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DOUBLE EQUIVALENTIAL ALGEBRAS

Abstract

In this paper, we introduce the notion of double equivalential algebras, defined as subreducts of double Heyting algebras with respect to the equivalence and dual equivalence operations. We establish the fundamental properties of these structures and investigate several distinguished classes related to them, including the class of double equivalential subreducts of Boolean algebras and the variety generated by the three-element chain, which is shown to be semisimple.

Keywords: equivalential algebras, double Heyting algebras, linear order, semisimple.

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1. Introduction

This article introduces the concept of double equivalential algebras, which are specific subreducts of double Heyting algebras. Double Heyting algebras are Heyting algebras whose dual lattices also form Heyting algebras. In these structures, in addition to the lattice operations and the constants 0 and 1, there are also two distinct binary operations interpreted as relative pseudocomplements. Double Heyting algebras — also known as semi-Boolean algebras, bi-Heyting algebras, or Heyting–Brouwer algebras — first appeared in the early 1970s ([20, 21]) and were later studied by the following authors: R. Beazer [1], S. Ghilardi [6], L. Iturrioz [11], P. Köhler [13]. These structures are also related to other algebraic systems, such as dually pseudocomplemented Heyting algebras, which were introduced in [22]. Over the past decade, double Heyting algebras have been revisited in Taylor's doctoral dissertation and publications ([28, 27, 29]). This line of research falls within the broader field of studying various subreducts of Heyting algebras, such as Brouwerian semilattices and Hilbert algebras.

In this article, we investigate another class of subreducts, namely those equipped with two binary operations: one corresponding to equivalence and the other to dual equivalence. Equivalential subreducts of Heyting algebras are referred to as equivalential algebras. This notion was introduced by J. K. Kabziński and A. Wroński in [12] as an algebraic counterpart of the equivalential fragment of intuitionistic propositional logic. These algebras were subsequently studied in depth by K. Słomczyńska ([24, 25]). In recent years, structures closely related to equivalential algebras have also been examined, particularly those with additional operations ([15, 17, 18]).

Equivalential algebras constitute an important example of Fregean algebras, that is, algebras whose language contains a constant term 1 and which are both 1-regular and congruence orderable. Indeed, it turns out that every congruence permutable Fregean variety has a binary term that turns each of its algebras into an equivalential algebra [9, p. 608]. Fregean varieties have been studied primarily by P. Idziak, K. Słomczyńska and A. Wroński ([9, 10]). Other examples of Fregean varieties include the varieties of Boolean algebras, Heyting algebras, Hilbert algebras, Brouwerian semilattices, and other algebras closely related to the algebraic counterparts of classical, intuitionistic, and intermediate logics. Congruence orderability and 1-regularity allow for the introduction of a natural partial order on the universe of every Fregean algebra.

The purpose of this paper is to introduce and investigate the basic properties of double equivalential algebras. These structures are $(\leftrightarrow, \circ)$-subreducts of double Heyting algebras equipped with the operation $\leftrightarrow$, defined by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$, and the operation $\circ$, defined by $x \circ y = (x \leftarrow y) \vee (y \leftarrow x)$, where $\leftarrow$ is the dual operation to $\to$.

The study of these structures presents several challenges. In particular, they are not Fregean, as they are not congruence orderable. Moreover, the class of double equivalential algebras is not locally finite. For this reason, in what follows, we consider varieties generated by finite double equivalential chains. This line of investigation is closely related to the study of the equivalential fragment of one of the most important intermediate logics: Gödel–Dummett (or chain) intermediate logic LC, which was introduced by Kurt Gödel in 1932 [7] and later axiomatized by Michael Dummett [3] by adding the linearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$ to intuitionistic propositional logic (see [25, p. 1341]).

In Section 2, we recall the basic concepts related to double Heyting algebras and equivalential algebras. Next, in Section 3, we introduce the notion of double equivalential algebras. We provide examples of these structures, outline the difficulties associated with their study, and describe their fundamental properties. Subsequently, we consider two classes of algebras related to double equivalential algebras. In Subsection 3.4, we focus on subreducts of Boolean algebras, and in Subsection 3.5, we examine the variety generated by the three-element chain.

2. Double Heyting algebras and equivalential algebras

First, we recall some fundamental notions and facts about double Heyting algebras and equivalential algebras.

Definition 2.1 ([27, p. 6]). An algebra $\mathbf{A} = (A, \vee, \wedge, \to, \leftarrow, 0, 1)$ is called a **double Heyting algebra** if $(A, \vee, \wedge, \to, 0, 1)$ is a Heyting algebra and

$(A, \vee, \wedge, \leftarrow, 0, 1)$ is a dual Heyting algebra. More precisely, $\mathbf{A}$ is a double Heyting algebra if $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and the operations $\rightarrow$ and $\leftarrow$ satisfy the following equivalences:

$$x \wedge y \leq z \Longleftrightarrow y \leq x \rightarrow z,$$

$$x \vee y \geq z \Longleftrightarrow y \geq z \leftarrow x.$$

Basic information about Heyting algebras can be found in [19] and [4]. Below, we present some of their most important properties.

Lemma 2.2 ([19, pp. 59–60]). *Let $\mathbf{L}$ be a Heyting algebra and let $x, y, z \in L$. Then:*

1. $x \leq y$ *if and only if* $x \rightarrow y = 1$,
2. $y \leq x \rightarrow y$,
3. $x \wedge (x \rightarrow y) = x \wedge y$,
4. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
5. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
6. *if* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$ *and* $x \rightarrow z \geq y \rightarrow z$.

We denote the class of double Heyting algebras by $\mathcal{DH}$.

Definition 2.3. By an **equivalential algebra** we mean an algebra $(A, \leftrightarrow)$ that is a $(\leftrightarrow)$-subreduct of a Heyting algebra, where the operation $\leftrightarrow$ is defined by:

$$x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x).$$

We adopt the convention of associating to the left and replacing with $\cdot$ (or ignoring) the symbol of the equivalence operation.

In 1975, Kabziński and Wroński proved that the class of all equivalential algebras is equationally definable by the following identities: $xxy = y$, $xyzz = xz(yz)$, $xy(xzz)(xzz) = xy$, and thus forms a variety (see [12]). We denote this class by $\mathcal{E}$.

Remark 2.4 ([12, p. 420]). Let $\mathbf{A} \in \mathcal{E}$. For all $a, b, c \in A$, we have

1. $ab = ba$,
2. $aa = bb$,
3. $abbb = ab$,
4. $abbaa = abb$,
5. $abcc = acc(bcc)$,
6. $abbcc = accbb$,
7. $abb(bc)(bc) = abbcc$,
8. $abcb = a(bc)b$.

We can define the unit element by $1 := aa$, for an arbitrary $a \in A$ (see [24, p. 526]). We note that $xy = 1$ if and only if $x = y$. If A is an algebra equipped with a binary operation satisfying the conditions of Definition 2.3, then we denote the equivalential reduct of $\mathbf{A}$ by $\mathbf{A}^e$.

Lemma 2.5 ([28, p. 4]). *Let* $\mathbf{H}$ *be a Heyting algebra and let* $x, y \in H$*. Then* $x \cdot y$ *is the greatest* $z \in H$ *such that* $x \wedge z = y \wedge z$*. That is:*

$$x \wedge z = y \wedge z \Longleftrightarrow z \leq x \cdot y$$

Equivalential algebras are important examples of Fregean algebras.

Definition 2.6 ([9, p. 597]). An algebra $\mathbf{A}$ with a distinguished constant term 1 is called **Fregean** if $\mathbf{A}$ is:

1. **1-regular**, i.e., $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$,
2. **congruence orderable**, i.e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$.
   A variety $\mathcal{V}$ is said to be Fregean if all its algebras are Fregean.

These two properties of congruences allow us to introduce a natural partial order on the universe of every Fregean algebra $\mathbf{A}$ in the following way: $a \leq b$ iff $\Theta_{\mathbf{A}}(1, b) \subseteq \Theta_{\mathbf{A}}(1, a)$. Clearly, 1 is the greatest element in this order. From 1-regularity it follows that the Fregean varieties are congruence modular (see [8]).

Definition 2.7 ([26, p. 2]). An algebra $(A, \cdot, 0)$ with one binary operation $\cdot$ and one constant 0 is called an **equivalential algebra with zero** if $(A, \cdot)$ is an equivalential algebra and $\mathbf{A}$ satisfies the identity:

$$x00yy = x00.$$

Dually, we introduce the following definition and notation:

Definition 2.8. By a **dual equivalential algebra** we mean an algebra $(A, \circ)$ that is a $(\circ)$-subreduct of a dual Heyting algebra, where the operation $\circ$ is defined by:

$$x \circ y := (x \leftarrow y) \vee (y \leftarrow x).$$

As with the operation $\cdot$, we adopt the convention of associating to the left, that is, instead of $(x \circ y) \circ z$ we use the more concise notation: $x \circ y \circ z$. This convention is also adopted when the operations $\cdot$ and $\circ$ appear together. Moreover, we assume that the operators $\cdot$ and $\circ$ bind more strongly than the standard operations in a Heyting algebra (i.e., $\wedge, \vee$ and $\rightarrow$).
We can define the zero element by $0 := a \circ a$, for an arbitrary $a \in A$. Furthermore:

Lemma 2.9. *If $\mathbf{H}$ is a dual Heyting algebra and $x, y \in H$, then $x \circ y$ is the smallest $z \in H$ such that $x \vee z = y \vee z$, that is:*

$$x \vee z = y \vee z \Longleftrightarrow x \circ y \leq z.$$

Dually to Definition 2.7, we define a **dual equivalential algebra with zero** (in this case, the role of zero is played by one) as an algebra $(A, \circ, 1)$, with one binary operation $\circ$ and one constant 1, such that $(A, \circ)$ is a dual equivalential algebra and A satisfies the identity: $x \circ 1 \circ 1 \circ y \circ y = x \circ 1 \circ 1$.

We say that $x, y \in A$ are **orthogonal** if $xyy = x$ and $yxx = y$. Dually, we say that $x, y \in A$ are **dually orthogonal** if $x \circ y \circ y = x$ and $y \circ x \circ x = y$.

We now introduce the following notation:

$$d(x) := x \circ 1 \cdot 0,$$

$$g(x) := x \cdot 0 \circ 1.$$

LEMMA 2.10. *Let* $\mathbf{H}$ *be a double Heyting algebra and let* $x, y \in H$*. Then:*

1. $x \cdot 0 = x \to 0 = \max\{c \in H : c \wedge x = 0\}$*, that is,* $x \cdot 0$ *is the pseudocomplement of* x *in* H*,*
2. $x \circ 1 = 1 \leftarrow x = \min\{c \in H : c \vee x = 1\}$*, that is,* $x \circ 1$ *is the dual pseudocomplement of* x *in* H*,*
3. $x \cdot 0 \leq x \circ 1$*,*
4. $d(x) \leq x \circ 1 \circ 1 \leq x \leq x \cdot 0 \cdot 0 \leq g(x)$*,*
5. $x \cdot 0 = x \circ 1$ *if and only if* $d(x) = x$*,*
6. $x \cdot y \circ 1 \geq x \circ y$ *and* $x \circ y \cdot 0 \leq x \cdot y$*.*

PROOF: Items (1–2) follow from the definitions of $\cdot$ and $\circ$, items (3–5) follow from [28, p. 5], and item (6) from [28, p. 74]. □

2.1. Filters in equivalential algebras

Let $\Theta(a, b)$ denote the congruence generated by the pair (a, b). Let $\mathbf{A} \in \mathcal{E}$. By a **filter**, we mean a non-empty subset $F \subseteq A$ such that for all $a, b, x \in A$: (i) if $a \in F$, then $axx \in F$ and (ii) if $a \in F$ and $ab \in F$, then $b \in F$. It follows from [12, p. 421] that there is a one-to-one correspondence between the filters of $\mathbf{A}$ and its congruence relations. Namely, if $F \subseteq A$, then F is a filter if and only if the relation φ given by $a\varphi b \iff ab \in F$ is a congruence on A. Moreover, $\Theta(1, ab) = \Theta(a, b)$ (see [9, Theorem 3.8]).

Let $x \in A$. We denote by $[x)_e$ the smallest filter containing x. The relation $\leq_e$ defined by $x \leq_e y$ whenever $y \in [x)_e$ is a partial order on $\mathbf{A}$. We also introduce the relation $<<$ on A, defined by $x << y$ if $xy = x$ (i.e., $yxx = 1$). Its restriction to $A \setminus \{1\}$ is a strict order, while $<< \cup \mathrm{Id}$ is a partial order on A. We note that if $\mathbf{A}$ is a Heyting algebra and $x << y$, then $x \to y = 1$.

Dually, by a **dual filter** we mean a non-empty subset $F \subseteq A$ such that for all $a, b, x \in A$: (i) if $a \in F$, then $a \circ x \circ x \in F$ and (ii) if $a \in F$ and $a \circ b \in F$, then $b \in F$. Let $x \in A$. We denote by $(x]_d$ the smallest dual filter containing x. The relation $\leq_d$, defined by $x \leq_d y$ whenever $x \in (y]_d$ is a partial order on A. We also introduce the relation $<<_d$ on A defined by $x <<_d y$ if $x \circ y = y$ (i.e., $x \circ y \circ y = 0$).

3. Double equivalential algebras

3.1. Definition and examples

Definition 3.1. Let $\mathbf{A} = (A, \cdot, \circ, 1, 0)$ be a $(\cdot, \circ)$-subreduct of a double Heyting algebra. Then $\mathbf{A}$ is called a **double equivalential algebra**.

We denote the class of double equivalential algebras by $\mathcal{DE}$. This class forms a quasivariety (see [14]). Note that if $\mathbf{A} = (A, \cdot, \circ, 1, 0) \in \mathcal{DE}$, then $(A, \cdot, 0)$ is an equivalential algebra with zero and $(A, \circ, 1)$ is a dual equivalential algebra with zero. Moreover, $1 = x \cdot x$ and $0 = x \circ x$ for all $x \in A$. Every algebra in $\mathcal{DE}$ has permutable congruences, since $(xyz)(xzzx)$ is a Mal'cev term for equivalential algebras (see [10, p. 332]). Consequently, every variety generated by algebras from $\mathcal{DE}$ is congruence permutable. Both the variety of equivalential algebras and the variety of dual equivalential algebras are Fregean – that is, they are congruence orderable, with the former being 1-regular and the latter 0-regular. Therefore, the class $\mathcal{DE}$ is both 1-regular and 0-regular (since point regularity is preserved under expansions of a language, see [9, p. 600]). However, the class $\mathcal{DE}$ is not Fregean, as it is not congruence orderable. Let us consider the following example.

Example 3.2. Let $\mathbf{A}$ be the $(\cdot, \circ)$-reduct of the double Heyting algebra $\mathbf{3} \times \mathbf{2}$ (see Figure 1). It is easily seen that $\Theta(1, a) = \Theta(1, c)$ and $\Theta(1, 0) = A^2$. Therefore, $\operatorname{Con} \mathbf{A}$ is a four-element lattice (in addition to the aforementioned ones, it also contains Id and $\Theta(1, b)$). Since $\Theta(1, a) = \Theta(1, c)$, the algebra $\mathbf{A}$ is not congruence orderable.

This presents a significant challenge in the analysis of these structures,

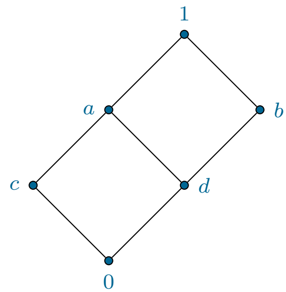

Figure 1: Double Heyting algebra $\mathbf{3} \times \mathbf{2}$.

specifically due to the lack of an inherent order. Although it is possible to define orderings such as $\leq_e, \leq_d, <<$, and $<<_d$ on the algebra $A \in \mathcal{DE}$ — and in the case where $\mathbf{A}$ is a Heyting algebra, also the standard lattice order (denoted by $\leq$) — these orderings do not coincide and fail to satisfy many of the properties that hold for the lattice order in double Heyting algebras. Moreover, in the case where $\mathbf{A}$ is a Boolean algebra, all of these orderings — except for $\leq$ — reduce to the identity relation on the set $A \setminus \{1, 0\}$.

The following diagrams (Figure 2) illustrate the aforementioned orderings for the algebra presented in Example 3.2 (from left to right: $\leq_e$, $<<$, $\leq_d$, and $<<_d$). Note that even in this simple example, some of the laws satisfied by the lattice order in double Heyting algebras do not hold.

1. $d(x) \leq_e x$ (counterexample: $d(d) = 0 \not\leq_e d$),
2. $x \leq_e y \Longrightarrow d(x) \leq_e d(y)$ (counterexample: $0 \leq_e a$, but $0 = d(0) \not\leq_e d(a) = c$; a similar situation occurs if we replace $\leq_e$ with $<<$),
3. $x \leq_e y \Longrightarrow y0 \leq_e x0$ (counterexample: $c \leq_e a$, but $a0 = 0 \not\leq_e b = c0$; a similar situation occurs if we replace $\leq_e$ with $<<$).

Moreover, we have the following theorem.

THEOREM 3.3. *$\mathcal{DE}$ is not locally finite.*

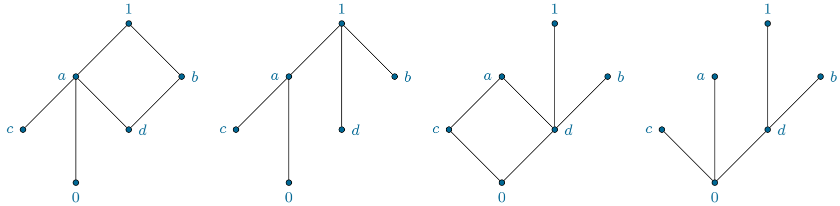

Figure 2: Different orderings for the algebra from Example 3.2 (from left to right: $\leq_e$, $<<$, $\leq_d$, and $<<_d$).

Proof: We proceed analogously to [16, p. 142]. Let us consider the free Heyting algebra over one generator x, i.e., the Rieger–Nishimura lattice $\mathbf{R}$ (see Figure 3). First, observe that $\mathbf{R}$ is a double Heyting algebra. The universe of $\mathbf{R}$ has the form $R = \{0,1\} \cup \{r_k : k \in \mathbb{N}\}$, where $r_1 = x$, $r_2 = x \cdot 0$, $r_{2n+1} = r_{2n} \circ r_{2n-1}$, $r_{2n+2} = r_{2n+1} \cdot r_{2n-1}$ for $n = 1, 2, \ldots$. Therefore, the $(\cdot, \circ)$-reduct of $\mathbf{R}$ is a one-generated infinite double equivalential algebra, hence $\mathcal{DE}$ is not locally finite. $\square$

3.2. Basic properties

Lemma 3.4. *Let $\mathbf{H} \in \mathcal{DH}$ and let $a, b \in H$. Then:*

1. $a \cdot b \geq a \wedge b$ *and* $a \circ b \leq a \vee b$,
2. $a0 \circ a = a0 \vee a$ *and* $(a \circ 1)a = (a \circ 1) \wedge a$.

Proof: **Ad 1.** This follows from Lemmas 2.5 and 2.9.

Ad 2. From Lemma 2.9, we have: $(a0 \circ a) \vee a = (a0 \circ a) \vee a0$. Taking the meet of both sides with $a0$, we obtain: $((a0 \circ a) \vee a) \wedge a0 = ((a0 \circ a) \vee a0) \wedge a0 = a0$. By distributivity, $((a0 \circ a) \wedge a0) \vee (a \wedge a0) = a0$, which implies that $(a0 \circ a) \wedge a0 = a0$, and hence $a0 \circ a \geq a0$. Next, taking the meet of both sides of the original equality $(a0 \circ a) \vee a = (a0 \circ a) \vee a0$ with a, and proceeding

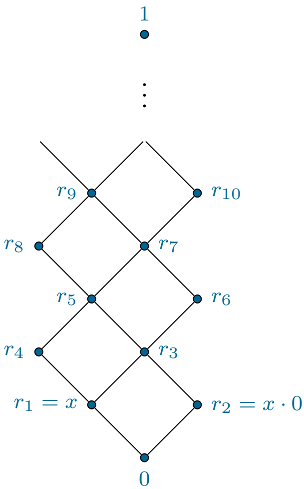

Figure 3: Rieger–Nishimura lattice $\mathbf{R}$.

analogously, we obtain $a0 \circ a \geq a$. Therefore, $a0 \circ a \geq a0 \vee a$. Since, by item (1), we know that $a0 \circ a \leq a0 \vee a$, it follows that $a0 \circ a = a0 \vee a$. The second equality is proved analogously. $\square$

THEOREM 3.5. *Let $\mathbf{A} \in \mathcal{DE}$ and let $a, b \in A$. Then:*

1. $(a \circ 1)aa = a \circ 1, \quad a0 \circ a \circ a = a0,$
2. $a(a \circ 1)(a \circ 1) = a, \quad a \circ (a0) \circ (a0) = a,$
3. $(a0 \circ a) \cdot 0 = 0, \quad (a \circ 1 \cdot a) \circ 1 = 1,$
4. $a0 \circ a \cdot a = a00, \quad a \circ 1 \cdot a \circ a = a \circ 1 \circ 1,$
5. $(a0) \cdot (a \circ 1) = a \cdot d(a), \quad (a0) \circ (a \circ 1) = a \circ g(a),$
6. *if $a << b$, then $b0 = 0$ and $g(b) = 1$.*

Proof: If a given item contains two dual identities, we prove the first one (the second, being its dual, can be shown analogously). It suffices to show that the given identities hold in double Heyting algebras.

Ad 1. Using item (2) from Lemma 3.4, we know that $(a\circ 1)aa \geq a\circ 1 \geq (a\circ 1)\wedge a = (a\circ 1)a$. From this and Lemma 2.5, it follows that $(a\circ 1)aa \wedge a = (a\circ 1)aa \wedge (a\circ 1)a = (a\circ 1)a$. Since $a\circ 1 \leq (a\circ 1)aa$, by modularity we obtain:

$$((a\circ 1)\vee a)\wedge (a\circ 1)aa = (a\circ 1)\vee (a\wedge (a\circ 1)aa). \tag{3.1}$$

Because $(a\circ 1)\vee a = 1$, the left-hand side of (3.1) equals $(a\circ 1)aa$. On the other hand, the right-hand side of (3.1), by the previously established equality, can be rewritten as $(a\circ 1)\vee (a\circ 1)a = a\circ 1$. Hence, the claim follows.

Ad 2. The proof proceeds analogously to item (1).

Ad 3. Using item (2) from Lemma 3.4 and the properties of operations in Heyting algebras, we obtain: $(a0\circ a)\cdot 0 = (a0\vee a)\cdot 0 = [(a0\vee a)\to 0]\wedge [0\to (a0\vee a)] = (a0\vee a)\to 0 = (a0\to 0)\wedge (a\to 0) = (a00)\wedge a0 = 0$.

Ad 4. $a0\circ a\cdot a = (a0\vee a)\cdot a = [(a0\vee a)\to a]\wedge [a\to (a0\vee a)] = (a0\vee a)\to a = (a0\to a)\wedge (a\to a) = a0\to a = (a\to 0)\to a$.
Since $axx = (a\to x)\to a$, the final expression equals $a00$.

Ad 5. This follows from Lemma 2.10 (item (5)).

Ad 6. From the assumption that $a\cdot b = a$, it follows that $b\to a = a$. Thus, $b0 = b\to 0 = b\to (a\wedge 0) = (b\to a)\wedge (b\to 0) = a\wedge (b\to 0)$. Hence, $b\to 0\leq a\leq b$, and therefore $b\to 0 = (b\to 0)\wedge b = b\wedge 0 = 0$. $\square$

Note that, based on items (1)–(2) of Theorem 3.5, it follows that the elements a and $a\circ 1$ are orthogonal, while the elements a and $a0$ are dually orthogonal.

Definition 3.6. Let $\mathbf{A}\in \mathcal{DE}$, and let $a\in A$. The element a is called **central** if $d(a) = a$.

THEOREM 3.7. *Let* $\mathbf{H} \in \mathcal{DH}$ *and let* $a, b \in H$*. Then, if* $a < b$*, it follows that* $b0 << (a0) \circ a$*.*

PROOF: Assume that $a < b$. Then $a0 \geq b0$. Since, by Lemma 3.4, item (2), we have $a0 \vee a = a0 \circ a$, it remains to show that

$$(a0 \vee a) \cdot (b0) = b0. \tag{3.2}$$

We first show that $[(a0 \vee a) \cdot (b0)] \wedge a = 0$. From the properties of operations in a Heyting algebra, we have:

$$[(a0 \vee a) \cdot (b0)] \wedge a = [((a0 \vee a) \to b0) \wedge (b0 \to (a0 \vee a))] \wedge a. \tag{3.3}$$

From the assumption, we have $a0 \vee a \geq b0$, so $b0 \to (a0 \vee a) = 1$. Therefore, expression (3.3) is equal to $[(a0 \vee a) \to b0] \wedge a = (a \to b0) \wedge (a0 \to b0) \wedge a = [(a \to b0) \wedge a] \wedge (a0 \to b0) = (a \wedge b0) \wedge (a0 \to b0) = a \wedge b0$. Since $b \wedge b0 = 0$ and $a < b$, it follows from Lemma 2.5 that expression (3.3) equals 0. By Lemma 2.5, we obtain $(a0 \vee a) \cdot (b0) \leq a0$.
Once again, by Lemma 2.5, we have:

$$[(a0 \vee a) \cdot (b0)] \wedge (a0 \vee a) = [(a0 \vee a) \cdot (b0)] \wedge b0. \tag{3.4}$$

From the previously proven inequality, the left-hand side of (3.4) equals $(a0 \vee a) \cdot (b0)$. As for the right-hand side, since $a0 \geq b0$, and by Lemma 3.4, item (1), we obtain $(a0 \vee a) \cdot (b0) \geq (a0 \vee a) \wedge (b0) \geq b0$. Therefore, the right-hand side of (3.4) is equal to $b0$, and we conclude that: $(a0 \vee a) \cdot (b0) = b0$.
□

COROLLARY 3.8. Let $\mathbf{H} \in \mathcal{DH}$ and let $c, x \in H$. Then, if $c < x0$, it follows that $x << (c \cdot 0) \circ c$.

PROOF: Using Theorem 3.7 and the fact that $x \leq x00$, the claim follows.
□

Let $\mathbf{A} \in \mathcal{DE}$, and let $x \in A$. The element x is called **regular** if $x00 = x$, and **dense** if $x00 = 1$.

Remark 3.9. Let $\mathbf{A} \in \mathcal{DE}$, and let $d \in A$. If d is a dense element, then $g(d) = 1$. Moreover, if there exist elements $x, d \in A$ such that $x << d$, then

d is dense.

PROOF: From the density of d, we have $d0 = d000 = 1 \cdot 0 = 0$. Hence, $g(d) = d \cdot 0 \circ 1 = 0 \circ 1 = 1$. The second part follows from item (6) of Theorem 3.5. □

THEOREM 3.10. *Let* $\mathbf{A} \in \mathcal{DE}$. *For any* $a \in A$, *the element* $(a \cdot 0) \circ a$ *is dense, and* $g((a \cdot 0) \circ a) = 1$. *Dually,* $d((a \circ 1) \cdot a) = 0$.

PROOF: If $a = 1$, then clearly $(a \cdot 0) \circ a$ is dense and $g(a) = 1$. Let us therefore assume that $a \neq 1$. Then $a << 1$. It follows from Theorem 3.7 that $1 \cdot 0 = 0 << (a \cdot 0) \circ a$. By Remark 3.9, we conclude that $(a \cdot 0) \circ a$ is dense and $g((a \cdot 0) \circ a) = 1$. □

3.3. Congruences of double equivalential algebras

THEOREM 3.11. *Let* $\mathbf{A} \in \mathcal{DE}$, *and let* $\varphi \in \operatorname{Con} \mathbf{A}$. *Then, for all* $x, y \in A$:

1. $x \in 1/\varphi$ *iff* $x \circ 1 \in 0/\varphi$,
2. $xy \in 1/\varphi$ *iff* $x \circ y \in 0/\varphi$ *iff* $(x \circ y) \cdot 0 \in 1/\varphi$,
3. $x \in 1/\varphi$ *iff* $d(x) \in 1/\varphi$,
4. $0/\varphi = \{x \in A : x0 \in 1/\varphi\}$.

PROOF: Let $\varphi \in \operatorname{Con} \mathbf{A}$. Observe that $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}^e \cap \operatorname{Con} \mathbf{A}^\circ$, where $\mathbf{A}^\circ$ denotes the $(\circ)$-reduct of $\mathbf{A}$. Hence, $(x, y) \in \varphi$ if and only if $(1, xy) \in \varphi$, and this is equivalent to $(0, x \circ y) \in \varphi$.

Ad 1. This follows from the equivalences:

$$x \in 1/\varphi \iff (x, 1) \in \varphi \iff (x \circ 1, 0) \in \varphi \iff x \circ 1 \in 0/\varphi.$$

Ad 2. The following equivalences hold:

$$xy \in 1/\varphi \iff (1, xy) \in \varphi \iff (x, y) \in \varphi \iff (x \circ y, 0) \in \varphi \iff$$
$$(x \circ y) \in 0/\varphi \iff x \circ y \cdot 0 \in 1/\varphi.$$

Ad 3. From item (1) and its dual, it follows that:

$$x \in 1/\varphi \iff x \circ 1 \in 0/\varphi \iff d(x) = x \circ 1 \cdot 0 \in 1/\varphi.$$

Ad 4. Since $x = x \cdot 1 = x \circ 0$, item (2) implies that $x \in 0/\varphi$ if and only if $x0 \in 1/\varphi$. $\square$

The following theorem, whose proof is due to Katarzyna Słomczyńska, specifies the conditions under which an equivalential congruence is also a double equivalential congruence.

THEOREM 3.12 ([23]). *Let* $\mathbf{A} \in \mathcal{DE}$, *and let* $\alpha \in \operatorname{Con} \mathbf{A}^e$. *Denote* $F := 1/\alpha$. *Then,* $\alpha \in \operatorname{Con} \mathbf{A}$ *iff the following conditions are satisfied:*

1. $ab \in F$ *iff* $(a \circ b) \cdot 0 \in F$,
2. *if* $a0 \in F$, *then* $(a \circ z) \cdot z \in F$ *for all* $z \in A$.

PROOF: It suffices to show that α is compatible with $\circ$. Let $(a, b) \in \alpha$ and $z \in A$. Then $ab \in F$, so by condition (1), $(a \circ b) \cdot 0 \in F$. By condition (2), we have $(a \circ b \circ z) \cdot z \in F$, and again, by condition (1), $((a \circ b \circ z) \circ z) \cdot 0 \in F$. Hence, $((a \circ z) \circ (b \circ z)) \cdot 0 \in F$. Finally, applying condition (1) once more yields $(a \circ z) \cdot (b \circ z) \in F$, i.e., $(a \circ z, b \circ z) \in \alpha$. $\square$

3.4. Double equivalential subreducts of Boolean algebras

Recall that for $(A, \cdot) \in \mathcal{E}$, the operation $\cdot$ is associative if and only if the identity $abb = a$ holds for all $a, b \in A$.

Let $\mathrm{S}\,\mathbb{B}^r$ denote the class of all $(\cdot, \circ)$-subreducts of Boolean algebras.

THEOREM 3.13. $(A, \cdot, \circ, 1, 0) \in \mathrm{S}\,\mathbb{B}^r$ *if and only if* $(A, \cdot)$ *and* $(A, \circ)$ *are Boolean groups,* $x \circ y = x \cdot y \cdot 0$, *and the following quasi-identity holds:* $0 = 1 \Longrightarrow x = y$.

PROOF: The implication from left to right is straightforward. So assume that $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, that $x \circ y = x \cdot y \cdot 0$, and that the quasi-identity $0 = 1 \Longrightarrow x = y$ holds. By [26, Theorem 3.13], it follows that $(A, \cdot, 0)$ can be embedded into an algebra $\mathbf{B}$ belonging to the quasivariety of $(\cdot, 0)$-subreducts of Heyting algebras. Since the operation $\cdot$

is associative, it follows from the proof of Theorem 3.13 in [26] that **B** is a Boolean algebra. Moreover, because $x \circ y = x \cdot y \cdot 0$, this embedding also preserves the operation $\circ$. □

Note that in the above theorem, instead of assuming that $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, we could have assumed only that they are equivalential algebras with zero. The associativity of both operations follows from the condition $x \circ y = x \cdot y \cdot 0$, since in that case we have, for any element $x \in A$: $x00 = x \circ 0 = x$. However, the converse does not hold, as demonstrated by the following example.

Example 3.14. Let $(A, \cdot, \circ, 1, 0)$ be a structure where the operations $\cdot$ and $\circ$ are defined by the following tables:

$\cdot$	1	a	b	c	d	e	f	0
1	1	a	b	c	d	e	f	0
a	a	1	d	e	b	c	0	f
b	b	d	1	f	a	0	c	e
c	c	e	f	1	0	a	b	d
d	d	b	a	0	1	f	e	c
e	e	c	0	a	f	1	d	b
f	f	0	c	b	e	d	1	a
0	0	f	e	d	c	b	a	1

$\circ$	1	a	b	c	d	e	f	0
1	0	d	e	f	a	b	c	1
a	d	0	c	b	1	f	e	a
b	e	c	0	a	f	1	d	b
c	f	b	a	0	e	d	1	c
d	a	1	f	e	0	c	b	d
e	b	f	1	d	c	0	a	e
f	c	e	d	1	b	a	0	f
0	1	a	b	c	d	e	f	0

Thus, $(A, \cdot)$ is the $(\cdot)$-reduct of the Boolean algebra shown in Figure 4 on the left, while $(A, \circ)$ is the $(\circ)$-reduct of the Boolean algebra depicted in Figure 4 on the right (observe that this algebra differs from the first one by swapping the positions of the elements d and f).

In this case, $(A, \cdot)$ and $(A, \circ)$ are Boolean groups, but $(A, \cdot, \circ, 1, 0)$ is not a $(\cdot, \circ)$-subreduct of a Boolean algebra, since the condition $x \circ y = x \cdot y \cdot 0$ does not hold (for example, $a \cdot f \cdot 0 = 1$, whereas $a \circ f = e$).

From Theorem 3.13, we obtain the following corollary.

Corollary 3.15. If $(A, \cdot, \circ, 1, 0) \in \mathrm{S}\,\mathbb{B}^r$, then

$$\mathrm{Con}(A, \cdot, \circ, 1, 0) = \mathrm{Con}(A, \cdot, 0) = \mathrm{Con}(A, \circ, 1).$$

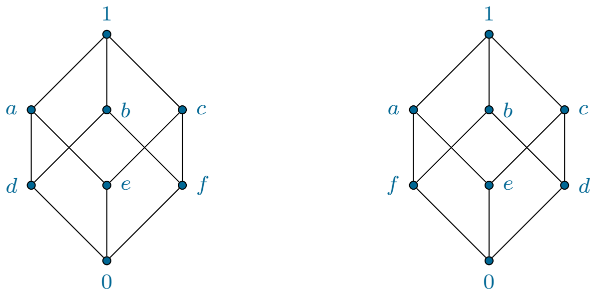

Figure 4: Boolean algebras whose reducts are $(A,\cdot)$ and $(A,\circ)$, respectively.

Now consider the case where the condition $x \circ y = x \cdot y$ holds.

Remark 3.16. Let $(A,\cdot,\circ,1,0) \in \mathcal{DE}$ and suppose that $x \circ y = x \cdot y$ for all $x, y \in A$. Then $0 = 1$ and $(A,\cdot)$ is a Boolean group.

PROOF: Notice that $1 = 1 \cdot 1 = 1 \circ 1 = 0$. Furthermore, for any $x, y \in A$, we have $xyy = x11yy = x00yy = x00 = x11 = x$. $\square$

3.5. V(3)

Let $n \in \mathbb{N}$. We denote by $\mathbf{n}$ the $(\cdot,\circ)$-reduct of an n-element double Heyting algebra whose underlying set forms a chain.

In particular, $\mathbf{3} = (\{0 < * < 1\},\cdot,\circ,1,0)$ and $\mathbf{2} = (\{0 < 1\},\cdot,\circ,1,0)$. Let $V(n)$ denote the variety generated by $\mathbf{n}$.

Remark 3.17. If $\mathbf{A} \in V(\mathbf{n})$, then the following identities hold:

1. $(x \circ y) \cdot 0 = (x \cdot y) \circ 1 \cdot 0$,
2. $(x \cdot y) \circ 1 = (x \circ y) \cdot 0 \circ 1$.

PROOF: These identities can be readily verified in $\mathbf{n}$ and hence hold in $V(\mathbf{n})$. $\square$

Using the above remark, Theorem 3.12 can be simplified for $\mathbf{A} \in V(\mathbf{n})$, taking the following form.

THEOREM 3.18. *Let* $\mathbf{A} \in V(\mathbf{n})$, *and let* $\alpha \in \operatorname{Con} \mathbf{A}^e$. *Denote* $F := 1/\alpha$. *Then,* $\alpha \in \operatorname{Con} \mathbf{A}$ *iff the following conditions are satisfied:*

1. $a \in F$ *iff* $d(a) \in F$,
2. *if* $a0 \in F$, *then* $(a \circ z) \cdot z \in F$ *for all* $z \in A$.

PROOF: If $A \in V(\mathbf{n})$, then, using Remark 3.17, we can rewrite condition (1) from Theorem 3.12 as follows:

$$ab \in F \text{ iff } (a \circ b) \cdot 0 \in F \text{ iff } (a \cdot b) \circ 1 \cdot 0 \in F.$$

Since $a = a \cdot 1$, the assertion follows. □

Let $2_{0=1}$ denote the two-element algebra $(A, \cdot, \circ, 1, 0)$ in which the constants 0 and 1 have the same interpretation. Note that, similarly to [26, p. 3], $\mathbf{2}_{0=1} \in \mathrm{H}(\mathbf{2} \times \mathbf{2})$, and therefore $\mathbf{2}_{0=1} \in V(\mathbf{3})$. From Remark 3.17, it follows that if $x, y \in \mathbf{2_{0=1}}$, then $x \cdot y = x \circ y$.

Observe that in the case of $\mathbf{3}$, we have $d(*) = 0$. Hence, Theorem 3.18 implies the following remark.

Remark 3.19. The algebras $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$ are simple.

We will use the following theorem:

THEOREM 3.20 (Foster–Pixley [5]). *Let* $\mathbf{S}_1, \ldots, \mathbf{S}_n$ *be simple algebras in a congruence-permutable variety* V. *If*

$$C \leq \mathbf{S}_1 \times \cdots \times \mathbf{S}_n$$

is a subdirect product, then

$$C \cong \mathbf{S}_{i_1} \times \cdots \times \mathbf{S}_{i_k}$$

for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

THEOREM 3.21. *Let* $\mathbf{A} \in \mathrm{SP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$ *with* $1 < |A| < \infty$. *Then* $\mathbf{A} \in \mathrm{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$.

Proof: Suppose $\mathbf{A} \in \mathrm{S}(\mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m)$ for some $n, k, m \in \mathbb{N}$. Since the algebras $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$ are simple, and all of their nontrivial subalgebras are also simple, it follows that $\mathbf{A}$ is a subdirect product of a finite algebra from $\mathrm{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$. By the Foster–Pixley Theorem, $\mathbf{A} \cong \mathbf{3}^i \times \mathbf{2}^j \times \mathbf{2}_{0=1}^l$ for some $i, j, l \in \mathbb{N}$ such that $i \leq n$, $j \leq k$, and $l \leq m$. □

We now describe the structure of homomorphic images of algebras of the form $\mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$.

Clearly, $\mathrm{Con}\,\mathbf{A} \subseteq \mathrm{Con}\,\mathbf{A}^e$. Using the fact that, by Theorem 3.18, $a \in 1/\varphi$ if and only if $d(a) \in 1/\varphi$, we can effectively restrict the set of congruences in $\mathrm{Con}\,\mathbf{A}^e$ to those that also belong to $\mathrm{Con}\,\mathbf{A}$.

Let $\mathbf{A} = \mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$, for some $n, k, m \in \mathbb{N}$, and let $x \in A$. We note that x is central if and only if $x_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n + k + m\}$. Moreover, for each such i, we have $d(x)_i = 1$ if $x_i = 1$, and $d(x)_i = 0$ if $x_i \in \{0, *\}$. Observe that the neutral element of the operation $\circ$, that is, 0, has the following form: $0_i = 0$ for all $i \in \{1, \ldots, n + k\}$, and 0_i=1, for all $i \in \{n + k + 1, \ldots, n + k + m\}$. By a double equivalential filter of $\mathbf{A}$ we mean an equivalential filter F whose corresponding congruence belongs to $\mathrm{Con}\,\mathbf{A}$.

Theorem 3.22. *Let $\mathbf{A} = \mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}_{0=1}^m$, for some $n, k, m \in \mathbb{N}$, and let $x \in A$. Then:*

1. *$F = \{1, x, d(x)\}$, where $x_l = *$ for exactly one $l \in \{1, \ldots, n\}$, and $x_i = 1$ for all $i \in \{1, \ldots, n+k+m\}$ with $i \neq l$, is a double equivalential filter.*

2. *$H = \{1, x\}$, where $x \neq 1$ is such that $x_i = 1$ for all $i \in \{1, \ldots n\}$, and $x_j \in \{0, 1\}$ for all $j \in \{n+1, \ldots, n+k+m\}$, is a double equivalential filter.*

Proof: First, note that both F and H are equivalential filters. Obviously, F and H satisfy condition (1) of Theorem 3.18. We will show that they also satisfy condition (2) of Theorem 3.18.

Ad 1. Without loss of generality, we may assume that $l = 1$. Let $a0 \in F$. If $a0 = 1$, then $a = 0$, and in this case, $a \circ z \cdot z = 0 \circ z \cdot z = z \cdot z = 1 \in F$, for all $z \in A$. The case $a0 = x$ is impossible. Thus, let $a0 = d(x)$. Then $a_1 \in \{*, 1\}$, $a_i = 0$ for all $i \in \{2, \ldots, n+k\}$, and $a_i = 1$ for all $i \in \{n+k+1, \ldots, n+k+m\}$. Let $z \in A$. Then $a_i \circ z_i \cdot z_i = 0 \circ z_i \cdot z_i = z_i \cdot z_i = 1$ for all $i \in \{2, \ldots, n+k\}$, and $a_i \circ z_i \cdot z_i = 1 \circ z_i \cdot z_i = z_i \cdot z_i = 1$ for all $i \in \{n+k+1, \ldots, n+k+m\}$. Hence, $a \circ z \cdot z \in F$.

Ad 2. Let $a0 \in H$. As before, if $a0 = 1$, then $a \circ z \cdot z = 1 \in H$. Suppose now that $a0 = x$, and define $K = \{j \in \{n+1, \ldots, n+k\} : x_j = 1\}$ and $M = \{j \in \{n+k+1, \ldots, n+k+m\} : x_j = 0\}$. Then a is such that $a_i = 0$ for all $i \in \{1, \ldots, n\} \cup K \cup M$, and $a_j = 1$ for all $j \in \{n+1, \ldots, n+k+m\} \setminus (K \cup M)$. Let $z \in A$. Then $a_i \circ z_i \cdot z_i = 0 \circ z_i \cdot z_i = 1 = x_i$ for all $i \in \{1, \ldots n\} \cup K$, and $a_i \circ z_i \cdot z_i = 0 \circ z_i \cdot z_i = 0 = x_i$ for all $i \in M$. Now let $j \in \{n+1, \ldots, n+k+m\} \setminus (K \cup M)$. Two cases are possible:
1) If $z_j = 1$, then $a_j \circ z_j \cdot z_j = 1 \circ 1 \cdot 1 = 0 = x_j$ for all $j \in \{n+1, \ldots, n+k\} \setminus K$, and $a_j \circ z_j \cdot z_j = 1 \circ 1 \cdot 1 = 1 = x_j$ for all $j \in \{n+k+1, \ldots, n+k+m\} \setminus M$,
2) If $z_j = 0$, then $a_j \circ z_j \cdot z_j = 1 \circ 0 \cdot 0 = 0 = x_j$ for all $j \in \{n+11, \ldots, n+k\} \setminus K$, and $a_j \circ z_j \cdot z_j = 1 \circ 0 \cdot 0 = 1 = x_j$ for all $j \in \{n+k+1, \ldots, n+k+m\} \setminus M$.
Hence, $a_j \circ z_j \cdot z_j = x_j$ for all $j \in \{1, \ldots, n+k+m\}$. Therefore, $a \circ z \cdot z = x \in H$, which shows that condition (2) of Theorem 3.18 is satisfied. □

THEOREM 3.23. *Let* $\mathbf{A} \in \mathrm{HSP}(\mathbf{3})$ *with* $1 < |A| < \infty$. *Then* $\mathbf{A} \in \mathrm{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$.

PROOF: Let $\mathbf{A} \in \mathrm{HSP}(\mathbf{3})$ be such that $1 < |A| < \infty$. By Theorem 3.21, it follows that $\mathbf{A} \in \mathrm{HP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$. By the Second Isomorphism Theorem, it suffices to show that if $\mathbf{B} = \mathbf{3}^n \times \mathbf{2}^k \times \mathbf{2}^m_{0=1}$ for some $n, k, m \in \mathbb{N}$, and α is an atom in $\mathrm{Con}\,\mathbf{B}$, then $\mathbf{B}/\alpha \in \mathrm{P}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$. We will show that the only atoms in $\mathrm{Con}\,\mathbf{B}$ are those congruences corresponding to the filters specified in Theorem 3.22. By item (1) of Theorem 3.18, it follows that these are indeed atoms in $\mathrm{Con}\,\mathbf{B}$.
Assume, for the sake of contradiction, that there exists another filter G corresponding to a congruence α which is an atom in $\mathrm{Con}\,\mathbf{B}$. From item (1) of Theorem 3.18, it follows that there exists a central element y such that $y \in G$. If y is such that $y_i = 1$ for all $i \in \{1, \ldots, n\}$ and $y_j \in \{0, 1\}$ for all $j \in \{n+1, \ldots, n+k+m\}$, then G contains a subset corresponding to the filter H from Theorem 3.22.

We will show that otherwise there exists an element $x \in G$ such that $x_l = *$ for exactly one $l \in \{1, \ldots, n\}$ and $x_i = 1$ for all $i \in \{1, \ldots, n+k+m\}$ with $i \neq l$. Let $l \in \{1, \ldots, n\}$ be the smallest index such that $y_l = 0$. Notice that if x is such that $x_l = *$ and $x_j = 1$ for all $j \in \{1, \ldots, n+k+m\} \setminus \{l\}$, then $x << y$. Hence, $x \in G$, so such an element x indeed exists, as desired. From item (1) of Theorem 3.18, it follows that $d(x)$ also belongs to G. Consequently, G contains a subset corresponding to the filter F from Theorem 3.22. Therefore, G is not an atom in the lattice of filters — a contradiction.
Hence, the only atoms in $\operatorname{Con}\mathbf{B}$ are the congruences corresponding to the filters described in Theorem 3.22.
Finally, observe that if F is as in item (1) of Theorem 3.22, then $B/F \cong \mathbf{3}^{n-1} \times \mathbf{2}^k \times \mathbf{2}^m_{0=1}$. If H is as in item (2) of Theorem 3.22, then $B/H \cong \mathbf{3}^n \times \mathbf{2}^i \times \mathbf{2}^j_{0=1}$, where $i + j = k + m - 1$. □

Corollary 3.24. The only finite subdirectly irreducible algebras in $V(\mathbf{3})$ are $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$.

From Quackenbush's Theorem (see [2, p. 226]), we obtain the following corollary:

Corollary 3.25. The only subdirectly irreducible algebras in $V(\mathbf{3})$ are $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$.

Theorem 3.26. *The variety $V(\mathbf{3})$ is:*

1. *semisimple,*
2. *directly representable (i.e., it is finitely generated and has only finitely many finite directly indecomposable members).*

Moreover, if $\mathbf{A} \in V(\mathbf{3})$ and $|A| < \infty$, then $\mathbf{A}$ is congruence uniform (i.e., for every $\varphi \in \operatorname{Con}\mathbf{A}$ and all $a, b \in A$, we have $|a/\varphi| = |b/\varphi|$).

Proof: Item (1) follows from Corollary 3.25 and Remark 3.19. Item (2) follows from Theorem 3.23. Congruence uniformity follows from McKenzie's Theorem [2, p. 188]. □

Remark 3.27. Let $n \geq 2$. Then $\mathbf{n} = (\{0 < x_1 < x_2 < \ldots < x_{n-2} < 1\}, \cdot, \circ, 1, 0)$ is a simple algebra. Moreover, for any $a, b \in \mathbf{n}$, and any $k \in$

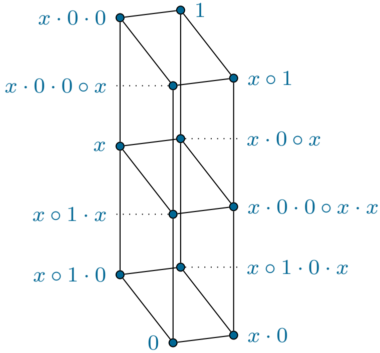

Figure 5: Double Heyting algebra $\mathbf{3} \times \mathbf{2}^2$ with generator x.

$\{1, \ldots, n-2\}$, we have $a \cdot b = x_k$ if and only if $a = x_k$ and $b > x_k$ (or vice versa), and $a \circ b = x_k$ if and only if $a = x_k$ and $b < x_k$ (or vice versa). Consequently, the element x_k can be generated only by using itself. Therefore, the results and theorems from this subsection can be generalized from the case $n = 3$ to any $n \geq 2$.

Example 3.28. Let $\mathcal{F}_{\mathbf{n}}(k)$ be the k-generated free algebra in the variety $V(\mathbf{n})$. Then, for every $n \geq 3$, the algebra $\mathcal{F}_{\mathbf{n}}(1)$ is the $(\cdot, \circ)$-reduct of the double Heyting algebra $\mathbf{3} \times \mathbf{2}^2$, with generator x. This algebra is illustrated in Figure 5.

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