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REVISITING THE ADEQUACY THEOREM FOR FRAGMENTS OF ŁUKASIEWICZ LOGIC

Abstract

A. V. Figallo introduced the 3-valued Super Łukasiewicz logic expanded with the Δ operator, denoted as $\mathcal{C}_3^{\rightarrowtail,\Delta}$, in 1990. This operator is used in the definition of 3-valued Łukasiewicz algebras, and it is not possible to recover Δ through implication and top in Super Łukasiewicz logic. On the other hand, Baaz introduced the Δ operator in Gödel logic, both in its propositional and quantified versions. Subsequently, this operator was extensively studied in the field of fuzzy logic.

In this paper, we prove a strong version of the Adequacy Theorem for $\mathcal{C}_3^{\rightarrowtail,\Delta}$. As a consequence, we demonstrate that the Deduction Theorem does not hold in this calculus. Furthermore, we introduce the first-order version of $\mathcal{C}_3^{\rightarrowtail,\Delta}$ and establish soundness and completeness results by adapting a recently developed

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algebraic technique. In this context, our presentation differs from others in the literature because we need to construct a special homomorphism, brought from the algebraic study of $\mathcal{C}_3^{\rightarrow,\Delta}$, in the syntactic setting. This homomorphism is also necessary to determine the generating algebras. While we can ascertain that the logical system is algebraizable by a (quasi-)variety of algebras, we cannot know a priori which are the subdirectly irreducible algebras.

Keywords: implicational fragment of Łukasiewicz logic, 3-valued Łukasiewicz logic, Δ operator, first-order logics.

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1. Introduction

Many-valued logics, and in particular Łukasiewicz logic, have played a central role in the development of non-classical reasoning. From its origin, Łukasiewicz's insight of assigning intermediate truth-values between truth and falsity opened the door to formalizing vagueness and uncertainty. Over the decades, this logic has influenced diverse areas such as fuzzy logic, artificial intelligence, and philosophical logic. The semantic richness of Łukasiewicz logic and its algebraic counterpart, MV-algebras, continue to serve as a testbed for deep foundational questions in logic.

However, one of the fundamental theorems of logic—the Adequacy Theorem—has a peculiar status in the context of Łukasiewicz logic and its fragments. While completeness and soundness are often provable through elegant Hilbert-style or algebraic methods, the lack of a Deduction Theorem in some fragments challenges the traditional equivalence between syntactic and semantic consequence. This invites a re-examination of what ”adequacy” means in systems without full deductive strength, and how proof theory and algebra can still be aligned.

Łukasiewicz logic is perhaps the oldest and most studied many-valued logic in the literature. Its semantics was first studied by Łukasiewicz himself and formalized by Chang using the structure of MV-algebras. MV-algebras are now the standard semantics for Łukasiewicz logic and have played a central role in the development of algebraic logic. Furthermore, the connection between MV-algebras and fuzzy logic has opened new areas of application in artificial intelligence and computer science.

In this paper, we revisit the Adequacy Theorem for fragments of Łukasiewicz logic, both in the propositional and quantified case. These fragments are obtained by restricting the set of connectives to subsets that do not validate the deduction theorem. We provide new proofs of adequacy that avoid classical dependence on the deduction theorem, and we analyze how these fragments behave semantically through algebraic techniques. Our results not only clarify the status of adequacy in these fragments but also suggest possible directions for extending the concept to other non-classical logics lacking standard deductive mechanisms.

The techniques developed herein provide a new framework for establishing adequacy theorems in systems lacking the Deduction Theorem, offering a unified algebraic perspective that complements and extends existing approaches.

2. Preliminaries

In this section, we will provide the necessary background to present our paper. To this end, we discuss some algebraic properties of the class of 3-valued Łukasiewicz residuation algebras expanded by the Δ operator, which will be relevant to the Hilbert system associated with them.

First, let's recall that a 3-valued Łukasiewicz residuation algebra is an algebra $\langle A, \rightarrowtail, 1\rangle$ of type $(2, 0)$ (briefly, $\text{Ł}_3^{\rightarrowtail}$-algebras) that satisfies the following identities (see, e.g., [25, 21]):

(Ł1) $x \rightarrowtail (y \rightarrowtail x) = 1$,

(Ł2) $(x \rightarrowtail y) \rightarrowtail ((y \rightarrowtail \gamma) \rightarrowtail (x \rightarrowtail \gamma)) = 1$,

(Ł3) $(x \rightarrowtail y) \rightarrowtail y = (y \rightarrowtail x) \rightarrowtail x$,

(Ł4) $((x \rightarrowtail y) \rightarrowtail (y \rightarrowtail x)) \rightarrowtail (y \rightarrowtail x) = 1$,

(Ł5) $1 \rightarrowtail x = x$,

(Ł6) $((x \rightarrowtail (x \rightarrowtail y)) \rightarrowtail x) \rightarrowtail x = 1$.

It is well known that the identities from Ł1 to Ł5 define Łukasiewicz residuation algebras. These structures were originally introduced and studied in connection with the implicative fragment of Łukasiewicz logic (see [25, 21]). Moreover, an order relation can be defined on every such algebra **A** as follows: $x \leq y$ iff $x \rightarrowtail y = 1$. We can also define a supremum for any $x, y \in A$ as $x \vee y := (x \rightarrowtail y) \rightarrowtail y$; and we also have that $z \leq 1$ for every $z \in A$. Hence, axiom Ł3 expresses the fact that for all $x, y \in A$, we have $(x \rightarrowtail y) \vee (y \rightarrowtail x) = 1$.

Now, let's move on to the class of algebras introduced and studied in [11].

Definition 2.1. An $Ł_3^{\rightarrowtail,\Delta}$-algebra is an algebra $(A, \rightarrowtail, \Delta, 1)$ of type $(2, 1, 0)$ such that $(A, \rightarrowtail, 1)$ is an $Ł_3^{\rightarrowtail}$-algebra, and the following identities are satisfied:

(ΔŁ1) $\Delta x \rightarrowtail y = x \rightarrowtail (x \rightarrowtail y)$,

(ΔŁ2) $\Delta(\Delta x \rightarrowtail y) = \Delta x \rightarrowtail \Delta y$.

In what follows, we will consider a new binary connective $\Rightarrow$ defined as follows: $x \Rightarrow y := \Delta x \rightarrowtail y$. With this definition, we can introduce the following concept:

Definition 2.2. For any $Ł_3^{\rightarrowtail,\Delta}$-algebra **A**, a subset D is considered an implicative filter of A if $1 \in D$, and if $x, x \Rightarrow y \in D$, then $y \in D$. This notion extends the classical concept of implicative filters in Łukasiewicz-type algebras (see [25]). We denote by $\mathcal{D}(A)$ the set of all implicative filters of A.

For any $Ł_3^{\rightarrowtail,\Delta}$-algebra **A**, we denote $Con(A)$ as the set of all congruences of **A**. Given an implicative filter D, the relation $R(D) = \{(x, y) \in A^2 : x \Rightarrow y, y \Rightarrow x \in D\}$ defines a congruence of **A**. Additionally, given a congruence Θ of **A**, $|1|_\Theta$ represents the class of 1 under Θ, and it is also an implicative filter. A crucial lemma in this context is:

Lemma 2.3. ([11]). *There exists a lattice isomorphism between $Con(A)$ and $D(A)$.*

Now, let's introduce a definition by A. Monteiro:

DEFINITION 2.4. (A. Monteiro). For an $\text{Ł}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$, $D \in \mathcal{D}(A)$, and $p \in A$, we say that D is an implicative filter tied to p if $p \notin D$ and for any $D' \in \mathcal{D}(A)$ such that $D \subsetneq D'$, then $p \in D'$.

Here's a proposition along with some properties:

PROPOSITION 2.5. ([10, p. 106]). Let $\mathbf{A}$ be an $\text{Ł}_3^{\rightarrowtail,\Delta}$-algebra and for any $x, y, z \in A$, the following properties hold:

(Ł7) $1 \Rightarrow x = x$,
(Ł8) $x \Rightarrow x = 1$,
(Ł9) $x \Rightarrow (y \Rightarrow z) = (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$,
(Ł10) $x \Rightarrow (y \Rightarrow x) = 1$,
(Ł11) $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = 1$.

Recall that for a given $\text{Ł}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$, we say that an implicative filter M is maximal if M is proper and for any $D \in \mathcal{D}(A)$, $M \subseteq D$ implies $D = A$ or $M = D$. Note that the above proposition provides fundamental properties of the implication $\Rightarrow$, which will play a key role in what follows.

Lastly, let's consider maximal implicative filters and a related lemma:

LEMMA 2.6. ([17, Lemma 3.9]). *Let* $\mathbf{A}$ *be an* $\text{Ł}_3^{\rightarrowtail,\Delta}$*-algebra, and* M *is a maximal implicative filter of* A*. Then, for every* $x \in A \setminus M$*, we have that* $x \Rightarrow y \in A$ *for every* $y \in A$*.*

For an $\text{Ł}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and according to Lemma 2.6 and (Ł11), we can conclude the following corollary:

COROLLARY 2.7. ([17, Section 6]). For a given $\text{Ł}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$, each implicative filter tied to some element of A is maximal, and vice versa.

Finally, it is worth recalling that in [13], the authors studied n-valued Łukasiewicz residuation algebras expanded with Moisil operators. The class of $\text{Ł}_3^{\rightarrowtail,\Delta}$-algebras constitutes the particular case corresponding to $n = 3$, and was analyzed in detail within the broader context of the n-valued setting.

$\rightarrowtail$	0	$\frac{1}{2}$	1	Δx
0	1	1	1	0
$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
1	0	$\frac{1}{2}$	1	1

Table 1: The operations of the $Ł_3^{\rightarrowtail,\Delta}$-algebra $\mathbb{C}_3$.

THEOREM 2.8. ([13, Theorem 3.17]). *The variety of $Ł_3^{\rightarrowtail,\Delta}$-algebras is semisimple. Furthermore, the generating algebras are $\mathbb{C}_3$ and the unique subalgebra with support $\{0, 1\}$, where the support of $\mathbb{C}_3$ is the set $\{0, \frac{1}{2}, 1\}$, and the operations $\rightarrowtail$ and Δ are defined by Table 1.*

3. A Calculus for $Ł_3^{\rightarrowtail,\Delta}$-algebras: $CŁ_3^{\rightarrowtail,\Delta}$

In this section, we introduce a Hilbert-style calculus for $Ł_3^{\rightarrowtail,\Delta}$-algebras, which was presented in [11]. We will provide all necessary definitions and results to establish, first, a weak and, subsequently, a strong version of the Adequacy Theorem.

To this end, let us consider a denumerable set Var of propositional variables and the propositional signature $\{\rightarrowtail, \Delta\}$. The propositional language generated by this signature over Var will be denoted by For; recall that For is the absolutely free algebra of propositional formulas.

The three-valued implicative propositional calculus of Łukasiewicz, denoted $CŁ_3^{\rightarrowtail,\Delta}$, is defined by the following axiom schemes:

(Ax1) $\alpha \rightarrowtail (\beta \rightarrowtail \alpha)$,

(Ax2) $(\alpha \rightarrowtail \beta) \rightarrowtail ((\beta \rightarrowtail \gamma) \rightarrowtail (\alpha \rightarrowtail \gamma))$,

(Ax3) $((\alpha \rightarrowtail \beta) \rightarrowtail \beta) \rightarrowtail ((\beta \rightarrowtail \alpha) \rightarrowtail \alpha)$,

(Ax4) $((\alpha \rightarrowtail \beta) \rightarrowtail (\beta \rightarrowtail \alpha)) \rightarrowtail (\beta \rightarrowtail \alpha)$,

(Ax5) $((\alpha \rightarrowtail (\alpha \rightarrowtail \beta)) \rightarrowtail \alpha) \rightarrowtail \alpha$,

(Ax6) $(\Delta\alpha \rightarrowtail \Delta\beta) \rightarrowtail \Delta(\Delta\alpha \rightarrowtail \beta)$, (Ax7) $\Delta(\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\beta))$,

(Ax8) $(\alpha \rightarrowtail (\alpha \rightarrowtail \beta)) \rightarrowtail (\Delta\alpha \rightarrowtail \beta)$.

The only inference rule is Modus Ponens:

(MP) $\dfrac{\alpha, \quad \alpha \rightarrowtail \beta}{\beta}$.

Within this calculus, we define two non-primitive connectives, $\vee$ and ∇, as follows:

$$\alpha \vee \beta := (\alpha \rightarrowtail \beta) \rightarrowtail \beta,$$
$$\nabla\alpha := (\alpha \rightarrowtail \Delta\alpha) \rightarrowtail \alpha.$$

We write $\Gamma \vdash \alpha$ to denote that there exists a derivation of α in $\text{CŁ}_3^{\rightarrowtail,\Delta}$ from hypotheses in the set Γ. The following well-known results, which are valid in Super-Łukasiewicz logic, also hold in our calculus:

We briefly comment on the role of axiom (Ax5) in our system. Although it is natural from the algebraic perspective of ${}_3^{\rightarrowtail,\Delta}$-algebras, we have not investigated whether it is independent from the remaining axioms. Its inclusion ensures the validity of identity (Ł6) in the associated Lindenbaum–Tarski algebra, which is used in several key arguments throughout the paper. A detailed study of its possible redundancy is left for future work.

Proposition 3.1. The following theorems and rules hold in $\text{CŁ}_3^{\rightarrowtail,\Delta}$:

T1. $\vdash ((\alpha \rightarrowtail \beta) \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \gamma)$,

R1. $\dfrac{\alpha \rightarrowtail \beta, \beta \rightarrowtail \gamma}{\alpha \rightarrowtail \gamma}$,

T2. $\vdash \alpha \rightarrowtail \alpha \vee \beta$,

T3. $\vdash ((\alpha \vee \gamma) \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta)$;

T4. $\vdash ((\alpha \vee \gamma) \rightarrowtail (\beta \rightarrowtail \gamma)) \rightarrowtail (\alpha \rightarrowtail (\beta \rightarrowtail \gamma))$, T5. $\vdash (\alpha \rightarrowtail (\beta \rightarrowtail \gamma)) \rightarrowtail ((\beta \vee \gamma) \rightarrowtail (\alpha \rightarrowtail \gamma))$,

T6. $\vdash (\alpha \rightarrowtail (\beta \rightarrowtail \gamma)) \rightarrowtail (\beta \rightarrowtail (\alpha \rightarrowtail \gamma))$,

T7. $\vdash \beta \rightarrowtail (\alpha \rightarrowtail \alpha)$,

T8. $\vdash \alpha \rightarrowtail \alpha$,

R2. $\dfrac{\alpha \rightarrowtail \beta}{(\gamma \rightarrowtail \alpha) \rightarrowtail (\gamma \rightarrowtail \beta)}$,

T9. $\vdash \big(((\beta \rightarrowtail \beta) \rightarrowtail \alpha) \rightarrowtail \alpha\big)$,

R3. $\dfrac{\alpha \rightarrowtail \beta}{(\beta \rightarrowtail \gamma) \rightarrowtail (\alpha \rightarrowtail \gamma)}$.

Proof:

T1: Follows from Ax1 and MP.

R1: Follows from Ax2 and MP.

T2: Follows from Ax1, Ax2, R1, and MP.

T3: Follows from Ax2, T2, and MP.

T4: Follows from Ax2, T2, and MP.

T5: Follows from Ax2 and the definition of $\vee$.

T6: Follows from T4, T3, and MP.

T7: Follows from T6, Ax1, and R1.

T8: Follows from T7, Ax1, and MP.

R2: Follows from T6, Ax2, and MP.

T9: Follows from T8, Ax2, Ax1, Ax3, and MP. □

In what follows, we present a lemma required for the remainder of the article. We include sketchy proofs for some theorems and rules that are derivable in $C_3^{\rightarrowtail,\Delta}$, whereas in other cases, we provide detailed proofs when the original ones are not entirely clear to us.

LEMMA 3.2. ([11]) *The following formulæ and rules hold in the logic* $C_3^{\rightarrowtail,\Delta}$*:*

(ΔT1): $\vdash \Delta(\Delta\alpha \rightarrowtail \alpha)$,

(ΔT2): $\vdash \alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha)$,

(ΔT3): $\vdash \Delta\alpha \rightarrowtail \alpha$,

(ΔT4): $\vdash (\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \beta))$,

(ΔT5): $\vdash \Delta(\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\Delta\alpha \rightarrowtail \Delta\beta)$,

(ΔR1): $\dfrac{\alpha}{\Delta\alpha}$,

(ΔR2): $\dfrac{\alpha \rightarrowtail \beta}{\Delta\alpha \rightarrowtail \Delta\beta}$,

(ΔT6): $\vdash \alpha \rightarrowtail \nabla\alpha$,

(ΔT7): $\vdash (\nabla\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta)$.

PROOF:

(ΔT1): It follows from Ax6, T8, and MP.

(ΔT2): It follows from Ax8, Ax1, and MP.

(ΔT3): It follows from Ax8, Ax1, and MP.

(ΔT4):

1.	$\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha)$	
2.	$(\beta \rightarrowtail \Delta\alpha) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$	1. (MP) (Ax1)
3.	$\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$	2. and (T6)

4. $((\beta \rightarrowtail \Delta\alpha) \rightarrowtail (\alpha \rightarrowtail \Delta\alpha)) \rightarrowtail (\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha))$ (T6)
5. $\alpha \rightarrowtail (\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha))$ 3., 4. and (R2)
6. $((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha) \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)$ (Ax3)
7. $(\alpha \rightarrowtail (\alpha \rightarrowtail ((\beta \rightarrowtail \Delta\alpha) \rightarrowtail \Delta\alpha))) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)))$ 6. and (R2)
8. $(\alpha \rightarrowtail (\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)))$ 5., 7., and MP
9. $(\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail \beta)) \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta))$ (T6)
10. $(\alpha \rightarrowtail ((\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \beta))$ (R5) and (T6)
11. $((\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \beta))$ 10., 9., and MP

(ΔT5): Follows from Ax3, Ax2, and MP.

(ΔR1):

1. $\vdash \alpha$ hyp.
2. $\vdash \Delta\alpha \rightarrowtail \Delta\alpha$ (T8)
3. $\vdash \Delta(\Delta\alpha \rightarrowtail \alpha) \rightarrowtail (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$ (Ax7)
5. $\vdash (\Delta\alpha \rightarrowtail \Delta\alpha) \rightarrowtail (\Delta(\Delta\alpha \rightarrowtail \alpha))$ (Ax6)
5. $\vdash \Delta(\Delta\alpha \rightarrowtail \alpha)$ 2., 4., and (MP)
6. $\vdash (\alpha \rightarrowtail (\alpha \rightarrowtail \Delta\alpha))$ 5., 3., and (MP)
7. $\vdash \Delta\alpha$ 1., 6., and (MP)

(ΔR2): Follows from R2, (ΔT3), (ΔR1), (ΔT5), and MP.

(ΔT6): Follows from Ax1 and the definition of ∇.

(ΔT7): Follows from (ΔT6) and $R3$.

□

Now, we will define a relation on formulas as follows: for the formulas α and β given, we write $\alpha \equiv \beta$ if and only if $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. Then, we have the following Lemma 3.3 and Theorem 3.5 that was given in [11] without proofs.

Lemma 3.3 ([11]). *$\equiv$ is a congruence relation on For.*

Proof: We will start by proving that $\equiv$ is an equivalence relation:

Reflexivity. Let us see that $\alpha \equiv \alpha$, but this is immediate from T8.

Symmetry. Let us see that $\alpha \equiv \beta$ if and only if $\beta \equiv \alpha$, but this is immediate from the very definitions.

Transitivity. Let us prove that if $\alpha \equiv \beta$ and $\beta \equiv \gamma$, then $\alpha \equiv \gamma$. Indeed, we have that $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$ as hypotheses. On the other hand, $\vdash \beta \rightarrowtail \gamma$ and $\vdash \gamma \rightarrowtail \beta$. Applying Ax2, we have that $\vdash \alpha \rightarrowtail \gamma$ and $\vdash \gamma \rightarrowtail \alpha$. Therefore, $\alpha \equiv \gamma$.

We finish by verifying that $\equiv$ is a congruential relation. Indeed:

1. If $\alpha \equiv \beta$, then $\Delta\alpha \equiv \Delta\beta$. Indeed:

    1. $\alpha \equiv \beta$ (hypothesis)
    2. $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$
    3. $\vdash \Delta\alpha \rightarrowtail \Delta\beta$ and $\vdash \Delta\beta \rightarrowtail \Delta\alpha$ (ΔR2)
    4. $\Delta\alpha \equiv \Delta\beta$

2. If $\alpha \equiv \beta$ and $\gamma \equiv \xi$, then $\alpha \rightarrowtail \gamma \equiv \beta \rightarrowtail \xi$. Indeed:

    1. $\alpha \equiv \beta$ (hypothesis)
    2. $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$
    3. $\gamma \equiv \xi$ (hypothesis)
    4. $\vdash \gamma \rightarrowtail \xi$ and $\vdash \xi \rightarrowtail \gamma$
    5. $\vdash (\beta \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \xi)$ (Ax2), 4., (MP)
    6. $\vdash (\alpha \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \gamma)$ (2. and (R2))
    7. $\vdash (\alpha \rightarrowtail \gamma) \rightarrowtail (\beta \rightarrowtail \xi)$ (5., 6., and (R1))

With a similar argument, we can see that $\vdash (\beta \rightarrowtail \xi) \rightarrowtail (\alpha \rightarrowtail \gamma)$ as desired. □

Lemma 3.4. *The following identities hold in $For/_{\equiv}$:*

(ΔŁ1) $\Delta|\alpha| \rightarrowtail |\beta| = |\alpha| \rightarrowtail (|\alpha| \rightarrowtail |\beta|)$;

(ΔŁ2) $\Delta(\Delta|\alpha| \rightarrowtail |\beta|) = \Delta|\alpha| \rightarrowtail \Delta|\beta|$.

Proof: Both identities follow from (ΔT4), (ΔT5), together with axioms (Ax6), (Ax8), and the definition of the operations on equivalence classes. □

Theorem 3.5 ([11]). *The Lindenbaum-Tarski algebra $\langle For/_{\equiv}, \rightarrowtail, \Delta, 1\rangle$ is an $Ł_3^{\rightarrowtail,\Delta}$-algebra, where $|\alpha \rightarrowtail \beta| = |\alpha| \rightarrowtail |\beta|$, $|\Delta\alpha| = \Delta|\alpha|$, and $1 = |\alpha \rightarrowtail \alpha| = \{\phi \in For : \vdash \phi\}$. Moreover, the relation $|\alpha| \leq |\beta|$, defined by $\vdash \alpha \rightarrowtail \beta$, is a partial order on $For/_{\equiv}$.*

Proof: • First, we prove that the relation $|\alpha| \leq |\beta|$ is a partial order on $For/_{\equiv}$. Indeed:

Reflexivity. From T8, we know that $\vdash \alpha \rightarrowtail \alpha$, and then $|\alpha| \leq |\alpha|$.

Antisymmetry. From the conditions $|\alpha| \leq |\beta|$ and $|\beta| \leq |\alpha|$, we have $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. So, $\alpha \equiv \beta$, and therefore $|\alpha| = |\beta|$.

Transitivity. From the conditions $|\alpha| \leq |\beta|$ and $|\beta| \leq |\gamma|$, we have $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \gamma$. Then, by applying (R1), we infer that $\vdash \alpha \rightarrowtail \gamma$ and therefore $|\alpha| \leq |\gamma|$.

• Next, we show that $|\beta| \leq |\alpha \rightarrowtail \alpha| = 1$, which is an immediate consequence of (T7).

• To demonstrate that $\langle For/_{\equiv}, \rightarrowtail, \Delta, 1\rangle$ is an $Ł_3^{\rightarrowtail,\Delta}$-algebra, let us consider $|\alpha|, |\beta|, |\gamma| \in For/_{\equiv}$ and recall that $\{\phi \in For : \vdash \phi\} = 1$. Then, we have:

(Ł1): To show that $|\alpha| \rightarrowtail \big(|\beta| \rightarrowtail |\alpha|\big) = 1$ holds, note that Ax1 $\in 1$, then:

$$
\begin{array}{lcl}
|\alpha \rightarrowtail (\beta \rightarrowtail \alpha)| & = & 1 \\
|\alpha| \rightarrowtail |(\beta \rightarrowtail \alpha)| & = & 1 \\
|\alpha| \rightarrowtail \big(|\beta| \rightarrowtail |\alpha|\big) & = & 1
\end{array}
$$

(Ł2): The identity $\big(|\alpha| \rightarrowtail |\beta|\big) \rightarrowtail \Big(\big(|\beta| \rightarrowtail |\gamma|\big) \rightarrowtail \big(|\alpha| \rightarrowtail |\gamma|\big)\Big) = 1$ is obtained from (Ax2).

(Ł3): The identity $(|\alpha| \rightarrowtail |\beta|) \rightarrowtail |\beta| = (|\beta| \rightarrowtail |\alpha|) \rightarrowtail |\alpha|$ follows from (Ax3) and the definition of $\leq$.

(Ł4): The identity $\big((|\alpha| \rightarrowtail |\beta|) \rightarrowtail (|\beta| \rightarrowtail |\alpha|)\big) \rightarrowtail (|\beta| \rightarrowtail |\alpha|) = 1$ is obtained from axiom (Ax4).

(Ł5): $1 \rightarrowtail |\alpha| = |\alpha|$.

(a) $|\alpha| \leq 1 \rightarrowtail |\alpha|$

1. $\vdash \alpha \rightarrowtail ((\beta \rightarrowtail \beta) \rightarrowtail \alpha)$ (Ax1)
2. $|\alpha| \leq |(\beta \rightarrowtail \beta) \rightarrowtail \alpha|$
3. $|\alpha| \leq |\beta \rightarrowtail \beta| \rightarrowtail |\alpha|$
4. $|\alpha| \leq 1 \rightarrowtail |\alpha|$

(b) $1 \rightarrowtail |\alpha| \leq |\alpha|$

1. $\vdash ((\beta \rightarrowtail \beta) \rightarrowtail \alpha) \rightarrowtail \alpha$ (T9)
2. $|((\beta \rightarrowtail \beta) \rightarrowtail \alpha)| \leq |\alpha|$
3. $|\beta \rightarrowtail \beta| \rightarrowtail |\alpha| \leq |\alpha|$
4. $1 \rightarrowtail |\alpha| \leq |\alpha|$

From (a) and (b), we conclude the proof.

(Ł6): To show that $\Big(\big(|\alpha| \rightarrowtail (|\alpha| \rightarrowtail |\beta|)\big) \rightarrowtail |\alpha|\Big) \rightarrowtail |\alpha| = 1$ holds, it suffices to use axiom (Ax5) and the very definitions.

Finally, the identities (ΔŁ1) and (ΔŁ2) follow from Lemma 3.4. □

We are now in a position to present the first soundness and completeness theorem in the weak sense. To that end, let us introduce the following definition:

Definition 3.6. A function $v : For \to A$ is said to be a *valuation* if it satisfies the following conditions:

(i) $v(\alpha \rightarrowtail \beta) = v(\alpha) \rightarrowtail v(\beta)$;

(ii) $v(\Delta\alpha) = \Delta v(\alpha)$;

(iii) $v(\top) = 1$.

Furthermore, a formula α is said to be *semantically valid*, denoted $\models \alpha$, if for every ${}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and every valuation $v : For \to A$, it holds that $v(\alpha) = 1$.

We now establish the first (weak) soundness and completeness result, whose proof follows standard lines using Theorem 3.5.

Theorem 3.7 (Weak Adequacy Theorem). *For every formula $\alpha \in For$, we have that $\vdash \alpha$ if and only if $\models \alpha$.*

Proof: *(Soundness)*: Let $\mathbf{A}$ be a fixed ${}_3^{\rightarrowtail,\Delta}$-algebra, and let $v : For \to A$ be any valuation. Suppose that $\alpha \in For$ admits a formal proof $\alpha_1, \ldots, \alpha_n$ such that $\alpha_n = \alpha$. We proceed by induction on n.

If $n = 1$, then $\alpha = \alpha_1$ is an axiom. By a direct verification using Definition 3.6, we obtain that $v(\alpha_1) = 1$.

Now assume that the result holds for all proofs of length less than k, and consider a proof of length k. We distinguish two cases:

1. If α_k is an axiom, then $v(\alpha_k) = 1$ by the same reasoning as in the base case.

2. If α_k results from applying Modus Ponens to α_i and $\alpha_i \rightarrowtail \alpha_k$, with $i < k$, then by the induction hypothesis we have $v(\alpha_i) = 1$ and $v(\alpha_i \rightarrowtail \alpha_k) = 1$. Thus, $v(\alpha_i) \rightarrowtail v(\alpha_k) = 1$, and since $v(\alpha_i) = 1$, by (L7) we obtain $v(\alpha_k) = 1$.

Hence, in all cases $v(\alpha) = 1$, and thus $\models \alpha$.

(Completeness): Suppose that $\models \alpha$. Then for every ${}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and every homomorphism $h : For \to \mathbf{A}$, we have $h(\alpha) = 1$. In particular, consider the canonical homomorphism $\pi : For \to For/_{\equiv}$, where $\pi(\gamma) = |\gamma|$

denotes the equivalence class of γ modulo the syntactic congruence $\equiv$. Since $\pi(\alpha) = 1$, it follows that $\alpha \in \{\beta \in For :\vdash \beta\}$. Therefore, $\vdash \alpha$. □

It is worth noting that a weak version of the Adequacy Theorem is not explicitly stated in [11]; however, the following lemma is a consequence of it.

Before stating the result, let us fix the following notation:

$$\vdash \alpha \leftrightarrow \beta \text{ if and only if } \vdash \alpha \rightarrowtail \beta \text{ and } \vdash \beta \rightarrowtail \alpha.$$

LEMMA 3.8. *The following formulas and inference patterns are theorems of the logic* $\mathcal{C}_3^{\rightarrowtail,\Delta}$*:*

(ΔT8) $\vdash \Delta\alpha \rightarrowtail \nabla\alpha$;

(ΔT9) $\vdash \nabla\alpha \leftrightarrow \nabla\nabla\alpha$,

(ΔT10) $\vdash \nabla\Delta\alpha \leftrightarrow \Delta\alpha$,

(ΔT11) $\vdash \alpha \rightarrowtail \Delta\nabla\alpha$,

(ΔT12) $\vdash (\Delta\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\beta \rightarrowtail (\alpha \rightarrowtail \beta))$,

(ΔT13) $\vdash (\Delta\alpha \rightarrowtail \beta) \rightarrowtail \nabla(\alpha \rightarrowtail \beta)$,

(ΔT14) $\vdash \Delta(\alpha \rightarrowtail \beta) \rightarrowtail (\Delta\alpha \rightarrowtail \Delta\beta)$,

(ΔT15) $\vdash (\nabla\alpha \rightarrowtail \nabla\beta) \rightarrowtail \nabla(\alpha \rightarrowtail \beta)$,

(ΔT16) $\vdash ((\alpha \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \gamma)) \rightarrowtail (\alpha \rightarrowtail (\beta \rightarrowtail \gamma))$,

(ΔT17) $\vdash ((\alpha \rightarrowtail \beta) \rightarrowtail \beta) \rightarrowtail (\alpha \rightarrowtail \Delta((\alpha \rightarrowtail \beta) \rightarrowtail \beta))$,

(ΔT18) $\vdash \nabla\alpha \leftrightarrow \Delta\nabla\alpha$,

(ΔT19) $\vdash \alpha \rightarrowtail (\alpha \rightarrowtail (\nabla(\alpha \rightarrowtail \beta) \rightarrowtail \nabla\beta))$,

(ΔT20) $\vdash \Delta(\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\alpha \rightarrowtail \nabla\beta)$.

PROOF: Let $v : For \to A$ be any valuation, and let **A** be an arbitrary ${}_3^{\rightarrowtail,\Delta}$-algebra. For each formula ϕ among (ΔT8) to (ΔT20), we verify that

$v(\phi) = 1$ by applying the definitions of Δ and ∇ and the properties of $\mathbf{A}$. Since these equations hold in every $\overset{\rightarrowtail,\Delta}{3}$-algebra and the variety is equational, it follows that each ϕ is semantically valid. By the Weak Adequacy Theorem, we conclude that $\vdash \phi$. □

3.1. Strong Version of the Adequacy Theorem

Recall that a logic defined over a language $\mathcal{S}$ is a system $\mathcal{L} = \langle For, \vdash_{\mathbf{L}} \rangle$, where For is the set of formulas over $\mathcal{S}$, and the relation $\vdash_{\mathbf{L}} \subseteq \mathcal{P}(For) \times For$, where $\mathcal{P}(A)$ is the set of all subsets of A. This general framework follows the standard approach to abstract consequence relations (see, e.g., [3]). We adopt the standard assumption that $\vdash_{\mathbf{L}}$ is closed under uniform substitution.

The logic $\mathcal{L}$ is said to be Tarskian if it satisfies the following properties for every set $\Gamma \cup \Omega \cup \{\varphi, \beta\}$ of formulas:

(1) if $\alpha \in \Gamma$, then $\Gamma \vdash_{\mathbf{L}} \alpha$,

(2) if $\Gamma \vdash_{\mathbf{L}} \alpha$ and $\Gamma \subseteq \Omega$, then $\Omega \vdash_{\mathbf{L}} \alpha$,

(3) if $\Omega \vdash_{\mathbf{L}} \alpha$ and $\Gamma \vdash_{\mathbf{L}} \beta$ for every $\beta \in \Omega$, then $\Gamma \vdash_{\mathbf{L}} \alpha$.

A logic $\mathcal{L}$ is said to be finitary if it satisfies the following:

(4) if $\Gamma \vdash_{\mathbf{L}} \alpha$, then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash_{\mathbf{L}} \alpha$.

Let $\mathcal{L}$ be a Tarskian logic, and let Γ be a set of formulas; we say that Γ is a theory. A theory Γ is said to be *consistent* if there exists a formula φ such that $\Gamma \nvdash_{\mathbf{L}} \varphi$. We also say that Γ is a maximal consistent theory if $\Gamma, \psi \vdash_{\mathbf{L}} \varphi$ for any $\psi \notin \Gamma$, and in this case, we say Γ is non-trivial maximal with respect to φ.

On the other hand, a logic is said to be standard if it is Tarskian and a finitary system. Furthermore, let $\mathscr{L}$ be a Tarskian logic. A set of formulas Γ is said to be closed in $\mathscr{L}$, or a closed theory of $\mathscr{L}$, if the following holds for every formula ψ: $\Gamma \vdash_{\mathbf{L}} \psi$ if and only if $\psi \in \Gamma$.

LEMMA 3.9. *Any non-trivial maximal set of formulas with respect to φ in $\mathcal{L}$ is closed, provided that $\mathcal{L}$ is Tarskian.*

Proof: This is a direct consequence of the definition of maximality and the Tarskian conditions, in particular the transitivity (3) and reflexivity (1) of the consequence relation. □

Lemma 3.10. (Lindenbaum-Łoś Lemma) *Let $\mathcal{L}$ be a standard logic and let $\Gamma \cup \{\varphi\}$ be a set of formulas such that $\Gamma \nvdash_{\mathbf{L}} \varphi$. Then, there exists a set of formulas Ω such that $\Gamma \subseteq \Omega$ with Ω maximal non-trivial with respect to φ in $\mathcal{L}$.*

Proof: See Theorem 2.22 of [28]. □

Proposition 3.11. The calculus $\mathcal{C}_3^{\rightarrowtail,\Delta}$ is a Tarskian and finitary logic.

Lemma 3.12. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that Γ is non-trivial maximal with respect to φ in $\mathcal{C}_3^{\rightarrowtail,\Delta}$. Then, if $\phi \notin \Gamma$, then $\Gamma \vdash \Delta\phi \rightarrowtail \beta$ for every $\beta \in For$.*

Proof: Let us consider the set $|\Gamma| = \{|\alpha| : \alpha \in \Gamma\}$ and suppose that $\alpha \in \Gamma$ such that $\alpha \equiv \beta$. Then, $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. Therefore, $\beta \in \Gamma$ and then we have that $|\Gamma|$ is closed under equivalence: if $\alpha \in \Gamma$ and $|\alpha| = |\beta|$, then $\beta \in \Gamma$.

Moreover, it is not hard to see that the conditions of Definition 2.2 are verified by $|\Gamma|$. Thus, $|\Gamma|$ is an implicative filter.

Recall that $\mathbf{For}/_{\equiv}$ is an ${}_3^{\rightarrowtail,\Delta}$-algebra in virtue of Theorem 3.5. Now, let $D \subseteq For/_{\equiv}$ be an implicative filter that properly contains $|\Gamma|$. Then there is $|\gamma| \in D$ such that $|\gamma| \notin |\Gamma|$, so $\gamma \notin \Gamma$ and therefore $\Gamma \cup \{\gamma\} \vdash \varphi$. From the latter and taking $D' = \{\alpha : |\alpha| \in D\}$, we can infer that $D' \vdash \varphi$. Since D' is closed, we obtain that $|\varphi| \in D$. This contradicts the maximality of Γ, hence $|\Gamma|$ must be a maximal implicative filter below $|\varphi|$.

So, if $\phi \notin \Gamma$, then $|\phi| \notin |\Gamma|$. From the latter and Lemma 2.6, we have that $\Delta|\phi| \rightarrowtail |\beta| \in |\Gamma|$. By definition of $|\Gamma|$, we have that $\Delta\phi \rightarrowtail \beta \in \Gamma$ as desired. □

The last Lemma is central for the following Theorem, as it allows us to construct the special homomorphism. It is worth noting that its proof requires Lemma 2.6 and certain algebraic properties of the class of ${}_3^{\rightarrowtail,\Delta}$-algebras. It would be of independent interest to obtain a purely syntactic proof of Lemma 3.12, avoiding the use of maximal implicative filters and the underlying algebraic machinery. This problem remains open and is left for future research.

Proposition 3.13. Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that Γ is non-trivial and maximal with respect to φ in $\mathcal{C}_3^{\rightarrowtail,\Delta}$. Then, the function defined for every $\gamma \in For$ as follows:

$$v(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma \\ \frac{1}{2} & \text{if } \gamma \in \Gamma_{1/2} \\ 0 & \text{if } \gamma \in \Gamma_0 \end{cases}$$

is a homomorphism from For into $\mathbb{C}_3$ such that $v^{-1}(\{1\}) = \Gamma$, where $\Gamma_{\frac{1}{2}} = \{\alpha \notin \Gamma : \Delta\alpha \notin \Gamma \text{ and } \nabla\alpha \in \Gamma\}$, $\Gamma_0 = \{\alpha \notin \Gamma : \nabla\alpha \notin \Gamma\}$, and $\mathbb{C}_3$ is the 3-element chain ${}_3^{\rightarrowtail,\Delta}$-algebra.

Proof: We show that $v(\alpha \rightarrowtail \beta) = v(\alpha) \rightarrowtail v(\beta)$. Indeed:

- If $v(\beta) = 1$, then $\beta \in \Gamma$. By (Ax1), we have $\beta \rightarrowtail (\alpha \rightarrowtail \beta) \in \Gamma$, and by (MP), $\alpha \rightarrowtail \beta \in \Gamma$. Thus, $v(\alpha \rightarrowtail \beta) = 1$.

- If $v(\alpha) = 0$ and $v(\beta) = 1/2$, then by Lemma 3.12, $\Delta\alpha \rightarrowtail \beta \in \Gamma$. Since $\nabla\beta \in \Gamma$, using (ΔT12) and (MP), we obtain $\alpha \rightarrowtail \beta \in \Gamma$, hence $v(\alpha \rightarrowtail \beta) = 1 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = v(\beta) = 0$, then $\nabla\alpha \notin \Gamma$. By Lemma 3.12, $\Delta\nabla\alpha \rightarrowtail \beta \in \Gamma$. By (Ax2), $\vdash (\nabla\alpha \rightarrowtail \Delta\nabla\alpha) \rightarrowtail ((\Delta\nabla\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\alpha \rightarrowtail \beta))$. From this, together with (ΔT18) and (MP), we infer that $(\Delta\nabla\alpha \rightarrowtail \beta) \rightarrowtail (\nabla\alpha \rightarrowtail \beta) \in \Gamma$, and hence $\nabla\alpha \rightarrowtail \beta \in \Gamma$. Using ($\Delta$T7) and (MP), it follows that $\alpha \rightarrowtail \beta \in \Gamma$, so $v(\alpha \rightarrowtail \beta) = 1 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = 1/2$ and $v(\beta) = 0$, then $\alpha, \nabla\beta \notin \Gamma$ and $\nabla\alpha \in \Gamma$. Since $\alpha \notin \Gamma$, Lemma 3.12 gives $\Delta\alpha \rightarrowtail \beta \in \Gamma$. Thus, ($\Delta$T13) and (MP) yield $\nabla(\alpha \rightarrowtail \beta) \in \Gamma$. Suppose, for contradiction, that $\alpha \rightarrowtail \beta \in \Gamma$. Then, using ($\Delta$T2) and (MP), we derive $\Delta(\alpha \rightarrowtail \beta) \in \Gamma$. From this, using ($\Delta$T20) and (MP), we obtain $\nabla\alpha \rightarrowtail \nabla\beta \in \Gamma$. Since $\nabla\alpha \in \Gamma$, it follows that $\nabla\beta \in \Gamma$, a contradiction. Therefore, $\alpha \rightarrowtail \beta \notin \Gamma$ and $v(\alpha \rightarrowtail \beta) = 1/2 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = v(\beta) = 1/2$, then $\alpha \notin \Gamma$ and $\nabla\beta \in \Gamma$. Lemma 3.12 ensures that $\Delta\alpha \rightarrowtail \beta \in \Gamma$. Then, ($\Delta$T12) and (MP) yield $\alpha \rightarrowtail \beta \in \Gamma$, and thus $v(\alpha \rightarrowtail \beta) = 1 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = 1$ and $v(\beta) = 0$, then $\alpha \in \Gamma$ and $\nabla\beta \notin \Gamma$. Suppose, for contradiction, that $\nabla(\alpha \rightarrowtail \beta) \in \Gamma$. Then, by ($\Delta$T19) and (MP), we get $\nabla\beta \in \Gamma$, a contradiction. Therefore, $\nabla(\alpha \rightarrowtail \beta) \notin \Gamma$ and $v(\alpha \rightarrowtail \beta) = 0 = v(\alpha) \rightarrowtail v(\beta)$.

- If $v(\alpha) = 1$ and $v(\beta) = 1/2$, then $\alpha, \nabla\beta \in \Gamma$ and $\beta \notin \Gamma$. By (Ax1), we have $\nabla\beta \rightarrowtail (\nabla\alpha \rightarrowtail \nabla\beta) \in \Gamma$, so by (MP), $\nabla\alpha \rightarrowtail \nabla\beta \in \Gamma$. Then, using ($\Delta$T15), we conclude that $\nabla(\alpha \rightarrowtail \beta) \in \Gamma$. Suppose $\alpha \rightarrowtail \beta \in \Gamma$. Since $\alpha \in \Gamma$, we would get $\beta \in \Gamma$, which contradicts the hypothesis. Thus, $\alpha \rightarrowtail \beta \notin \Gamma$ and $v(\alpha \rightarrowtail \beta) = 1/2 = v(\alpha) \rightarrowtail v(\beta)$.

We now show that $v(\Delta\alpha) = \Delta v(\alpha)$. Indeed:

- If $v(\alpha) = 0$, then $\alpha, \nabla\alpha \notin \Gamma$. Suppose, for contradiction, that $\nabla\Delta\alpha \in \Gamma$. Then, using ($\Delta$T10) and (MP), we derive $\Delta\alpha \in \Gamma$, and from (ΔT3) and (MP), $\alpha \in \Gamma$, a contradiction. Thus, $\nabla\Delta\alpha \notin \Gamma$ and $v(\Delta\alpha) = 0 = \Delta v(\alpha)$.

- If $v(\alpha) = 1/2$, then $\alpha \notin \Gamma$ and $\nabla\alpha \in \Gamma$. Suppose $\nabla\Delta\alpha \in \Gamma$. Then, using ($\Delta$T10) and (MP), we get $\Delta\alpha \in \Gamma$, and by (ΔT3) and (MP), $\alpha \in \Gamma$, a contradiction. Thus, $v(\Delta\alpha) = 0 = \Delta v(\alpha)$.

- If $v(\alpha) = 1$, then $\alpha \in \Gamma$. By (ΔT2) and (MP), we have $\Delta\alpha \in \Gamma$, hence $\Delta v(\alpha) = 1 = v(\Delta\alpha)$. □

Theorem 3.13 is the key ingredient in the statement of the following Completeness Theorem. It is worth noting that we were able to prove Theorem 3.13 without relying on Lemma 3.12. To conclude this section, we define the semantic entailment symbol $\Gamma \vDash \alpha$ to mean that, for every ${}_3^{\rightarrowtail,\Delta}$-algebra $\mathbf{A}$ and every valuation v, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$, then $v(\alpha) = 1$.

Theorem 3.14. (Strong Soundness and Completeness of $C_3^{\rightarrowtail,\Delta}$ w.r.t. the class of ${}_3^{\rightarrowtail,\Delta}$-algebras). *Let $\Gamma \cup \{\varphi\} \subseteq For$, $\Gamma \vdash \varphi$ if and only if $\Gamma \vDash \varphi$.*

Proof: Soundness: It is not hard to see that every axiom is valid for every $_3^{\rightarrowtail,\Delta}$-algebra A. In addition, satisfaction is preserved by the inference rules.

Completeness: Suppose $\Gamma \vDash \varphi$ and $\Gamma \nvdash \varphi$. According to Lemma 3.10, there is a maximal consistent theory Ω such that $\Gamma \subseteq \Omega$ and $\Omega \nvdash \varphi$. From the latter and Proposition 3.13, there is a valuation $\mu : For \to \mathbb{C}_3$ such that $\mu(\Omega) = \{1\}$ but $\mu(\varphi) \neq 1$. Since $\Gamma \subseteq \Omega$, we have $\mu(\gamma) = 1$ for every $\gamma \in \Gamma$, This contradicts the assumption that $\Gamma \vDash \varphi$, and thus $\Gamma \vdash \varphi$ must hold. □

The result above establishes the strong version of completeness for our calculus, syntactically characterized and algebraically sound. It is worth mentioning that Theorem 3.14 could be obtained from [13, Theorem 4.12] for taking $n = 3$, but our contribution lies in providing a purely syntactic proof, which is independent from the general framework and therefore more elementary and self-contained. As an important consequence of this Theorem, we have that $C_3^{\rightarrowtail,\Delta}$ does not enjoy Deduction Theorem as we will see in the next Corollary.

Corollary 3.15. In the logic $C_3^{\rightarrowtail,\Delta}$, Deduction Theorem does not hold.

Proof: In virtue of Theorem 3.14 and the rule ΔR1 of Lemma 3.2, we have that $\varphi \vDash \Delta\varphi$, but it is not hard to see that $\nvDash \varphi \rightarrowtail \Delta\varphi$. Indeed, it is enough to take a valuation $v(\varphi) = \frac{1}{2}$, and so, $v(\varphi \rightarrowtail \Delta\varphi) = 0$. Hence, the implication fails in the semantics even when the entailment holds, showing that the Deduction Theorem is not valid. □

4. First-order version of $C_3^{\rightarrowtail,\Delta}$: the logic $\forall_3^{\rightarrowtail,\Delta}$

In this section we introduce the first-order extension of the logic $C_3^{\rightarrowtail,\Delta}$, denoted $\forall_3^{\rightarrowtail,\Delta}$. Our main goal is to extend the propositional framework to the first-order level, providing an appropriate semantic setting and a corresponding deductive system that preserves the essential features of the original logic. In particular, we develop the notion of valuation over first-order structures and adapt the key algebraic tools to this richer setting. However, we would like to stress that the shift from the propositional to the first-order level requires a careful reinterpretation of the semantics and the expansion of the language to accommodate variables, terms, quantifiers, and substitution mechanisms. While some notions inevitably mirror the propositional framework, they are now defined within a richer language and semantic context that substantially changes their scope and treatment.

Let us begin by fixing the propositional signature Θ of $C_3^{\rightarrowtail,\Delta}$, and extending it with two quantifier symbols $\forall$ and $\exists$, as well as the usual punctuation symbols. We consider a countable set Var of individual variables and denote by $\mathfrak{Fm}_\Sigma$ the set of formulas over a first-order signature $\Sigma = \langle \mathcal{P}, \mathcal{F}, \mathcal{C} \rangle$, where $\mathcal{P}$ is a non-empty set of predicate symbols, $\mathcal{F}$ a set of function symbols, and $\mathcal{C}$ a set of individual constants. The set Ter denotes the absolutely free term algebra over $\mathcal{F}$ and $\mathcal{C}$.

As customary, we define the notions of free and bound variables, substitution, closed terms, and sentences. Given a formula φ, we denote by $\varphi(x/t)$ the result of simultaneously replacing all free occurrences of the variable x by the term t, provided t is free for x in φ.

A Σ-structure $\mathfrak{A}$ for $\forall_3^{\rightarrowtail,\Delta}$ is a pair $\langle \mathbf{A}, \mathbf{S} \rangle$ where $\mathbf{A}$ is a complete $_3^{\rightarrowtail,\Delta}$-algebra and $\mathbf{S}$ provides the standard first-order interpretation over a non-empty domain S. That is:

- every constant $c \in \mathcal{C}$ is assigned an element $c^{\mathfrak{A}} \in S$,
- each n-ary function symbol $f \in \mathcal{F}$ is interpreted as a function $f^{\mathfrak{A}} : S^n \to S$,
- each n-ary predicate symbol $P \in \mathcal{P}$ is interpreted as a function $P^{\mathfrak{A}} : S^n \to A$.

Truth values of terms and formulas in a structure $\mathfrak{A}$ under a valuation $v : Var \to S$ are defined recursively in the usual way, with logical connectives interpreted via the algebraic operations of $\mathbf{A}$. Notably, quantifiers are interpreted through meet and join operations:

$$||\forall x\alpha||_v^{\mathfrak{A}} = \bigwedge_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{A}}, \qquad ||\exists x\alpha||_v^{\mathfrak{A}} = \bigvee_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{A}}.$$

Satisfaction and semantic consequence are defined analogously to the propositional case, with $\mathfrak{A} \models \varphi[v]$ meaning that $||\varphi||_v^{\mathfrak{A}} = 1$, and $\Gamma \models \varphi$ holding when every model that satisfies all formulas in Γ also satisfies φ.

The deductive system of $\forall_3^{\rightarrowtail,\Delta}$ builds on the propositional calculus $C_3^{\rightarrowtail,\Delta}$ by adding a collection of standard axiom schemas and inference rules for the quantifiers, adapted to the semantics of the operator Δ. In particular, we include two additional equivalences involving the distribution of Δ over the quantifiers, which are central to the algebraic treatment of the system and have no direct analogue in classical logic.

Axiom Schemas

- ($\forall$1) $\varphi(x/t) \rightarrowtail \exists x\varphi$, if t is free for x in φ,
- ($\forall$2) $\forall x\varphi \rightarrowtail \varphi(x/t)$, if t is free for x in φ,
- ($\forall$3) $\Delta\exists x\varphi \leftrightarrow \exists x\Delta\varphi$,
- ($\forall$4) $\Delta\forall x\varphi \leftrightarrow \forall x\Delta\varphi$,

Inference Rules

- ($\forall$R1) $\dfrac{\alpha \rightarrowtail \beta}{\exists x\alpha \rightarrowtail \beta}$, provided x does not occur free in β,
- ($\forall$R2) $\dfrac{\alpha \rightarrowtail \beta}{\alpha \rightarrowtail \forall x\beta}$, provided x does not occur free in α.

The design of $\forall_3^{\rightarrowtail,\Delta}$ is based on a conservative and modular extension of the propositional core, preserving its non-classical features while allowing for standard model-theoretic techniques in the first-order setting. Observe that, although the domain of interpretation may be infinite, the set of truth values is finite, namely $\mathbb{C}_3 = \{0, \frac{1}{2}, 1\}$. Hence, all required infima and suprema exist, and the interpretation of the quantifiers is well-defined without requiring additional completeness assumptions on the underlying algebra. In what follows, we will develop the fundamental metatheorems of this logic and establish its soundness and completeness with respect to the class of first-order ${}_3^{\rightarrowtail,\Delta}$-structures.

LEMMA 4.1. [16] *Let* $\mathbf{A}$ *be a complete* ${}_3^{\rightarrowtail,\Delta}$*-algebra and the set* $\{a_i\}_{i\in I}$ *of elements of* A *for any non-empty set* I*. Then, if there exists* $\bigvee_{i\in I} a_i$ $(\bigwedge_{i\in I} a_i)$*, then there exists* $\bigvee_{i\in I} \Delta a_i$ $(\bigwedge_{i\in I} \Delta a_i)$*, and also* $\bigvee_{i\in I} \Delta a_i = \Delta \bigvee_{i\in I} a_i$ *and* $\bigwedge_{i\in I} \Delta a_i = \Delta \bigwedge_{i\in I} a_i$ *hold.*

THEOREM 4.2. (Soundness Theorem). *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$*, if* $\Gamma \vdash \varphi$ *then* $\Gamma \vDash \varphi$*.*

PROOF: Let us consider the fixed structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{S}\rangle$. Let φ be a formula such that $\Gamma \vdash \varphi$. Then, there exists $\alpha_1, \cdots, \alpha_n$ a derivation of φ from Γ. If $n = 1$ then φ is an axiom or $\varphi \in \Gamma$. If $\varphi \in \Gamma$, then it is easy to see that $\Gamma \vDash \varphi$. If φ is an axiom, then the truth of (Ax1) to (Ax8) is obtained at the propositional level.

Unlike the propositional case, we now deal with formulas containing quantifiers, and some axioms require additional semantic justification. Let us observe that on $\mathfrak{Fm}_\Sigma$ we can define an order relation $\leq$ in a similar way as was done in Theorem 3.5. So, let us suppose that φ is $\alpha(x/t) \rightarrowtail \exists x\alpha$. Then, $||\varphi||_v^{\mathfrak{M}} = ||\alpha||^{\mathfrak{M}}_{v[x\to||t||^{\mathfrak{M}}_v]} \rightarrowtail ||\exists x\alpha||^{\mathfrak{M}}_v$. It is clear that $||\alpha||^{\mathfrak{M}}_{v[x\to||t||^{\mathfrak{M}}_v]} \leq \bigvee_{a\in S} ||\alpha||^{\mathfrak{M}}_{v[x\to a]}$, then $||\alpha||^{\mathfrak{M}}_{v[x\to||t||^{\mathfrak{M}}_v]} \leq ||\exists x\alpha||^{\mathfrak{M}}_v$. Therefore, we have that $||\alpha(x/t) \rightarrowtail \exists x\alpha||^{\mathfrak{M}}_v = 1$, and so axiom (∀1) is valid on $\mathfrak{M} = \langle \mathbf{A}, \mathbf{S}\rangle$. Analogously, axiom (∀2) is also valid.

For axioms (∀3) and (∀4), which explicitly involve the operator Δ, the proof of validity requires a structural property of the algebra captured by Lemma 4.1. This is a key point where the first-order setting reveals its specific algebraic nuances, in contrast with the propositional case.

Besides, it is not difficult to see that satisfaction is preserved by the inference rules. □

To proceed with the completeness proof, we now adapt the construction of the canonical model to the first-order setting. This transition requires us to restrict the attention to closed formulas, reflecting the semantic role of sentences in classical model theory.

Let us first take the set of closed formulas denoted by $C\mathfrak{Fm}_\Sigma$ and consider the relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash \alpha \rightarrowtail \beta$ and $\vdash \beta \rightarrowtail \alpha$. Thus, we have that the algebra $C\mathfrak{Fm}_\Sigma/_{\equiv}$ is a ${}_3^{\rightarrowtail,\Delta}$-algebra, as in the propositional case, but now involving quantifier-free equivalence classes of closed formulas.

Let us consider the system $C\forall Ł_3^{\rightarrowtail,\Delta}$, which is obtained from $\forall Ł_3^{\rightarrowtail,\Delta}$ but is defined over sentences (closed formulas). It is clear that $C\forall Ł_3^{\rightarrowtail,\Delta}$ is a Tarskian and finitary logic, as discussed in Section 3.1. Additionally, we can introduce the notion of the set of formulas that are maximal non-trivial with respect to some closed formula φ. The concept of closed theories is defined in the same way as in the propositional case. However, unlike the purely propositional scenario, we now handle a richer language with quantifiers, and this impacts the structure of the Lindenbaum algebra. Therefore, Lindenbaum-Łoś's Theorem holds for $C\forall Ł_3^{\rightarrowtail,\Delta}$. Consequently, we have the following lemma:

Lemma 4.3. *Let $\Gamma \cup \{\varphi\}$ be a set of closed formulas, such that Γ is non-trivial and maximal with respect to φ in $C\forall Ł_3^{\rightarrowtail,\Delta}$. If $\phi \notin \Gamma$, then $\Gamma \vdash \Delta\phi \rightarrowtail \beta$ for every $\beta \in C\mathfrak{Fm}_\Sigma$.*

Proof: While the reasoning parallels that of Lemma 3.12, the key difference lies in the domain of discourse and the interpretation of closed terms. Now we consider that $C\mathfrak{Fm}_{\mathbf{\Sigma}}/_{\equiv}$ is an $Ł_3^{\rightarrowtail,\Delta}$-algebra. □

With this setting in place, we are ready to define a canonical first-order model based on closed terms. Unlike the earlier propositional model, the interpretation of function and predicate symbols must respect arities and term construction, thus requiring an explicit definition over a term domain.

Let us now consider the structure:

$$\mathfrak{M} = \langle \mathbb{C}_3, CTer, \cdot^{CTer} \rangle,$$

where $CTer$ is a set of closed terms. We can define the interpretation as follows:

- If $\hat{c}$ is a constant, then $||\hat{c}||_\mu^{\mathfrak{M}} := c$.
- If $f \in \mathcal{F}$, then $||f(t_1, \cdots, t_n)||_\mu^{\mathfrak{M}} = f(t_1, \cdots, t_n)$.

- If $P \in \mathcal{P}$, then $||P(t_1, \cdots, t_n)||^{\mathfrak{M}}_{\mu} = P^{\mathfrak{M}}(t_1, \cdots, t_n)$.

We recall that $P^{\mathfrak{M}} : (CTer)^n \to \mathbb{C}_3$ is a function that allows us to define $|| \cdot ||^{\mathfrak{M}}_{\mu}$ correctly. Our interpretation is defined for atomic closed formulas, but it is easy to see that $||\alpha||^{\mathfrak{M}}_{\mu}$ is correctly defined for every quantifier closed formula α, as we will see in the following Proposition.

PROPOSITION 4.4. Let $\Gamma \cup \{\varphi\}$ be a set of closed formulas (sentences) such that Γ is non-trivial and maximal with respect to φ in $\forall Ł_3^{\rightarrowtail,\Delta}$. Then, the function defined by

$$||\phi||^{\mathfrak{M}}_{\mu} = \begin{cases} 0 & \text{if } \phi \in \Gamma_0 \\ 1/2 & \text{if } \phi \in \Gamma_{1/2} \\ 1 & \text{if } \phi \in \Gamma \end{cases}$$

is a homomorphism from $C\mathfrak{Fm}$ into $\mathbb{C}_3$, where $\Gamma_{\frac{1}{2}} = \{\alpha \notin \Gamma : \Delta\alpha \notin \Gamma \text{ and } \nabla\alpha \in \Gamma\}$, $\Gamma_0 = \{\alpha \notin \Gamma : \nabla\alpha \notin \Gamma\}$, and $\mathbb{C}_3$ is the 3-element chain $Ł_3^{\rightarrowtail,\Delta}$-algebra. Moreover, $|| \cdot ||^{\mathfrak{M}}_{\mu}$ is a $\mathfrak{M}$-valuation, and $C\mathfrak{Fm}$ is the set of closed formulas.

PROOF: From Proposition 3.13, we can affirm that $||\varphi \rightarrowtail \phi||^{\mathfrak{M}}_{\mu} = ||\varphi||^{\mathfrak{M}}_{\mu} \rightarrowtail ||\phi||^{\mathfrak{M}}_{\mu}$ and $||\Delta\phi||^{\mathfrak{M}}_{\mu} = \Delta||\phi||^{\mathfrak{M}}_{\mu}$, which has the same proof as in the propositional case, but now using Lemma 4.3.

From ($\forall$1) and ($\forall$R1), we have:

$$||\forall x\alpha||^{\mathfrak{M}}_{\mu} = \bigwedge_{a \in T_{\Theta}} ||\alpha||^{\mathfrak{M}}_{\mu[x \to a]}$$

Then, by applying ($\forall$R2) (used twice), we obtain:

$$||\exists x\alpha||^{\mathfrak{M}}_{\mu} = \bigvee_{a \in T_{\Theta}} ||\alpha||^{\mathfrak{M}}_{\mu[x \to a]}$$

Hence, the proof is complete. □

We are now in a position to prove the following central theorem:

THEOREM 4.5. (Completeness Theorem for Sentences). *Let* $\Gamma \cup \{\varphi\}$ *be a set of closed formulas (sentences). If* $\Gamma \vDash \varphi$*, then* $\Gamma \vdash \varphi$*.*

PROOF: Suppose that $\Gamma \nvdash \varphi$. Then, there exists a theory Ω, maximal and consistent (with respect to closed formulas) in $C\forall Ł_3^{\rightarrowtail,\Delta}$ with respect to φ, such that $\Gamma \subseteq \Omega$, as proven in Lemma 3.10. According to Proposition 4.4, there exists an interpretation map $||\cdot||_\mu^{\mathfrak{M}}$ such that $||\alpha||_\mu^{\mathfrak{M}} = 1$ if and only if $\alpha \in \Omega$. Therefore, $\mathfrak{M} \vDash \gamma$ for every $\gamma \in \Gamma$, but $\mathfrak{M} \nvDash \varphi$, which contradicts our hypothesis. □

To present a completeness theorem for arbitrary formulas, we now introduce some auxiliary notions. Given a formula α, let $\{x_1, \ldots, x_n\}$ be the set of variables that occur freely in α. The *universal closure* of α is the closed formula $(\forall\alpha)$, defined as α itself if $n = 0$, and otherwise as $\forall x_1 \ldots \forall x_n \alpha$. The completeness theorem for arbitrary formulas in $\forall Ł_3^{\rightarrowtail,\Delta}$ now follows easily from the previous result:

THEOREM 4.6. (Completeness of $\forall Ł_3^{\rightarrowtail,\Delta}$ with respect to the class of $Ł_3^{\rightarrowtail,\Delta}$-algebras). *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas. Then:* $\Gamma \vDash \varphi$ *implies that* $\Gamma \vdash \varphi$*.*

PROOF: By $(\forall 2)$ and $(\forall R2)$, it is easy to prove that $\alpha \vdash (\forall\alpha)$ and $(\forall\alpha) \vdash \alpha$, for every formula α. On the other hand, by the definition of $\vDash$, it is straightforward to verify that $\alpha \vDash \alpha$ and $(\forall\alpha) \vDash \alpha$, for every formula α. Then, for every $\Gamma \cup \{\varphi\}$, we have that $\Gamma \vdash \varphi$ if and only if $(\forall\Gamma) \vdash (\forall\varphi)$, and $\Gamma \vDash \varphi$ if and only if $(\forall\Gamma) \vDash (\forall\varphi)$, where $(\forall\Gamma) = \{(\forall\beta) : \beta \in \Gamma\}$. Thus, the desired result follows immediately from Theorem 4.5. □

5. Final Remarks and Conclusions

In the book [5], the authors presented Adequacy Theorems for several paraconsistent logics and Logics of Formal Inconsistency at the propositional level. In Chapter 4, they constructed a homomorphism for each of the three-valued logics studied in it; these logics are algebraizable with Blok-Pigozzi's method as established in the mentioned chapter. Clearly, this homomorphism can be constructed because of the algebraizability of the logics. This idea is present in our paper in Proposition 3.13 for the propositional case. Interestingly, we have taken this homomorphism from [13, Theorem 3.17], but our homomorphism is, in fact, the three-valued syntactic version of their presentation; in this setting, we have given a new syntactic proof for Proposition 3.13. The authors of [13] needed this homomorphism to determine the generating algebras of the variety; in our case, see Lemma 2.6, we could prove Theorem 2.8 of Section 1 using the algebraic version of our homomorphism. This homomorphism was also constructed in other classes of algebras, see, for instance, [10, 17, 13].

From a broader perspective, our work can be contrasted with several well-established approaches in the literature. In particular, while the algebraic theory of consequence developed by Blok and Pigozzi [4] and further expanded in [18] provides a general framework for algebraizable logics, our contribution focuses on a specific implicational fragment enriched with the Δ operator and emphasizes a direct syntactic treatment. Moreover, although the Δ operator has been extensively studied in fuzzy logics following Baaz [1], our setting is strictly finite-valued and algebraically simpler, which allows for a more explicit construction of the corresponding homomorphisms. In this sense, our results complement these general approaches by providing a concrete and self-contained analysis in the three-valued case.

We remark that, although the propositional homomorphism used in Proposition 4.4 is strongly inspired by the one defined in Proposition 3.13, the first-order extension involves non-trivial syntactic adjustments, particularly in the interpretation of quantifiers. These steps are not a mere repetition but rather reflect the adaptation of the method to a different logical level, and as such, constitute an original contribution.

Another issue that deserves a brief comment is the technical result given in Lemma 3.12; this is essential to construct the mentioned homomorphism, and this is a powerful syntactic property that we are only able to prove using algebraic arguments. In fact, this Lemma is a syntactic version of an algebraic one given by A. Monteiro; in the paper [17], it was established how it holds in a family of the semisimple class of algebras. In our case, see Lemma 2.6. On the other hand, this kind of homomorphisms cannot be constructed for da Costa's systems C_n ($n < \omega$) because these logics are not algebraizable, see, for instance, [26].

On the first-order side, other kinds of proofs of Adequacy Theorems were given in [16, 17], where this technique used in them is strongly based on the study of algebraic properties of Lindenbaum-Tarski algebras for some first-order logics. This technique was recently applied to the first-order version of the logic $G'3$ ([7]) because it is not possible to apply the technique given in [5, Chapter 7]. Recall that the proofs of Adequacy Theorems for first-order logics given in [5, Chapter 7] are based on the fact that the propositional levels enjoy the Deduction Theorem, but it is not the case for the logic $G'3$ as it was proved in [7, Corollary 3.28].

In this context, the logic $\forall\mathcal{C}_3^{\rightarrowtail,\Delta}$ presents a particularly interesting case, since—despite being algebraizable—it lacks a standard Deduction Theorem, and thus the syntactic proof of adequacy had to be carefully adapted to circumvent this issue. This responds directly to the concern raised by one referee regarding the originality and necessity of our approach.

Additionally, we have proved Adequacy for the first-order version of the logic $\text{CŁ}_3^{\rightarrowtail,\Delta}$ by using the homomorphism given in Proposition 3.13, now in Proposition 4.4. The novelty here is to show that it is possible to present a more "syntactic" proof without the necessity of using algebraic properties of the corresponding first-order Lindenbaum-Tarski algebra.

This more syntactic presentation, as opposed to previous algebraic ones such as those in [17], may facilitate future extensions and proof-theoretic analyses in the study of first-order paraconsistent logics.

Another positive outcome is that our presentation can be used for the algebraizable three-valued logics studied in Carnielli and Coniglio's Chapter 4 of the book [5], where the interpretation map for quantified formulas should be given by $||\forall x\alpha||_v^{\mathfrak{S}} = \inf\{||\alpha||_{v[x\to a]}^{\mathfrak{S}} : a \in S\}$ and $||\exists x\alpha||_v^{\mathfrak{S}} = \sup\{||\alpha||_{v[x\to a]}^{\mathfrak{S}} : a \in S\}$. These kinds of interpretations are present in the celebrated Rasiowa's Book ([27]), in the first-order version of fuzzy and Δ-fuzzy logics given in [9, 19, 20], and in D'Ottaviano's work [8].

Finally, we hope that the explicit consideration of the issues raised by the reviewers—concerning the originality of our approach, the differences between propositional and first-order levels, and the role of syntactic techniques—will strengthen the clarity and value of this contribution.

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References

[1] M. Baaz, *Infinite-valued Gödel logics with 0–1-projections and relativizations*, [in:] P. Hájek (ed.), **Gödel '96: Logical Foundations of Mathematics, Computer Science and Physics**, vol. 6 of Lecture Notes in Logic, Association for Symbolic Logic, Berlin (1996), pp. 23–33, DOI: https://doi.org/10.1007/978-3-662-21963-8_2.

[2] J. Berman, W. J. Blok, *Free Łukasiewicz and hoop residuation algebras*, **Studia Logica**, vol. 77(2) (2004), pp. 153–180, DOI: https://doi.org/10.1023/B:STUD.0000037125.49866.50.

[3] J.-Y. Béziau, *From Consequence Operator to Universal Logic: A Survey of General Abstract Logic*, [in:] J.-Y. Béziau (ed.), **Logica Universalis**, Birkhäuser, Basel (2007), pp. 3–17, DOI: https://doi.org/10.1007/978-3-7643-8354-1_1.

[4] W. J. Blok, D. Pigozzi, **Algebraizable Logics**, vol. 77 of Memoirs of the American Mathematical Society, American Mathematical Society, Providence, RI (1989), DOI: https://doi.org/10.1090/memo/0396.

[5] W. Carnielli, M. E. Coniglio, **Paraconsistent Logic: Consistency, Contradiction and Negation**, vol. 40 of Logic, Epistemology, and the

Unity of Science, Springer, Cham (2016), DOI: https://doi.org/10.1007/978-3-319-33205-5.

[6] R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, **Algebraic Foundations of Many-Valued Reasoning**, vol. 7 of Trends in Logic, Springer, Dordrecht (2000), DOI: https://doi.org/10.1007/978-94-015-9480-6.

[7] M. E. Coniglio, A. Figallo-Orellano, A. Hernández-Tello, M. Pérez-Gaspar, *G'3 as the logic of modal 3-valued Heyting algebras*, **IfCoLog Journal of Logics and their Applications**, vol. 9(1) (2022), pp. 175–197.

[8] I. M. L. D'Ottaviano, **Sobre uma teoria de modelos trivalente**, Ph.D. thesis, Universidade Estadual de Campinas, Campinas, Brazil (1982), DOI: https://doi.org/10.47749/t/unicamp.1982.47358.

[9] F. Esteva, L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, **Fuzzy Sets and Systems**, vol. 124(3) (2001), pp. 271–288, DOI: https://doi.org/10.1016/S0165-0114(01)00098-7.

[10] A. V. Figallo, I_3-∇ *algebras*, **Revista Colombiana de Matemáticas**, vol. 17(3–4) (1983), pp. 105–116.

[11] A. V. Figallo, $I\Delta_3$-*algebras*, **Reports on Mathematical Logic**, vol. 24 (1990), pp. 3–16.

[12] A. V. Figallo, M. Figallo, *An algebraic construction of Moisil operators in* $(n+1)$*-valued Łukasiewicz propositional calculus*, **Journal of Multiple-Valued Logic and Soft Computing**, vol. 21(1–2) (2013), pp. 131–145.

[13] A. V. Figallo, A. Figallo-Orellano, M. Figallo, *Super-Łukasiewicz logics expanded by* Δ, **Fuzzy Sets and Systems**, vol. 465 (2023), p. 108549, DOI: https://doi.org/10.1016/j.fss.2023.108549.

[14] A. V. Figallo, A. F. Jr., M. Figallo, A. Ziliani, *Łukasiewicz residuation algebras with infimum*, **Demonstratio Mathematica**, vol. 40(4) (2007), pp. 751–758, DOI: https://doi.org/10.1515/dema-2007-0402.

[15] A. Figallo-Orellano, M. Pérez-Gaspar, J. M. Ramírez-Contreras, *Paraconsistent and paracomplete logics based on k-cyclic modal pseudocomplemented De Morgan algebras*, **Studia Logica**, vol. 110(5) (2022), pp. 1291–1325, DOI: https://doi.org/10.1007/s11225-022-10004-7.

[16] A. Figallo-Orellano, J. S. Slagter, *An algebraic study of the first-order version of some implicational fragments of three-valued Łukasiewicz logic*, **Computación y Sistemas**, vol. 26(2) (2022), pp. 801–813, DOI: https://doi.org/10.13053/cys-26-2-4246.

[17] A. Figallo-Orellano, J. S. Slagter, *Monteiro's algebraic notion of maximal consistent theory for Tarskian logics*, **Fuzzy Sets and Systems**, vol. 445 (2022), pp. 90–122, DOI: https://doi.org/10.1016/j.fss.2022.04.007.

[18] J. M. Font, **Abstract Algebraic Logic: An Introductory Textbook**, College Publications, London (2016).

[19] P. Hájek, **Metamathematics of Fuzzy Logic**, vol. 4 of Trends in Logic, Springer, Dordrecht (1998), DOI: https://doi.org/10.1007/978-94-011-5300-3.

[20] P. Hájek, P. Cintula, *On theories and models in fuzzy predicate logics*, **Journal of Symbolic Logic**, vol. 71(3) (2006), pp. 863–880, DOI: https://doi.org/10.2178/jsl/1154698581.

[21] L. Iturrioz, O. Rueda, *Algebras implicatives trivalentes de Łukasiewicz libres*, **Discrete Mathematics**, vol. 18(1) (1977), pp. 35–44, DOI: https://doi.org/10.1016/0012-365X(77)90004-8.

[22] A. F. Jr., M. Figallo, A. Ziliani, *Free $(n+1)$-valued Łukasiewicz BCK-algebras*, **Demonstratio Mathematica**, vol. 37(2) (2004), pp. 245–254, DOI: https://doi.org/10.1515/dema-2004-0202.

[23] Y. Komori, *The separation theorem of the $\aleph_0$-valued Łukasiewicz propositional logic*, **Reports of the Faculty of Science, Shizuoka University**, vol. 12 (1978), pp. 1–5.

[24] Y. Komori, *Super-Łukasiewicz implicational logics*, **Nagoya Mathematical Journal**, vol. 72 (1978), pp. 127–133, DOI: https://doi.org/10.1017/S0027763000018249.

[25] A. Monteiro, *Algebras implicativas trivalentes de Łukasiewicz* (1968), lecture notes, Universidad Nacional del Sur, Bahía Blanca.

[26] M. Osorio, A. Figallo-Orellano, M. Pérez-Gaspar, *A family of genuine and non-algebraisable C-systems*, **Journal of Applied Non-Classical**

Logics, vol. 31(1) (2021), pp. 56–84, DOI: https://doi.org/10.1080/11663081.2021.1885167.

[27] H. Rasiowa, **An Algebraic Approach to Non-Classical Logics**, vol. 78 of Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam (1974).

[28] R. Wójcicki, **Lectures on Propositional Calculi**, Ossolineum, Wrocław (1984).

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