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AN ELEMENTARY PROOF OF THE CHARACTERIZATION THEOREM FOR CONJUNCTIVE MULTIPLE-CONCLUSION CONSEQUENCE RELATIONS

Abstract

We give a characterization theorem for multiple-conclusion consequence relations with the conjunctive reading of conclusions. As in the case of disjunctive multiple-conclusion consequence relations, we define consequence relations in terms of sets of two-set partitions of formulae. We see that a binary relation between sets of formulae is a conjunctive multiple-conclusion consequence relation if it is closed under the properties of inclusion, transitivity and reducibility. To prove this result we use only the definition and some basic properties of conjunctive multiple-conclusion consequence relations.

Keywords: multiple-conclusion consequence relation, conjunctive set of conclusions, characterization theorem.

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1. Introduction

Smiley [3, p. 30] proves a characterization theorem for multiple-conclusion consequence relations. There a set of conclusions Y is said to follow from a set X of premises if at least one formula in Y is true whenever all formulae in X are true. In other words, the set of conclusions is read disjunctively. Later, Šikić [4] improved Smiley's result by characterizing disjunctive consequence relations in terms of properties that explicitly refer to the relevant formal language.

Here we deal with the characterization problem for multiple-conclusion consequence relations with a conjunctive set of conclusions. On this reading, a set of conclusions Y is said to follow from a set X of premises if all formulae in Y are true whenever all formulae in X are true. The characterization theorem (See Theorem 3.2 in Section 3) provides a list of necessary and sufficient conditions for a binary relation between sets of formulas to be a conjunctive multiple-conclusion consequence relation.

Nowak [2, p. 1141] also gives a characterization theorem as a corollary in his study on disjunctive and conjunctive multiple-conclusion consequence relations. He obtains that result within the theory of Galois connections and closure systems. Here we present an elementary proof of a characterization theorem using solely the definition of conjunctive multiple-conclusion consequence relations and some of their basic properties (Compare Theorem 2.1 and its proof in [3, p. 30]). We also give a proof of the equivalence of Nowak's characterization theorem with our result.

2. Terminology, Notation and Preliminaries

In this section we review some notions and previous results from the theory of multiple-conclusion logic. Those who are already familiar with this material may skip this section and go directly to Section 3.

Every formal language $\mathcal{L}$ will be considered as identical with its set of formulae. We write $T, U, X, Y, Z, \ldots, T', U', X', Y', Z', \ldots$ for sets of formulae and $A, B, \ldots$ for formulae. For every language $\mathcal{L}$, there are mainly two natural multiple-conclusion consequence relations between sets of $\mathcal{L}$-formulae: Given a language $\mathcal{L}$, for all sets of $\mathcal{L}$-formulae X and Y:

- Y is a *disjunctive set of conclusions of* X if and only if at least one formula in Y is true whenever all formulae in X are true.
- Y is a *conjunctive set of conclusions of* X if and only if all formulae in Y are true whenever all formulae in X are true.

Here we will be dealing mainly with conjunctive consequence relations. The classic reference on disjunctive consequence relations is [3]. See also [1] for an order-theoretic study of consequence relations that connects them with closure operations on complete lattices. Following the works cited in this paper, to give a formal definition of a multiple-conclusion consequence relation for a language $\mathcal{L}$, we make use of sets of two-set partitions of $\mathcal{L}$.[1] Let T represent the set of all true $\mathcal{L}$-formulae and U the set of all untrue $\mathcal{L}$-formulae according to a possible states of affairs. Since every formula is either true or untrue, and no formula is both true and untrue, $T \cup U = \mathcal{L}$ and $T \cap U = \emptyset$. In other words, the sets T and U form a two-set partition (T, U) of $\mathcal{L}$. Considered as the collection of all possible states of affairs, every set of two-set partitions of $\mathcal{L}$ enables us to give formal definitions of the two types of multiple-conclusion consequence relations mentioned above:

Definition 2.1. Let $\mathcal{L}$ be a formal language and $\mathcal{I}$ be a set of two-set partitions of $\mathcal{L}$. Let $\models^{d}_{\mathcal{I}}$ and $\models^{c}_{\mathcal{I}}$ symbolize, respectively, the disjunctive and conjunctive multiple-conclusion consequence relations with regard to $\mathcal{I}$:

- $X \models^{d}_{\mathcal{I}} Y$ if and only if there is no partition (T, U) in $\mathcal{I}$ such that all formulae in X are true and all formulae in Y are untrue with regard to that partition:

[1] As remarked by an anonymous reviewer, in the setting of classical logic it would be more convenient to use families of subsets of the set of all formulae: once we are given a set T consisting of the true formulae of $\mathcal{L}$ in any particular states of affairs, the corresponding two-set partition would be $(T, (\mathcal{L} - T))$. We have opted for following the notation settled in this area starting from Smiley's monograph [3] and using pairs (T, U) to represent the set of all true and untrue formulae in a possible states of affairs.

$$X \models^{d}_{\mathcal{I}} Y \Leftrightarrow \neg\exists((T,U) \in \mathcal{I})\,(X \subseteq T \;\&\; Y \subseteq U) \tag{2.1}$$

- $X \models^{c}_{\mathcal{I}} Y$ if and only if whenever all formulae in X are true with regard to a partition (T,U) in $\mathcal{I}$, all formulae in Y are also true with regard to that partition:

$$X \models^{c}_{\mathcal{I}} Y \Leftrightarrow \forall((T,U) \in \mathcal{I})(X \subseteq T \Rightarrow Y \subseteq T) \tag{2.2}$$

For every set X and every formula A, we may abbreviate $X \models^{c}_{\mathcal{I}} \{A\}$ as $X \models^{c}_{\mathcal{I}} A$. We may use the comma as the symbol of set-theoretic union. Thus, for example, we may write $X, Y \models^{c}_{\mathcal{I}} Z, A$ for $X \cup Y \models^{c}_{\mathcal{I}} Z \cup \{A\}$. Moreover, for every set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$, we let:

$$\begin{aligned} \mathcal{T} &= \{T \subseteq \mathcal{L} : \exists(U \subseteq \mathcal{L})(T,U) \in \mathcal{I}\} \\ \mathcal{U} &= \{U \subseteq \mathcal{L} : \exists(T \subseteq \mathcal{L})(T,U) \in \mathcal{I}\} \end{aligned} \tag{2.3}$$

Note that in Definition 2.1 we impose nothing either on the structure of formulae, or on the semantic relations among formulae based on their structure. Therefore, *every* set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$ gives us a disjunctive and a conjunctive multiple-conclusion consequence relation:

DEFINITION 2.2. A relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a *disjunctive multiple-conclusion consequence relation* if $\vdash = \models^{d}_{\mathcal{I}}$ for some set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$. Similarly, a relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a *conjunctive multiple-conclusion consequence relation* if $\vdash = \models^{c}_{\mathcal{I}}$ for some set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$.

To establish our results we will need the following basic properties of conjunctive consequence relations:

FACT 2.3. For every language $\mathcal{L}$ and every set $\mathcal{I}$ of partitions of $\mathcal{L}$,

(a) If $Y \subseteq X$, then $X \models^{c}_{\mathcal{I}} Y$.

(b) If $X' \models^{c}_{\mathcal{I}} Y$ and $X' \subseteq X$, then $X \models^{c}_{\mathcal{I}} Y$.

(c) If $X \mathrel{|\!\frac{c}{\mathcal{I}}} Y'$ and $Y \subseteq Y'$, then $X \mathrel{|\!\frac{c}{\mathcal{I}}} Y$.

(d) If $X \mathrel{|\!\frac{c}{\mathcal{I}}} Y$ and $Y \mathrel{|\!\frac{c}{\mathcal{I}}} Z$, then $X \mathrel{|\!\frac{c}{\mathcal{I}}} Z$.

(e) Let $T \in \mathcal{T}$. Then $T \mathrel{|\!\frac{c}{\mathcal{I}}} X$ if and only if $X \subseteq T$.

(f) $X \mathrel{|\!\frac{c}{\mathcal{I}}} Y$ if and only if $\forall A(A \in Y \Rightarrow X \mathrel{|\!\frac{c}{\mathcal{I}}} A)$

PROOF: Proofs of the first four clauses are immediate from the definitions. We only prove the less trivial 2.3 and 2.3 below:

2.3 Assume that $T \in \mathcal{T}$ and $T \mathrel{|\!\frac{c}{\mathcal{I}}} X$. Thus, for all $T' \in \mathcal{T}$, if $T \subseteq T'$, then $X \subseteq T'$. From the assumption that $T \in \mathcal{T}$ and the fact that $T \subseteq T$, it follows that $X \subseteq T$. The converse is an immediate result of the clause 2.3 of this Fact.

2.3 The left-to-right conditional is due to the clauses 2.3 and 2.3 of this Fact. To prove the converse, assume that $\forall A(A \in Y \Rightarrow X \mathrel{|\!\frac{c}{\mathcal{I}}} A)$ and let $T \in \mathcal{T}$. If $X \subseteq T$, then $\{A\} \subseteq T$ for every $A \in Y$. Therefore, $Y \subseteq T$. □

It is remarked in [4] that for every set $\mathcal{I}$ of two-set partitions of $\mathcal{L}$ and for every two-set partition (T, U):

$$(T, U) \in \mathcal{I} \text{ if and only if } \neg(T \mathrel{|\!\frac{d}{\mathcal{I}}} U)$$

If we consider conjunctive consequence relations and look for a similar result, we first deduce from Fact 2.3.2.3 that $T \mathrel{|\!\frac{c}{\mathcal{I}}} U$ if and only if $(T, U) = (\mathcal{L}, \emptyset)$. Moreover, we may deduce the following proposition from this observation:

FACT 2.4. Let $\mathcal{L}$ be a language and $\mathcal{I} = \{(T, U) : T \cap U = \emptyset \ \& \ T \cup U = \mathcal{L} \ \& \ \exists X(T \subseteq X \ \& \ X \neq \mathcal{L})\}$. For every (T, U):

$$T \mathrel{|\!\frac{c}{\mathcal{I}}} U \Leftrightarrow (T, U) \notin \mathcal{I}$$

In the next section we give a characterization theorem for conjunctive multiple-conclusion consequence relations. We finish this section by recalling the characterization theorem for disjunctive consequence relations:

Theorem 2.5 ([3, p. 30]). *A binary relation $\vdash$ on $\mathcal{P}(\mathcal{L})$ is a disjunctive multiple-conclusion consequence relation if and only if it is closed under the following properties:*

(Overlap) *If $X \cap Y \neq \emptyset$, then $X \vdash Y$.*

(Dilution) *If $X' \vdash Y'$ where $X' \subseteq X$ and $Y' \subseteq Y$, then $X \vdash Y$.*

(Cut for sets) *If $X, Z_1 \vdash Z_2, Y$ for every partition (Z_1, Z_2) of Z, then $X \vdash Y$.*

3. The Characterization Theorem

In this section we prove a characterization theorem for conjunctive multiple-conclusion consequence relations. We replace the overlap property of Theorem 2.5 with inclusion. We also replace dilution with weakening and split it into two statements, namely the left and right weakening properties. For the proof of our main theorem we only need the right weakening. As is shown below, each of these weakening properties results from inclusion and transitivity. Finally, we introduce a property that we call reducibility.

Lemma 3.1. *Let inclusion and transitivity be the following properties:*

(Inclusion) *If $Y \subseteq X$, then $X \vdash Y$.*

(Transitivity) *If $X \vdash Z$ and $Z \vdash Y$, then $X \vdash Y$.*

Then, any binary relation that satisfies inclusion and transitivity also satisfies the following properties of weakening:

(Left weakening) *If $X' \vdash Y$ and $X' \subseteq X$, then $X \vdash Y$.*

(Right weakening) *If $X \vdash Y'$ and $Y \subseteq Y'$, then $X \vdash Y$.*

Proof: We first prove the left weakening. Assume that $X' \subseteq X$ and $X' \vdash Y$. By inclusion, $X \vdash X'$. It follows by transitivity that $X \vdash Y$. The proof of the right weakening from inclusion and transitivity is similar: Let

$X \vdash Y'$ and $Y \subseteq Y'$. Bu inclusion, $Y' \vdash Y$. It then follows by transitivity that $X \vdash Y$. □

THEOREM 3.2. *A binary relation $\vdash$ on $\mathcal{P}(\mathcal{L})$ is a conjunctive multiple-conclusion consequence relation if and only if it is closed under inclusion, transitivity and the following property of reducibility:*

(Reducibility) $X \vdash Y$ *if and only if* $\forall A(A \in Y \Rightarrow X \vdash A)$.

PROOF: Let $\vdash$ be a conjunctive multiple-conclusion consequence relation, that is, let $\vdash = \vDash^{c}_{\mathcal{I}}$ for some set of partitions $\mathcal{I}$. We already have seen that every conjunctive multiple-conclusion consequence relation is closed under inclusion, transitivity and reducibility (See the clauses 2.3, 2.3 and 2.3 of Fact 2.3).

To prove the converse, let $\vdash$ be closed under inclusion, transitivity and reducibility. Let:

$$\mathcal{I} = \{(T, U) : \forall Z(T \vdash Z \Leftrightarrow Z \subseteq T)\} \tag{3.1}$$

We claim that $\vdash = \vDash^{c}_{\mathcal{I}}$, where $\mathcal{I}$ is as in (3.1). To demonstrate this, we first assume that $X \vdash Y$. Let $(T, U) \in \mathcal{I}$ and $X \subseteq T$. Then, by inclusion $T \vdash X$. From this result and the assumption that $X \vdash Y$, it follows by transitivity that $T \vdash Y$. From the definition of $\mathcal{I}$ in (3.1), we conclude that $Y \subseteq T$. Therefore, $X \vDash^{c}_{\mathcal{I}} Y$.

Now let us assume that $X \nvdash Y$. Let $T = \{A : X \vdash A\}$ and $U = \{A : X \nvdash A\}$. One can easily see that (T, U) is a partition. To see that $(T, U) \in \mathcal{I}$, we must only prove that for all Z, if $T \vdash Z$, then $Z \subseteq T$. (The converse holds by inclusion. Note that we do not yet know that $T \in \mathcal{T}$. Thus, we could not use the clause 2.3 of Fact 2.3 to prove that $X \subseteq T$ from the assumption that $T \vDash^{c}_{\mathcal{I}} X$.) Let $T \vdash Z$ and $A \in Z$. Since $X \vdash A$ for all $A \in T$ (by the definition of T), by reducibility it follows that $X \vdash T$. By the assumption that $T \vdash Z$, it follows by transitivity that $X \vdash Z$. Since $A \in Z$, we conclude by right weakening that $X \vdash A$. Thus, $A \in T$, by the definition of T, and we conclude that $Z \subseteq T$. Therefore, $(T, U) \in \mathcal{I}$. We now prove that, $Y \not\subseteq T$, although $X \subseteq T$, thus proving that $\neg(X \mid\!\frac{c}{\mathcal{I}} Y)$. Since $X \vdash A$ for every $A \in X$, we conclude that $X \subseteq T$, by the definition of T. If $Y \subseteq T$ were also true, by inclusion it would follow that $T \vdash Y$. Since we have also seen above that $X \vdash T$, it would follow by transitivity that $X \vdash Y$, contrary to our assumption. Therefore $Y \not\subseteq T$. Since $X \subseteq T$, the partition $(T, U) \in \mathcal{I}$ demonstrates that $\neg(X \mid\!\frac{c}{\mathcal{I}} Y)$. □

As a corollary of Theorem 3.2, we now prove Nowak's proposition that also gives a characterization of conjunctive multiple-conclusion consequence relations:

Corollary 3.3 ([2, p. 1141]). A binary relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a conjunctive multiple-conclusion consequence relation if and only if it is closed under the following properties:

(Transitivity) If $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.

(Ext. of converse order) If $Y \subseteq X$, then $X \vdash Y$.

(Closure on sup) $X \vdash \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\}$.

Proof: Note that the property here called ext. of converse order is what we have named as inclusion. We only need to show that reducibility and closure on sup are deducible from each other in the presence of inclusion and transitivity.

We first assume that $\vdash$ is closed under reducibility together with inclusion and transitivity. If $A \in \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\}$, then $A \in Y$ for some Y such that $X \vdash Y$. Since $A \in Y$, it follows by inclusion $Y \vdash A$. By transitivity, $X \vdash A$. Therefore, $\forall A(A \in \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\} \Rightarrow X \vdash A)$. We can now conclude by reducibility that $X \vdash \bigcup\{Y \subseteq \mathcal{L} : X \vdash Y\}$.

We now assume closure on sup, together with inclusion and transitivity. If $X \vdash Y$, and $A \in Y$, then by means of inclusion and transitivity it follows that $X \vdash A$. Therefore, $\forall A(A \in Y \Rightarrow X \vdash A)$. To prove the converse of reducibility, assume that $X \vdash A$ for every formula A in Y. Then $\{A\} \in \{Z \subseteq \mathcal{L} : X \vdash Z\}$, for all $A \in Y$. Therefore, by basic set theory,

$Y = \bigcup_{A \in Y} \{A\} \subseteq \bigcup \{Z \subseteq \mathcal{L} : X \vdash Z\}$. From the property of closure on sup, $X \vdash \bigcup \{Z \subseteq \mathcal{L} : X \vdash Z\}$. Since $\vdash$ satisfies right weakening, we conclude that $X \vdash Y$. □

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