q=1 \;\Leftrightarrow\; f\leq q.$$

So if $f=1$, then $q=1$ and so ξ_f is 1-valuation and clearly, $\xi_f=\xi$. □

DEFINITION 3.10. [11] Let $\mathcal{L}$ and $\mathcal{K}$ be two L-algebras. Then a map $\Theta:\mathcal{L}\to\mathcal{K}$ is called an *L-homomorphism* if for any $q,v\in\mathcal{L}$ we have $\Theta(q*_{\mathcal{L}}v)=\Theta(q)*_H\Theta(v)$.

If Θ is an injective, then Θ is called a *monomorphism*, and if Θ is onto, then Θ is called an *epimorphism*. In addition, if Θ is a bijective function, then Θ is called an *isomorphism*.

Remark 3.11. Let $(\mathcal{L}_1,*_1,1_{\mathcal{L}_1})$ and $(\mathcal{L}_2,*_2,1_{\mathcal{L}_2})$ be two L-algebras. Then $\mathcal{L}=\mathcal{L}_1\times\mathcal{L}_2$ with the following operations is an L-algebra.

$$(q,v)*(f,y)=(q*_1 f,v*_2 y),\text{ and }\mathbf{1}=(1_{\mathcal{L}_1},1_{\mathcal{L}_2}),$$

for any $(q,v),(f,y)\in\mathcal{L}$.

PROPOSITION 3.12. Let $(\mathcal{L}_1,*_1,1_{\mathcal{L}_1})$ and $(\mathcal{L}_2,*_2,1_{\mathcal{L}_2})$ be two L-algebras, and $\mathcal{L}=\mathcal{L}_1\times\mathcal{L}_2$. We can say that $\mathcal{L}$ has a valuation ξ if each $\mathcal{L}_i$ has a valuation ξ_i for $i=1,2$, and $\xi=\xi_1+\xi_2$.

PROOF: Let ξ_1 and ξ_2 be valuations on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Define $\xi:\mathcal{L}\to\mathbb{R}$ by $\xi(q)=\xi_1(q_1)+\xi_2(q_2)$, for any $q=(q_1,q_2)$. Then $\xi(1)=\xi((1_{\mathcal{L}_1},1_{\mathcal{L}_2}))=\xi_1(1_{\mathcal{L}_1})+\xi_2(1_{\mathcal{L}_2})=0$. Now, let $q=(q_1,q_2),v=(v_1,v_2)\in\mathcal{L}$. Then, by Remark 3.11, we have

$$\begin{aligned}\xi(q*v) &= \xi(q_1 *_1 v_1, q_2 *_2 v_2)\\ &= \xi_1(q_1 *_1 v_1) + \xi_2(q_2 *_2 v_2)\\ &\geq \xi_1(v_1) - \xi_1(q_1) + \xi_2(v_2) - \xi_2(q_2)\\ &= \xi_1(v_1) + \xi_2(v_2) - (\xi_2(q_1) + \xi_2(q_2))\\ &= \xi(v) - \xi(q).\end{aligned}$$

Hence, $\xi \in \mathcal{V}(\mathcal{L})$. □

4. Quotient L-algebras induced by valuation maps

In this section, we will define a map that will allow us to introduce a congruence relation on $\mathcal{L}$. We will then explore various properties of the quotient L-algebras that arise from this congruence relation.

Note. Let's clarify that we will consider $\mathcal{L}$ as a CKL-algebra from now on unless mentioned otherwise.

According to Remark 3.11, we can define a mapping $d_\xi : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ for $\xi \in \mathcal{V}(\mathcal{L})$. For any $q, v \in \mathcal{L}$, the mapping is defined as follows:

$$d_\xi(q, v) = \xi(q * v) + \xi(v * q). \tag{4.1}$$

Remark 4.1. Clearly, $d_\xi(1, q) = \xi(q)$.

Proposition 4.2. The following inequalities are satisfied by every d_ξ:

(i) $d_\xi(q, q) = 0$.

(ii) $d_\xi(q, v) \geq 0$.

(iii) $d_\xi(q, v) = d_\xi(v, q)$.

(iv) $d_\xi(q, b) \leq d_\xi(q, v) + d_\xi(v, b)$.

(v) $d_\xi(q, v) \geq d_\xi(q * f, v * f)$.

(vi) $d_\xi(q, v) \geq d_\xi(f * q, f * v)$.

(vii) $d_\xi(q*v, f*y) \geq d_\xi(q*v, f*v) + d_\xi(f*v, f*y)$.

(viii) $\mid \xi(q) - \xi(v) \mid \leq d_\xi(q, v)$.

PROOF: It is evident that (i), (ii), and (iii) are true.

(iv) Assume $q, v, b \in \mathcal{L}$. Then

$$\begin{aligned} d_\xi(q,v) + d_\xi(v,b) &= \xi(q*v) + \xi(v*q) + \xi(v*b) + \xi(b*v) \\ &\geq \xi(q*b) + \xi(b*q) \qquad \text{By Proposition 3.4(v)} \\ &= d_\xi(q,b). \end{aligned}$$

Hence, $d_\xi(q,b) \leq d_\xi(q,v) + d_\xi(v,b)$.

(v) Let $q, v, f \in \mathcal{L}$. Then by Proposition 2.5(vi), $v*q \leq (q*f)*(v*f)$ and by Proposition 3.4(i), $\xi((q*f)*(v*f)) \leq \xi(v*q)$. By the similar way, we can see that $\xi((v*f)*(q*f)) \leq \xi(q*v)$, and so $d_\xi(q,v) \geq d_\xi(q*f, v*f)$.

(vi) By Propositions 2.5(vii) and 3.4(i), the proof is straightforward.

(vii) This is just a special case of (iv).

(viii) Let $q, v \in \mathcal{L}$. Then:

$$\xi(v) - \xi(q) \leq \xi(q*v) \leq \xi(v*q) + \xi(q*v) = d_\xi(q,v).$$

and

$$\xi(q) - \xi(v) \leq \xi(v*q) \leq \xi(q*v) + \xi(v*q) = d_\xi(q,v).$$

Hence $-d_\xi(q,v) \leq \xi(q) - \xi(v) \leq d_\xi(q,v)$. Thus $\mid \xi(q) - \xi(v) \mid \leq d_\xi(q,v)$. □

Let $\xi \in \mathcal{V}(\mathcal{L})$. Define $\equiv_\xi$ for any $q, v \in \mathcal{L}$ by

$$(q,v) \in \equiv_\xi \quad \Leftrightarrow \quad d_\xi(q,v) = 0. \tag{4.2}$$

PROPOSITION 4.3. The binary relation $\equiv_\xi$ is a congruence relation on $\mathcal{L}$, called the congruence relation induced by ξ.

PROOF: Proposition 4.2(i) and (iii) demonstrate that $d_\xi(q,v)$ possesses the properties of reflexivity and symmetry. Let $q, v, b \in \mathcal{L}$ such that $(q,v) \in \equiv_\xi$ and $(v,b) \in \equiv_\xi$. By Proposition 4.2(iv), we have $d_\xi(q,b) \leq d_\xi(q,v) + d_\xi(v,b) = 0$. Therefore, by Proposition 4.2(ii), we have $d_\xi(q,b) =$

0, and so $(q,b) \in \equiv_\xi$. Now, suppose that $d_\xi(q,v) = d_\xi(b,r) = 0$. By applying Proposition 4.2(v) and (vi) we get,

$$0 = d_\xi(q,v) \geq d_\xi(q*b, v*b), \quad 0 = d_\xi(b,r) \geq d_\xi(v*b, v*r),$$

Moreover, by utilizing Proposition 4.2(ii) and (vii), we can infer that:

$$d_\xi(q*b, v*b) = d_\xi(v*b, v*r) = 0,$$

Applying the principle of transitivity, we can thus conclude that $(q*b, v*r) \in \equiv_\xi$. Consequently, it can be inferred that $\equiv_\xi$ is a congruence relation. $\square$

Remark 4.4. Let q,v be elements of $\mathcal{L}$. Since ξ belongs to $\mathcal{V}(\mathcal{L})$, we obtain

$$\begin{aligned} \xi(q*v) = 0 \ &\Leftrightarrow \xi(1) + \xi(q*v) = 0 \ \Leftrightarrow \xi((q*v)*1) + \xi(1*(q*v)) = 0 \\ &\Leftrightarrow d_\xi(q*v, 1) = 0. \end{aligned}$$

By the similar discussion, we have $\xi(v*q) = 0 \ \Leftrightarrow d_\xi(v*q, 1) = 0$. Hence,

$$\begin{aligned} (q,v) \in \equiv_\xi \Leftrightarrow \ & d_\xi(q,v) = 0 \\ \Leftrightarrow \ & \xi(q*v) = \xi(v*q) = 0 \\ \Leftrightarrow \ & d_\xi(q*v, 1) = 0, \text{ and } d_\xi(v*q, 1) = 0. \end{aligned}$$

Remark 4.5. Let ξ be an element of $\mathcal{V}(\mathcal{L})$. Define $[q]_\xi$ as the set $\{v \in \mathcal{L} \mid (q,v) \in \equiv_\xi\}$, where $q \in \mathcal{L}$. The collection of all such $[q]_\xi$ is denoted by $\dfrac{\mathcal{L}}{\equiv_\xi}$. By Proposition 4.3, $\equiv_\xi$ is a congruence relation, the operation $*$ can be defined on $\dfrac{\mathcal{L}}{\equiv_\xi}$ as $[q]_\xi * [v]_\xi = [q*v]_\xi$ for all $q, v \in \mathcal{L}$. Hence, by Remark 4.4, $\langle \dfrac{\mathcal{L}}{\equiv_\xi}, *, [1]_\xi \rangle$ forms an L-algebra.

Proposition 4.6. Assuming $\xi \in \mathcal{V}(\mathcal{L})$. Then $\sim_{\ker(\xi)}$ is equivalent to $\equiv_\xi$.

Proof: By Proposition 3.6, $\ker(\xi) \in \mathcal{I}d(\mathcal{L})$. Also, if $\equiv_\xi$ is a congruence relation providing an L-algebra, then it is already defined by the ideal $[1]_\xi$ as ideals are in one-to-one correspondence to congruences whose quotient

is an L-algebra. In addition, $(q,v) \in \sim_{\ker(\xi)}$. Then, according to (2.2), we have $q*v, v*q \in \ker(\xi)$. Consequently, $\xi(q*v), \xi(v*q) = 0$. Therefore, $d_\xi(q,v) = 0$. Thus, using (4.2), we can conclude that $(q,v) \in \equiv_\xi$. Hence, $\sim_{\ker(\xi)} \subseteq \equiv_\xi$.
Conversely, if $(q,v) \in \equiv_\xi$, then $d_\xi(q,v) = 0$. Consequently, $\xi(q*v) + \xi(v*q) = 0$. According to Proposition 3.4(ii), this implies that $\xi(q*v) = \xi(v*q) = 0$, and therefore, $q*v, v*q \in \ker(\xi)$. Thus, by (2.2), we can deduce that $(q,v) \in \sim_{\ker(\xi)}$, which in turn implies that $\equiv_\xi \subseteq \sim_{\ker(\xi)}$. Therefore, we conclude that $\sim_{\ker(\xi)} = \equiv_\xi$. □

Remark 4.7. Suppose $q \in [1]_\xi$. This implies that $d_\xi(q,1) = 0$, and consequently, $(q,1) \in \equiv_\xi$. According to Proposition 4.6, it follows that $(q,1) \in \sim_{\ker(\xi)}$. Thus, $q \in \ker(\xi)$, leading to the conclusion that $\xi(q) = 0$.

THEOREM 4.8. *Assume $\xi_1, \xi_2 \in \mathcal{V}(\mathcal{L})$, where $[1]_{\xi_1} = [1]_{\xi_2}$. Then $\dfrac{\mathcal{L}}{\equiv_{\xi_1}} = \dfrac{\mathcal{L}}{\equiv_{\xi_2}}$.*

PROOF: Since $[1]_{\xi_1}$, $[1]_{\xi_2}$ are defining ideals for the congruences $\equiv_{\xi_1}$, $\equiv_{\xi_2}$, respectively, we get that if they are equal, the congruences are either. □

COROLLARY 4.9. Let $\xi \in \mathcal{V}(\mathcal{L})$. Then $\bar{\xi} \in \mathcal{V}\left(\dfrac{\mathcal{L}}{\equiv_\xi}\right)$, where $\bar{\xi} : \dfrac{\mathcal{L}}{\equiv_\xi} \to \mathbb{R}$ and $\bar{\xi}([q]_\xi) = \xi(q)$, for any $q \in \mathcal{L}$.

PROOF: Straightforward. □

In the following, we have two general facts in universal algebra: congruences of $\mathcal{L}/\equiv$ are in one-to-one correspondence with congruences of $\mathcal{L}$ that contain $\equiv$. Thus, we have:

PROPOSITION 4.10. Assuming that $\xi \in \mathcal{V}(\mathcal{L})$ and $\mathbb{I} \in \mathcal{I}d(\mathcal{L})$ such that $[1]_\xi \subseteq \mathbb{I}$, then we have:

(i) $q \in \mathbb{I}$ if and only if $[q]_\xi \in \dfrac{\mathbb{I}}{\equiv_\xi}$.

(ii) $\dfrac{\mathbb{I}}{\equiv_\xi} \in \mathcal{I}d\left(\dfrac{\mathcal{L}}{\equiv_\xi}\right)$.

PROOF: (i) Let $[q]_\xi \in \dfrac{\mathbb{I}}{\equiv_\xi}$. Then there exists $v \in \mathbb{I}$ such that $[q]_\xi = [v]_\xi$, which implies that $d_\xi(q, v) = 0$. By Remark 4.4, we have $d_\xi(q * v, 1) = d_\xi(v * q, 1) = 0$. Thus $q * v$ and $v * q$ belong to $[1]_\xi \subseteq \mathbb{I}$. So, $v * q \in \mathbb{I}$. Since $v \in \mathbb{I}$ and $\mathbb{I} \in \mathcal{I}d(\mathcal{L})$ we conclude that $q \in \mathbb{I}$. Conversely, if $q \in \mathbb{I}$, then $d_\xi(q, 1) = 0$ and so $q \in [1]_\xi$. Hence, $[q]_\xi = [1]_\xi \in \dfrac{\mathbb{I}}{\equiv_\xi}$. Therefore, $[q]_\xi \in \dfrac{\mathbb{I}}{\equiv_\xi}$.

(ii) By (i), the proof is clear. □

THEOREM 4.11. *There exists a one-to-one corresponding between ideals of $\mathcal{L}$ and ideals of $\dfrac{\mathcal{L}}{\equiv_\xi}$ that contain $[1]_\xi$.*

PROOF: By referring to Proposition 4.10, the proof becomes evident. □

THEOREM 4.12. *If $\mathcal{K}$ is an L-algebra, $\Theta : \mathcal{L} \to \mathcal{K}$ is an L-homomorphism and $\xi \in \mathcal{V}(\mathcal{K})$, then $\xi \circ \Theta \in \mathcal{V}(\mathcal{L})$.*

PROOF: The proof is simple and direct. □

THEOREM 4.13. *Suppose $\mathcal{K}$ is an L-algebra, $\Theta : \mathcal{L} \to \mathcal{K}$ is an L-epimorphism, and $\xi \in \mathcal{V}(\mathcal{K})$. It follows that $\dfrac{\mathcal{L}}{\equiv_{\xi\circ\Theta}}$ is isomorphic to $\dfrac{\mathcal{K}}{\equiv_\xi}$*

PROOF: According to Theorem 4.12, we know that $\xi \circ \Theta \in \mathcal{V}(\mathcal{L})$. Additionally, based on Proposition 4.3 and Remark 4.5, we can establish that $\dfrac{\mathcal{L}}{\equiv_{\xi\circ\Theta}}$ and $\dfrac{\mathcal{K}}{\equiv_\xi}$ are both L-algebras. We can define a mapping $\psi : \dfrac{\mathcal{L}}{\equiv_{\xi\circ\Theta}} \to \dfrac{\mathcal{K}}{\equiv_\xi}$ such that $\psi([q]_{\xi\circ\Theta}) = [\Theta(x)]_\xi$, for any $q \in \mathcal{L}$. Consequently,

$$\begin{aligned}
[q]_{\xi\circ\Theta} = [v]_{\xi\circ\Theta} &\Leftrightarrow (q, v) \in \equiv_{\xi\circ\Theta} \\
&\Leftrightarrow \ (\xi \circ \Theta)(q * v) = (\xi \circ \Theta)(v * q) = 0 \\
&\Leftrightarrow \ \xi(\Theta(q * v)) = \xi(\Theta(v * q)) = 0 \\
&\Leftrightarrow \ \xi(\Theta(q) * \Theta(v)) = \xi(\Theta(v) * \Theta(q)) = 0 \\
&\Leftrightarrow \ (\Theta(q), \Theta(v)) \in \equiv_{\xi} \\
&\Leftrightarrow \ [\Theta(q)]_{\xi} = [\Theta(v)]_{\xi} \\
&\Leftrightarrow \ \psi([q]_{\xi\circ\Theta}) = \psi([v]_{\xi\circ\Theta}).
\end{aligned}$$

Thus, we can conclude that ψ is both one-to-one and well-defined. Additionally, let us assume that $[q]_{\xi\circ\Theta}, [v]_{\xi\circ\Theta} \in \dfrac{\mathcal{L}}{\equiv_{\xi\circ\Theta}}$. Consequently, we have:

$$\begin{aligned}
\psi\left([q]_{\xi\circ\Theta} * [v]_{\xi\circ\Theta}\right) &= \psi\left([q * v]_{\xi\circ\Theta}\right) = [\Theta(q * v)]_{\xi} = [\Theta(q)]_{\xi} * [\Theta(v)]_{\xi} \\
&= \psi\left([q]_{\xi\circ\Theta}\right) * \psi\left([v]_{\xi\circ\Theta}\right).
\end{aligned}$$

Furthermore, if $[v]_{\xi} \in \dfrac{\mathcal{K}}{\equiv_{\xi}}$, then due to the fact that Θ is an L-epimorphism, there exists an $q \in \mathcal{L}$ such that $\Theta(q) = v$. Consequently, we can deduce that $\psi\left([q]_{\xi\circ\Theta}\right) = [\Theta(q)]_{\xi} = [v]_{\xi}$. Thus, ψ can be considered an L-epimorphism. As a result, we can conclude that $\dfrac{\mathcal{L}}{\equiv_{\xi\circ f\Theta}} \cong \dfrac{\mathcal{K}}{\equiv_{\xi}}$.

Second proof: By using the composition $\mathcal{L} \to \mathcal{K} \to \dfrac{\mathcal{K}}{\equiv_{\xi}}$, then it is easy too see that the defining congruence of the epimorphism $\mathcal{L} \to \dfrac{\mathcal{K}}{\equiv_{\xi}}$ is given by $\equiv_{\xi\circ\Theta}$. □

5. Topology on L-algebra induced by a valuation map

In the following, we utilize the concept of valuation and equation (4.1) to establish a pseudo metric and a topological L-algebra. Furthermore, we explore the properties of this structure.

We consider a list of elements of $\mathcal{L}$ such as $\{q_n\}$ is called a sequence on $\mathcal{L}$ where for any $\{q_n\}$ and $\{v_n\}$, we have $\{q_n\} * \{v_n\} = \{q_n * v_n\}$.

Remark 5.1. Let $\xi : \mathcal{L} \to \mathbb{R}$ be a continuous function, and let $\{q_n\}$ be a sequence in $\mathcal{L}$. We say that q_n converges to q, denoted as $q_n \longrightarrow q \in \mathcal{L}$, if $\lim\limits_{n\to\infty} d_\xi(q_n, q) = 0$.

THEOREM 5.2. *If $\xi \in \mathcal{V}(\mathcal{L})$, then d_ξ is a pseudo-metric* [1] *on $\mathcal{L}$ and so $(\mathcal{L}, d_\xi)$ is a pseudo-metric space.*

PROOF: Based on Proposition 4.2(i), (ii), (iii), and (iv), the proof is straightforward. □

THEOREM 5.3. *If ξ is a 1-valuation map on $\mathcal{L}$, then the pair $(\mathcal{L}, d_\xi)$ forms a metric space.*

PROOF: By Theorem 5.2, we know that $(\mathcal{L}, \xi)$ is a pseudo-metric space. Consider $q, v \in \mathcal{L}$ such that $d_\xi(q, v) = 0$. This implies that $\xi(q * v) + \xi(v * q) = 0$. According to Proposition 3.4(ii), we have $\xi(q * v) = \xi(v * q) = 0$. Since ξ is a 1-valuation map, it follows that $q * v = v * q = 1$. By property (L_3), we conclude that $q = v$. Therefore, $(\mathcal{L}, d_\xi)$ is a metric space. □

Note. If $(\mathcal{L}, d)$ is a pseudo-metric space, then for each $q \in \mathcal{L}$ and $\varepsilon > 0$, the set

$$N_\varepsilon(q) = \{v \in \mathcal{L} \mid \ d(q, v) < \varepsilon\},$$

is called a *ball of radius ε with center at q.*

PROPOSITION 5.4. Consider two L-algebras, $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\xi_1 \in \mathcal{V}(\mathcal{L}_1)$ and $\xi_2 \in \mathcal{V}(\mathcal{L}_2)$. Let $(q, v), (f, y) \in \mathcal{L}_1 \times \mathcal{L}_2$. We define the function $d((q, v), (f, y)) = d_{\xi_1}(q, f) + d_{\xi_2}(v, y)$, and we claim that d is a pseudo metric on $\mathcal{L}_1 \times \mathcal{L}_2$.

PROOF: Clearly, d is just the metric associated with the valuation $\xi_1 + \xi_2$-

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[1] By a pseudo-metric on $\mathcal{L}$, we mean a real-valued function $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ satisfying the following properties:

$$d(q, v) \geq 0,\ d(q, q) = 0, d(q, v) = d(v, q) \text{ and } d(q, b) \leq d(q, v) + d(v, b),$$

for every $q, v, b \in \mathcal{L}$. If $d(q, v) = 0$ implies $q = v$, then d is called a metric. The space $(\mathcal{L}, d)$ is called a pseudo-metric space and metric space, respectively.

which is a valuation by Proposition 3.12. Thus, by Propositions 3.12 and 4.2 and Theorem 5.2, the proof is clear. □

COROLLARY 5.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two L-algebras. If ξ_1 and ξ_2 are two 1-valuations on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, then the pair $(\mathcal{L}_1 \times \mathcal{L}_2, d)$ forms a metric space.

THEOREM 5.6. *Suppose we have a pseudo-metric space $(\mathcal{L}, d_\xi)$. We can then define a new pseudo-metric space $(\mathcal{L}^2, d_\xi^*)$, where $\mathcal{L}^2$ is the Cartesian product $\mathcal{L} \times \mathcal{L}$. For any $(q,v),(f,y) \in \mathcal{L}^2$, the pseudo-metric d_ξ^* is given by:*

$$d_\xi^*\left((q,v),(f,y)\right) = \max\left\{d_\xi(q,f), d_\xi(v,y)\right\}.$$

PROOF: By Proposition 4.2(i), (ii) and (iii), clearly $d_\xi^*\left((q,v),(f,y)\right) \geq 0$, $d_\xi^*\left((q,v),(q,v)\right) = 0$ and $d_\xi^*\left((q,v),(f,y)\right) = d_\xi^*\left((f,y),(q,v)\right)$. Suppose $(q,v),(b,r),(w,z) \in \mathcal{L}^2$. Then

$$\begin{aligned}
&d_\xi^*\left((q,v),(b,r)\right) + d_\xi^*\left((b,r),(w,z)\right)\\
&= \max\left\{d_\xi(q,b), d_\xi(v,r)\right\} + \max\left\{d_\xi(b,w), d_\xi(r,z)\right\}\\
&= \max\left\{d_\xi(q,b) + d_\xi(b,w), d_\xi(v,r) + d_\xi(r,z)\right\}\\
&\geq \max\left\{d_\xi(q,w), d_\xi(v,z)\right\} \qquad \text{by Proposition 4.2(iv)}\\
&= d_\xi^*\left((q,v),(w,z)\right).
\end{aligned}$$

Hence, $(\mathcal{L}^2, d_\xi^*)$ is a pseudo-metric space. □

THEOREM 5.7. *If ξ is a 1-valuation map on $\mathcal{L}$, then $(\mathcal{L}^2, d_\xi^*)$ forms a metric space.*

PROOF: The proof of this statement follows a similar approach to the proof of Theorem 5.3. □

THEOREM 5.8. *By referring to Remark 3.11, it can be observed that the operation $*$ is continuous.*

PROOF: Consider $(q,v),(f,y) \in \mathcal{L}^2$. For any $\varepsilon > 0$, if $d_\xi^*\left((q,v),(f,y)\right) < \frac{\varepsilon}{2}$, then it follows that $\max\left\{d_\xi(q,f), d_\xi(v,y)\right\} < \frac{\varepsilon}{2}$. Consequently, we have

$d_\xi(q,f), d_\xi(v,y) < \dfrac{\varepsilon}{2}$. By utilizing Proposition 4.2(v), (vi), and (vii), we can deduce that

$$d_\xi(q*v, f*y) \leq d_\xi(q*v, f*v)+d_\xi(f*v, f*y) \leq d_\xi(q,f)+d_\xi(v,y) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

As a result, we can conclude that the operation $*$ is continuous. □

COROLLARY 5.9. If τ_{d_ξ} is the topology induced by d_ξ, then $(\mathcal{L}, *, \tau_{d_\xi})$ is a topological L-algebra.

PROOF: By Theorem 5.2, $(\mathcal{L}, d_\xi)$ is a pseudo-metric space. Let $q*v \in N_\varepsilon(q*v)$. We claim that $N_{\frac{\varepsilon}{2}}(q) * N_{\frac{\varepsilon}{2}}(v) \subseteq N_\varepsilon(q*v)$. Assume $b \in N_{\frac{\varepsilon}{2}}(q) * N_{\frac{\varepsilon}{2}}(v)$. Then there exist $w \in N_{\frac{\varepsilon}{2}}(q)$ and $z \in N_{\frac{\varepsilon}{2}}(v)$ such that $b = w*z$. Hence, $d_\xi(q,w) \leq \dfrac{\varepsilon}{2}$ and $d_\xi(v,z) \leq \dfrac{\varepsilon}{2}$. By Proposition 4.2(v) and (vi) we have $d_\xi(q*v, w*v) \leq d_\xi(q,w)$ and $d_\xi(w*v, w*z) \leq d_\xi(v,z)$. By Proposition 4.2(vii),

$$d_\xi(q*v, w*z) \leq d_\xi(q*v, w*v) + d_\xi(w*v, w*z) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $b = w*z \in N_\varepsilon(q*v)$. Therefore, $(\mathcal{L}, *, \tau_{d_\xi})$ is a topological L-algebra. □

THEOREM 5.10. *We can define a function* $d^* : \frac{\mathcal{L}}{\equiv_\xi} \times \frac{\mathcal{L}}{\equiv_\xi} \to \mathbb{R}$, *where for any* $[q]_\xi, [v]_\xi \in \frac{\mathcal{L}}{\equiv_\xi}$, *we have* $d^*([q]_\xi, [v]_\xi) = d_\xi(q,v)$. *Then, if* ξ *is a* 1*-valuation map, the space* $\left(\frac{\mathcal{L}}{\equiv_\xi}, d^*\right)$ *forms a metric space.*

PROOF: By Theorem 5.7, the proof becomes clear. □

PROPOSITION 5.11. If $(\mathcal{L}, d_\xi)$ is a metric space, then ξ is a 1-valuation map.

PROOF: By Remark 4.1, $\xi(q) = d_\xi(1,q)$, so if d_ξ is a metric, we have $\xi(q) = 0$ if and only if $d_\xi(1,q) = 0$ if and only if $q = 1$. Hence, ξ is a 1-valuation map. □

Note. If $(\mathcal{L}, d)$ is a pseudo-metric space, then

(i) The set $U \subseteq \mathcal{L}$ is open in $(\mathcal{L}, d_\xi)$ if for each $q \in U$, there exists $\varepsilon > 0$

such that $N_\varepsilon(q) \subseteq U$,

(ii) The topology τ_{d_ξ} induced by d_ξ is the collection of all open sets in $(\mathcal{L}, d_\xi)$.

PROPOSITION 5.12. A valuation ξ on the topological L-algebra $(\mathcal{L}, \tau_{d_\xi})$ is continuous if and only if for each $\varepsilon > 0$, there exists a neighborhood U of 1 such that $\xi(b) < \varepsilon$, for each $b \in U$.

PROOF: Let $q \in \mathcal{L}$ and ε be a positive number. Consider a neighborhood U of 1 such that $\xi(b) < \varepsilon$ for every $b \in U$. By property $(L1)$, we have $q{*}q = 1$. Therefore, there exist open neighborhoods V and W of q such that $V{*}W \subseteq U$. Put $F = V{\cap}W$. For every $v \in F$, we have $q{*}v, v{*}q \in F{*}F \subseteq U$, which implies $\xi(q * v), \xi(v * q) < \varepsilon$. Thus, by Proposition 4.2(viii), we have $|\xi(v) - \xi(q)| < \varepsilon$. This shows that ξ is continuous.

Conversely, if ξ is continuous on $\mathcal{L}$, then it is continuous in 1. Let ε be a positive number. Since $\xi(1) = 0$, $(-\varepsilon, \varepsilon)$ is an open neighborhood of $\xi(1)$ in $\mathbb{R}$. There exists an open neighborhood U of 1 in $\mathcal{L}$ such that $\xi(U) \subseteq (-\varepsilon, \varepsilon)$. Therefore, $\xi(b) < \varepsilon$ for every $b \in U$.

Short proof: If $q \in N_\varepsilon(q)$, then $\xi(q) < \varepsilon$, so ξ should always be continuous w.r.t. d_ξ. □

A function between two metric spaces is called an *isometry* if it preserves distances. Let $\xi_{\mathcal{L}} \in \mathcal{V}(\mathcal{L})$ and $\xi_{\mathcal{K}} \in \mathcal{V}(\mathcal{K})$, where $\mathcal{K}$ is an L-algebra. Then, a function $f : \mathcal{L} \to \mathcal{K}$ is called valuation preserving if $\xi_{\mathcal{K}} \circ f = \xi_{\mathcal{L}}$.

PROPOSITION 5.13. Consider $\xi_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$ and $\xi_{\mathcal{K}} : \mathcal{K} \to \mathbb{R}$ as valuations, where $\mathcal{K}$ is an L-algebra. If $f : \mathcal{L} \to \mathcal{K}$ is an L-homomorphism, then the following statements are equivalent:

(i) f is valuation preserving,

(ii) f is an isometry.

PROOF: Assuming that f is valuation preserving, we have $\xi_{\mathcal{K}}(f(q)) = \xi_{\mathcal{L}}(q)$ for any $q \in \mathcal{L}$. Furthermore, for any $q, v \in \mathcal{L}$, we have the following:

$$\begin{aligned} d_{\xi_{\mathcal{K}}}(f(q), f(v)) &= \xi_{\mathcal{K}}(f(q) * f(v)) + \xi_{\mathcal{K}}(f(v) * f(q)) \\ &= \xi_{\mathcal{K}}(f(q * v)) + \xi_{\mathcal{K}}(f(v * q)) \\ &= \xi_{\mathcal{K}} \circ f(q * v) + \xi_{\mathcal{K}} \circ f(v * q) \\ &= \xi_{\mathcal{L}}(q * v) + \xi_{\mathcal{L}}(v * q) \\ &= d_{\xi_{\mathcal{L}}}(q, v). \end{aligned}$$

Hence, it follows that f is an isometry. Conversely, if f is an isometry, then for any $q \in \mathcal{L}$, we have

$$\xi_{\mathcal{L}}(q) = d_{\xi_{\mathcal{L}}}(q, 1) = d_{\xi_{\mathcal{K}}}(f(q), f(1)) = \xi_{\mathcal{K}}(f(q)) + \xi_{\mathcal{K}}(f(1)) = \xi_{\mathcal{K}}(f(q)).$$

Therefore, f is valuation preserving. □

Proposition 5.14. Consider $\xi \in \mathcal{V}(\mathcal{L})$, $X \subseteq \mathcal{L}$ is closed, and $q \in \mathcal{L}$. If there exists $v \in X$ such that $q \equiv_{\ker(\xi)} v$, then it follows that $q \in X$. Conversely, if $\ker(\xi)$ is a neighborhood of 0, the converse is also true.

Proof: Assume that there exists $v \in X$ such that $q \equiv_{\ker(\xi)} v$. This implies that $\xi(v * q) = \xi(q * v) = 0$. Consequently, for any $\varepsilon > 0$, we have $\xi(q * v) + \xi(v * q) < \varepsilon$. Therefore, for each $\varepsilon > 0$, the set $N_\varepsilon(q) \cap X$ is non-empty, which implies $q \in X$.

Conversely, let $\ker(\xi)$ be a neighborhood of 1 and $q \in X$. There exists a sequence $\{q_n\}$ in X such that $q_n \mapsto q$. Since $*$ is continuous, it follows that $\xi(q * q_n) \mapsto 0$ and $\xi(q_n * q) \mapsto 0$. Hence, there exists a positive integer n_0 such that $q_{n_0} * q \in \ker(\xi)$ and $q * q_{n_0} \in \ker(\xi)$. Therefore, $q_{n_0} \equiv_{\ker(\xi)} q$. □

Theorem 5.15. *Let $\xi \in \mathcal{V}(\mathcal{L})$. We define the relation $\approx$ as follows:*

$$\{q_n\} \approx \{v_n\} \;\Leftrightarrow\; d_\xi(q_n, v_n) \longmapsto 0,$$

for all sequences $\{q_n\}$ and $\{v_n\}$ in $\mathcal{L}$. We will show that $\approx$ is a congruence relation on the set of all sequences in $\mathcal{L}$.

Proof: Obviously, by Proposition 4.2(i), (iii) and (iv), $\approx$ is an equivalence relation on $\mathcal{L}$. Let $\{q_n\} \approx \{v_n\}$ and $\{f_n\} \approx \{y_n\}$. Then $d_\xi(q_n, v_n) \longmapsto 0$ and $d_\xi(f_n, y_n) \longmapsto 0$. By Proposition 4.2(v) and (vi), we have $d_\xi(q_n *$

$f_n, v_n * f_n) \longmapsto 0$ and $d_\xi(v_n * f_n, v_n * y_n) \longmapsto 0$. By Proposition 4.2(iv), we have $d_\xi(q_n * v_n, f_n * y_n) \longmapsto 0$ and so $\{q_n\} * \{v_n\} \approx \{f_n\} * \{y_n\}$. Therefore, $\approx$ is a congruence relation on the set of all sequences in $\mathcal{L}$. $\square$

Note. Let ξ be an element of $\mathcal{V}(\mathcal{L})$. The set of all equivalence classes, denoted by $\widetilde{\{q_n\}}$, is defined as $\{\{v_n\} \mid \{v_n\} \approx \{q_n\}\}$. Set $\widetilde{\mathcal{L}} = \{\widetilde{\{q_n\}} \mid \{q_n\}$ is a sequences in $\mathcal{L}\}$. By Theorem 5.15, it is clear that $(\widetilde{\mathcal{L}}, *, \widetilde{\{1\}})$ forms an L-algebra.

Proposition 5.16. Let ξ be a 1-valuation on $\mathcal{L}$. Then, the set $(\widetilde{\mathcal{L}}, *, \widetilde{\{1\}})$ forms an L-algebra, and the set $(\widetilde{\mathcal{L}}, \widetilde{d_\xi})$ forms a metric space. The pseudo-metric d_ξ induces a metric $\widetilde{d_\xi}$ on $\mathcal{L}$ in the following way:

$$\widetilde{d_\xi}\left(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}\right) = \lim_n d_\xi(q_n, v_n),$$

for all $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$.

Proof: It is evident that $\widetilde{d_\xi}$ is a pseudo-metric on $\widetilde{\mathcal{L}}$. For any $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$, if $d_\xi(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) = 0$, then it implies that $d_\xi(q_n, v_n) \to 0$. Consequently, $\{q_n\} \approx \{v_n\}$, which implies $\widetilde{\{q_n\}} = \widetilde{\{v_n\}}$. Hence, we can conclude that $(\widetilde{\mathcal{L}}, \widetilde{d_\xi})$ forms a metric space. $\square$

Proposition 5.17. Let $\xi \in \mathcal{V}(\mathcal{L})$. Then the mapping $\pi_\xi : \mathcal{L} \to \widetilde{\mathcal{L}}$ defined by $\widetilde{\pi}_\xi(q) = \widetilde{\{q\}}$, where $\widetilde{\{q\}}$ is the equivalence class of the constant sequence with any element equal to q, is a homomorphism.

Proposition 5.18. Consider $\xi \in \mathcal{V}(\mathcal{L})$. Then $\widetilde{\xi} \in \mathcal{V}(\widetilde{\mathcal{L}})$ where $\widetilde{\xi} : \widetilde{\mathcal{L}} \to \mathbb{R}$ by $\widetilde{\xi}(\widetilde{\{q_n\}}) = \lim_n \xi(q_n)$ for each sequence in $\mathcal{L}$.

Proof: It is evident that $\widetilde{\xi}(\widetilde{\{1\}}) = 0$. Now, let $\{q_n\}$ and $\{v_n\}$ be sequences in $\mathcal{L}$. Since both ξ and $*$ are continuous, we can conclude that,

$$\begin{aligned}\widetilde{\xi}(\{v_n\}) - \widetilde{\xi}(\{q_n\}) &= \lim_n \xi(v_n) - \lim_n \xi(q_n) = \lim_n \xi(v_n) - \xi(q_n) \\ &\leq \lim_n \xi(q_n * v_n) = \widetilde{\xi}(\{q_n\} * \{v_n\}).\end{aligned}$$

Hence $\widetilde{\xi}$ is a valuation on $\widetilde{\mathcal{L}}$. □

Note. A sequence $\{q_n\} \subseteq \mathcal{L}$ is said to be d_ξ-Cauchy if it is a Cauchy sequence with respect to the pseudo-metric d_ξ on $(\mathcal{L}, d_\xi)$. The space $(\mathcal{L}, d_\xi)$ is said to be d_ξ-complete if every d_ξ-Cauchy sequence converges to an element in $\mathcal{L}$. Now, let $\{q_n\}$ and $\{v_n\}$ be two d_ξ-Cauchy sequences. It follows that the sequence $\{d_\xi(q_n, v_n)\}$ is convergent, as it is a Cauchy sequence in the real numbers $\mathbb{R}$.

Corollary 5.19. The metric space $(\widetilde{\mathcal{L}}, \widetilde{d_\xi})$ is $\widetilde{d_\xi}$-complete.

Proposition 5.20. If $\widetilde{\mathcal{L}}, \widetilde{\xi}, \pi_\xi$ and d are defined as described above, then the following properties hold:

(i) $\widetilde{\xi} \circ \pi_\xi = \xi$, which implies that π_ξ is 1-valuation preserving.

(ii) ξ is a 1-valuation if and only if $\pi_\xi(q) = \widetilde{\{1\}}$ implies that $q = 1$.

(iii) $\widetilde{d_\xi} = d_{\widetilde{\xi}}$.

(iv) π_ξ is continuous.

Proof: (i) For any $q \in \mathcal{L}$, $\widetilde{\xi} \circ \pi_\xi(q) = \widetilde{\xi}(\pi_\xi(q)) = \lim_n \xi(q) = \xi(q)$.

(ii) Let ξ be a 1-valuation and $\pi_\xi(q) = \widetilde{\{1\}}$. Then $\widetilde{\{q\}} = \widetilde{\{1\}}$ and so $\{q\} \approx \{1\}$. Hence, $\xi(q) = d_\xi(q, 1) = 0$. Since ξ is a 1-valuation, $q = 1$. Conversely, if $\xi(q) = 0$ for any $q \in \mathcal{L}$, then $d_\xi(q, 1) = \xi(q) = 0$ and so $\pi_\xi(q) = \widetilde{\{q\}} = \widetilde{\{1\}}$. Hence $q = 1$. Thus ξ is a 1-valuation.

(iii) For any $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$ since ξ and $*$ are continuous, we have

$$\begin{aligned} d_{\widetilde{\xi}}(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) &= \widetilde{\xi}(\widetilde{\{q_n * v_n\}}) + \widetilde{\xi}(\widetilde{\{v_n * q_n\}}) = \lim_n \xi(q_n * v_n) + \lim_n \xi(v_n * q_n) \\ &= \lim_n (\xi(q_n * v_n) + \xi(v_n * q_n)) = \lim_n d_\xi(q_n, v_n) \\ &= d_\xi(\lim_n q_n, \lim_n v_n) = \widetilde{d_\xi}(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}). \end{aligned}$$

(iv) This follows from Proposition 5.13(i) as π_ξ is valuation-preserving and the fact that isometries are always continuous. □

THEOREM 5.21. *Let $\psi \in \mathcal{V}(\mathcal{K})$ such that $(\mathcal{K}, d_\psi)$ is a d_ψ-complete space, where $\mathcal{K}$ is an L-algebra. If $\xi \in \mathcal{V}(\mathcal{L})$ and $\Theta : \mathcal{L} \to \mathcal{K}$ is a valuation preserving homomorphism, then there exists a unique valuation preserving homomorphism $\widetilde{\Theta} : \widetilde{\mathcal{L}} \to \mathcal{K}$ such that $\widetilde{\Theta} \circ \pi_\xi = \Theta$.*

PROOF: Suppose $\Theta : \mathcal{L} \to \mathcal{K}$ is a valuation-preserving homomorphism. By Proposition 5.13, Θ is an isometry. If $\{q_n\}$ is a sequence in $\mathcal{L}$, then $\{\Theta(q_n)\}$ is a sequence in $\mathcal{K}$. Since $\mathcal{K}$ is d_ψ-complete, $\Theta(q_n)$ converges to v for some $v \in \mathcal{K}$. We define $\widetilde{\Theta}(\widetilde{\{q_n\}}) = v$. We will now show that $\widetilde{\Theta}$ is the unique isometry such that $\widetilde{\Theta} \circ \pi_\xi = \Theta$. Let $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$, $\Theta(q_n) \mapsto q$ and $\Theta(v_n) \mapsto v$. Then

$$\begin{aligned}
d_{\widetilde{\xi}}(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) &= \widetilde{d_\xi}(\widetilde{\{q_n\}}, \widetilde{\{v_n\}}) \\
&= \lim_n \xi(q_n * v_n) + \lim_n \xi(v_n * q_n) \\
&= \lim_n \psi \circ \Theta(q_n * v_n) + \lim_n \psi \circ \Theta(v_n * q_n) \\
&= \lim_n \psi(\Theta(q_n * v_n)) + \lim_n \psi(\Theta(v_n * q_n)) \\
&= \lim_n \psi(\Theta(q_n) * \Theta(v_n)) + \lim_n \psi(\Theta(v_n) * \Theta(q_n)) \\
&= \lim_n \psi(q * v) + \lim_n \psi(v * q) \\
&= \psi(q * v) + \psi(v * q) \\
&= \psi(\widetilde{\Theta}(\widetilde{\{q_n\}}) * \widetilde{\Theta}(\widetilde{\{v_n\}})) + \psi(\widetilde{\Theta}(\widetilde{\{y_n\}}) * \widetilde{\Theta}(\widetilde{\{q_n\}})) \\
&= d_\psi(\widetilde{\Theta}(\widetilde{\{q_n\}}), \widetilde{\Theta}(\widetilde{\{v_n\}})).
\end{aligned}$$

The uniqueness of $\widetilde{\Theta}$ is evident. Since the operation in $\mathcal{K}$ is continuous with respect to d_ψ, we can conclude that $\widetilde{\Theta}$ is a homomorphism. Since let $\widetilde{\{q_n\}}, \widetilde{\{v_n\}} \in \widetilde{\mathcal{L}}$, such that $\Theta(q_n) \mapsto q$ and $\Theta(v_n) \mapsto v$. From Θ is a homomorphism, we get $\Theta(q_n * v_n) = \Theta(q_n) * \Theta(v_n) \mapsto q * v$. Thus,

$$\begin{aligned}\widetilde{\Theta}(\widetilde{\{q_n\}} * \widetilde{\{v_n\}}) &= \widetilde{\Theta}(\widetilde{\{q_n * v_n\}}) = \widetilde{\Theta} \circ \pi_\xi(q_n * v_n) = \Theta(q_n * v_n)\\ &= \Theta(q_n) * \Theta(v_n) = \widetilde{\Theta} \circ \pi_\xi(\{q_n\}) * \widetilde{\Theta} \circ \pi_\xi(\{v_n\})\\ &= \widetilde{\Theta}(\widetilde{\{q_n\}}) * \widetilde{\Theta}(\widetilde{\{v_n\}}).\end{aligned}$$

Lastly, for each $q \in \mathcal{L}$, we have $\widetilde{\Theta} \circ \pi_\xi(q) = \widetilde{\Theta}(\widetilde{\{q\}}) = \Theta(q)$. Therefore, $\widetilde{\Theta} \circ \pi_\xi = \Theta$. $\square$

Note. A general remark, it is noteworthy that $\dfrac{\mathcal{L}}{\equiv_\xi}$ is the smallest quotient where $\bar{\xi}([q]) = \xi(q)$ is well-defined and defines a 1-valuation. Therefore, factoring out $\equiv_\xi$ provides a canonical L-algebra with a 1-valuation associated with ξ.

6. Conclusion

In this paper, the notion of valuations on L-algebras is defined, and related properties are investigated. Using a valuation map, a congruence relation is defined and proved that this congruence relation is equivalent with a congruence relation made by ideal. Also, it was shown that image and converse image of any valuation is a valuation. In addition, by using valuation, a metric space is introduced, and is was shown that the operation “ $*$” in an L-algebra is uniformly continuous.

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