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RADICAL AND JACOBSON RADICAL IN L-ALGEBRAS

Abstract

In this paper, inspired by the concepts of the radical of an ideal and the Jacobson radical in ring theory, we introduce these notions in the context of L-algebras. Our main goal is to explore the nature of the radical of an ideal and Jacobson radical in specific L-algebras. We successfully identify the elements of these radicals in special classes of L-algebras, including proper Glivenko algebras and finite Brouwerian semi-lattices using a closure operator. To achieve this goal, we introduce a new notion that is associated with any element $a \in L$ as well as with an ideal of an L-algebra L.

Keywords: L-algebra, CKL-algebra, ideal, radical of an ideal, Jacobson radical.

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1. Introduction

The quantum Yang-Baxter equation (QYBE), a fundamental concept in mathematical physics, was formulated independently by Zhenning Yang in 1967 and R.J. Baxter in 1972. The QYBE is connected to mathematical structures including quantum binomial algebras [7, 20], I-type semigroups [8], colorings of plane curves [6], dyeing of bijective cocycles, semimunipolar small triangular Hopf algebras, dynamic systems, and geometric crystals. Numerous initial solutions to the Quantum Yang-Baxter Equation have been found, and these solutions have spurred significant research. Furthermore, the inherent algebraic structures that are linked and related to these QYBE solutions have been the subject of extensive and widespread investigation.

In 2005, W. Rump demonstrated that any set X with a binary operation $\to$ satisfying

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z),$$

corresponds to a solution of the quantum Yang–Baxter equation if left multiplication is bijective [16]. Using this equation, he provided a precise definition for L-algebras and conducted an investigation into their properties. His work included showing that several established algebraic structures can be classified as L-algebras. More specifically, he proved that Hilbert algebras, locales, (left) hoops, (pseudo) MV-algebras, and l-group cones all satisfy the conditions necessary to be considered L-algebras according to his definition [17].

Rump introduced the concept of an ideal in an L-algebra [17]. We continued the work on ideals based on his definition and were able to define concepts such as the radical of an ideal, prime ideal, and primary ideal. We were also able to find the relationship between prime ideals and maximal ideals [14]. Significant results have been achieved in this area, as documented in [2, 4, 9, 10, 12, 15, 17, 22, 24, 25, 26]. For us, the question arose as to which elements in different L-algebras are members of the radical of an ideal. This was the motivation for writing this paper. Additionally, the Jacobson radical plays a crucial role in structural theorems of Ring Theory and Module Theory, such as Nakayama's lemma, and provides insight into the non-semisimple part of a ring. Therefore, we were motivated to extend these foundational ideas to L-algebras, exploring how analogous radicals can be defined and what structural information they reveal in this broader algebraic context.

In the present paper, our primary aim is to delve into the nature of the radical of an ideal and the Jacobson radical within the context of specific L-algebras. Initially, we provide an introduction to the radical of an ideal, outlining its properties. Following this, we define the Jacobson radical. The main thrust of our investigation then becomes pinpointing the elements that belong to these radicals.

In Sections 3 and 4, we introduce special L-algebras such as proper Glivenko algebras and Brouwerian semi-lattices and then find the Jacobson radical members in proper Glivenko algebras and bounded Brouwerian semi-lattices. In the following, considering the ideal I and an element x of an L-algebra, we define the notion of I_x and, based on it, characterize the radical elements of that ideal in finite Brouwerian semi-lattices.

2. Preliminaries

In this section, we recall the known default notions and proven results that are used in other sections.

An algebraic structure $(L; \to, 1)$ of type $(2, 0)$ is called an *L-algebra* if it satisfies these conditions.

(L_1) $x \to x = x \to 1 = 1$ and $1 \to x = x$,

(L_2) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,

(L_3) if $x \to y = y \to x = 1$, then $x = y$,

for any $x, y, z \in L$.

Condition (L_1) states that 1 is a logical unit, and (L_2) is related to the quantum Yang-Baxter equation [8]. The logical unit is always unique. If the binary operation $\rightarrow$ is taken as logical implicative, then there exists a partial order on L defined by

$$x \leqslant y \text{ if and only if } x \rightarrow y = 1. \tag{2.1}$$

You can see the proof in [18].

PROPOSITION 2.1 ([23]). label5 Let $(L; \rightarrow, 1)$ be an L-algebra. Then $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$, for any $x, y, z \in L$.

PROPOSITION 2.2 ([23]). For an L-algebra $(L; \rightarrow, 1)$ and any $x, y, z \in L$, the following are equivalent.

(i) $x \leq y \rightarrow x$,

(ii) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$,

(iii) $((x \rightarrow y) \rightarrow z) \rightarrow z \leq ((x \rightarrow y) \rightarrow z) \rightarrow ((y \rightarrow x) \rightarrow z)$.

DEFINITION 2.3 ([5]). An L-algebra $(L; \rightarrow, 1)$ which satisfies the following condition

$$x \rightarrow (y \rightarrow x) = 1, \qquad (K)$$

for any $x, y \in L$, is called a *KL-algebra*.

Notation. If $(L; \rightarrow, 1)$ is a KL-algebra, then the equivalent statements of Proposition 2.2 hold.

DEFINITION 2.4 ([5]). A CKL-algebra is an L-algebra which satisfies the following condition

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \qquad (C)$$

for any $x, y, z \in L$. In [19] bounded CKL-algebras are known as *Glivenko algebras*.

PROPOSITION 2.5 ([15]). Let $(L; \rightarrow, 1)$ be a CKL-algebra. Then the following properties hold for all $x, y, z \in L$.

(i) $x \leqslant y \rightarrow z$ if and only if $y \leqslant x \rightarrow z$.

(ii) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Note. Every CKL-algebra is a KL-algebra, since for any $x, y \in L$, we have

$$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1. \tag{2.2}$$

Definition 2.6 ([18]). A subset I of an L-algebra $(L; \to, 1)$ is called an *ideal of* L, if it satisfies the following conditions:

(I_1) $1 \in I$,

(I_2) if $x \in I$ and $x \to y \in I$, then $y \in I$,

(I_3) if $x \in I$, then $(x \to y) \to y \in I$,

(I_4) if $x \in I$, then $y \to x \in I$ and $y \to (x \to y) \in I$,

for any $x, y \in L$.

Clearly, $\{1\}$ and L are two trivial ideals of L. An ideal I of L is called a *proper ideal* if $I \neq L$. The set of all ideals of L is denoted by $\mathscr{I}_d(L)$ and the set of all proper ideals of L is denoted by $p\mathscr{I}_d(L)$. It is easy to see that $(\mathscr{I}_d(L), \subseteq)$ and $(p\mathscr{I}_d(L), \subseteq)$ are poset.

An L-algebra L is called simple if $\mathscr{I}_d(L) = \{\{1\}, L\}$.

Proposition 2.7 ([5]). Suppose L is a CKL-algebra and $\emptyset \neq I \subseteq L$. Then $I \in \mathscr{I}_d(L)$ if and only if (I_1) and (I_2) hold.

Let L be an L-algebra and $\emptyset \neq X \subseteq L$. Then the smallest ideal of L containing X is called *the generated ideal by* X and it is denoted by $\langle X \rangle$. Clearly, $\langle X \rangle = \bigcap_{\substack{I \in \mathscr{I}_d(L) \\ X \subseteq I}} I$.

Proposition 2.8 ([14]). Suppose that L is a CKL-algebra, $I \in \mathscr{I}_d(L)$ and $x \in L \setminus I$. Then

$$\langle I \cup \{x\} \rangle = \{a \in L \mid \exists n \in \mathbb{N} \text{ s.t } x \xrightarrow{n} a \in I\}.$$

where

$$x \xrightarrow{0} a = a, x \xrightarrow{1} a = x \to a, \ \cdots, \ x \xrightarrow{n} a = x \to (x \xrightarrow{n-1} a).$$

Note. Assume $(X;\leq)$ is a poset. For every $Y\subseteq X$, we define

$$\uparrow Y=\{x\in X\mid \exists y\in Y\ \text{ s.t }\ y\leq x\} \text{ and } \downarrow Y=\{x\in X\mid \exists y\in Y\ \text{ s.t }\ x\leq y\}.$$

The set Y is called *upset* if $\uparrow Y=Y$ and is called *downset* if $\downarrow Y=Y$. If $y\in X$, then $\downarrow\{y\}$ is denoted by $\downarrow y$.

If $(L;\to,1)$ is an L-algebra and $I\in\mathscr{I}_d(L)$, then I is upset.

Suppose that $I\in p\mathscr{I}_d(L)$. Then I is called a *maximal ideal* of L if it is not included in any other proper ideal of L. The set of all maximal ideals of L is denoted by $\mathscr{M}ax(L)$.

Proposition 2.9 ([14]). Assume $M\in p\mathscr{I}_d(L)$. Then the following statements are equivalent.

(i) $M\in\mathscr{M}ax(L)$.

(ii) $\langle M\cup\{x\}\rangle=L$, for any $x\in L\setminus M$.

Proposition 2.10 ([14]). Let L be a bounded L-algebra and $I\in p\mathscr{I}_d(L)$. Then there exists $M\in\mathscr{M}ax(L)$ such that $I\subseteq M$.

Note. From now on, we assume that $(L;\to,1)$ or simply L is an L-algebra unless otherwise stated.

3. Radical and Jacobson radical in CKL-algebras

This section commences with the definitions of the radical of an ideal and the Jacobson radical in L-algebras. Next, we introduce proper Glivenko algebras and identify the Jacobson radical elements within them. Finally, we conclude with a conjecture regarding the elements of the Jacobson radical in finite L-algebras.

Definition 3.1 ([14]). Let $I\in p\mathscr{I}_d(L)$. Then the intersection of all maximal ideals of L containing I is called a *radical* of I and it is denoted by $\mathcal{R}ad(I)$. So

$$\mathcal{R}ad(I)=\bigcap_{\substack{M\in\mathscr{M}ax(L)\\ I\subseteq M}} M.$$

Clearly, $\mathcal{R}ad(I) \in p\mathscr{I}_d(L)$.
In the following, we recall the notion of a closure operator and show that $\mathcal{R}ad$ is a closure operator.

Definition 3.2 ([1, 3]). Suppose P is an ordered set. A map $C : P \to P$ is called a *closure operator* if it satisfies

(i) $x \leq C(x)$,

(ii) $C(C(x)) = C(x)$,

(iii) $x \leq y$ implies $C(x) \leq C(y)$,

for any $x, y \in P$. An element x of P is called a *closed* if $C(x) = x$.

Proposition 3.3. The map $\mathcal{R}ad$ is a closure operator on $p\mathscr{I}_d(L)$.

Proof: It is clear that $\mathcal{R}ad : p\mathscr{I}_d(L) \longrightarrow p\mathscr{I}_d(L)$ is a map. Let $I, J \in p\mathscr{I}_d(L)$. Clearly, $I \subseteq \mathcal{R}ad(I)$. Furthermore, $\mathcal{R}ad(\mathcal{R}ad(I)) = \mathcal{R}ad(I)$ (See [14]). Suppose $I \subseteq J$ and $t \in \mathcal{R}ad(I)$. Then $t \in M$, for any $I \subseteq M \in \mathscr{M}ax(L)$. Since $I \subseteq J$, any maximal ideal containing J also contains I. Thus, if $J \subseteq S \in \mathscr{M}ax(L)$, then $t \in S$. Hence, $t \in \bigcap_{\substack{S \in \mathscr{M}ax(L)\\ J \subseteq S}} S = \mathcal{R}ad(J)$. So, $\mathcal{R}ad(I) \subseteq \mathcal{R}ad(J)$. □

Definition 3.4. The intersection of all maximal ideals of L is called *Jacobson radical* of L and it is denoted by $\mathcal{J}(L)$. In other words,

$$\mathcal{J}(L) = \bigcap_{M \in \mathscr{M}ax(L)} M.$$

Clearly, $\mathcal{J}(L) = \mathcal{R}ad(\{1\})$. Now, we verify these previously discussed notions when L is a chain.

Proposition 3.5. *If L is a bounded chain, then it has a unique maximal ideal.*

Proof: Since $\{1\} \in p\mathscr{I}_d(L)$, by Proposition 2.10, there exists $M \in \mathscr{M}ax(L)$ such that $\{1\} \subseteq M$. So $\mathscr{M}ax(L)$ is not empty. Now, we show that $p\mathscr{I}_d(L)$ is a chain. To see this, consider $I, J \in p\mathscr{I}_d(L)$ with $J \nsubseteq I$, indicating there exists $y \in J$ which is not in I. For every $x \in I$, since L is a chain, we have either $x \leqslant y$ or $y \leqslant x$. Let $x \leqslant y$. Since I is upset, we get $y \in I$ which is a contradiction. Hence, $y \leqslant x$. Since J is upset, we get $x \in J$. So $I \subseteq J$. Thus, $p\mathscr{I}_d(L)$ is a chain and M is the maximal element of $p\mathscr{I}_d(L)$, implying $\mathscr{M}ax(L) = \{M\}$. □

Corollary 3.6. If L is a bounded chain and $I \in p\mathscr{I}_d(L)$, then $\mathcal{R}ad(I) = \mathcal{J}(L)$.

Proof: The proof is clear by Proposition 3.5. □

Corollary 3.7. If L is a bounded chain and $I \in p\mathscr{I}_d(L)$, then I is closed with respect to the closure operator $\mathcal{R}ad$ if and only if it is a maximal ideal.

Now, we want to identify the elements of $\mathcal{J}(L)$, and the following proposition provides insight into $\mathcal{J}(L)$.

Proposition 3.8. If L is a Glivenko algebra, then

(i) $I \subseteq \{t \in L \mid \exists\ y \leqslant t \text{ such that } (t \xrightarrow{n} y) \to t \in I, \text{ for any } n \in \mathbb{N}\} \subseteq \mathcal{R}ad(I)$.

(ii) If $t, y \in L$ satisfy $y \leqslant t$ and t is an upper bound for the sequence $\{t \xrightarrow{n} y\}_{n \in \mathbb{N}}$, then $t \in \mathcal{J}(L)$.

Proof: (i) Let $A = \{t \in L \mid \exists\ y \leqslant t \text{ such that } (t \xrightarrow{n} y) \to t \in I, \text{ for any } n \in \mathbb{N}\} \nsubseteq \mathcal{R}ad(I)$. Then there exists $t \in A$ such that $t \notin \mathcal{R}ad(I)$. So, there exists $M \in \mathscr{M}ax(L)$ where $I \subseteq M$ and $t \notin M$. By Proposition 2.9, we get $\langle M \cup \{t\} \rangle = L$. Therefore, $y \in \langle M \cup \{t\} \rangle$. Using Proposition 2.8, there exists $n \in \mathbb{N}$ such that $t \xrightarrow{n} y \in M$. By the assumption, $(t \xrightarrow{n} y) \to t \in I \subseteq M$. So by (I_2), we conclude $t \in M$, which is a contradiction. Hence, $t \in \mathcal{R}ad(I)$. Now, we show that $I \subseteq A$. Let $t \in I$. Then $t \leqslant t$ and $(t \xrightarrow{n} t) \to t = t \in I$, for any $n \in \mathbb{N}$. So, $t \in A$ and $I \subseteq A$.
(ii) Assume that $M \in \mathscr{M}ax(L)$. Since t is an upper bound for this sequence, we have $t \xrightarrow{n} y \leqslant t$, for any $n \in \mathbb{N}$. So $(t \xrightarrow{n} y) \to t = 1 \in M$, for each $n \in \mathbb{N}$. By (i), we conclude that $t \in \mathcal{R}ad(M)$. Hence, $t \in M$ for any $M \in \mathscr{M}ax(L)$. Thus, $t \in \mathcal{J}(L)$. □

Definition 3.9. [21] Let L be an L-algebra. An element $x \in L$ is called *dense* if there exists $y \leqslant x$ such that $x \to y = y$. The set of all dense elements of L is denoted by $\mathcal{D}(L)$.

Using Proposition 3.8 and the concept of dense elements, we conclude that in Glivenko algebras, all dense elements belong to the Jacobson radical.

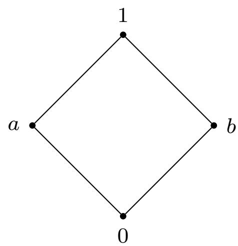
Corollary 3.10. If L is a Glivenko algebra, then $\mathcal{D}(L) \subseteq \mathcal{J}(L)$.

Proof: Consider $t \in \mathcal{D}(L)$. Then there exists $y \in L$ such that $y \leqslant t$ and $t \to y = y$. For every $n \in \mathbb{N}$, we have $t \xrightarrow{n} y = y \leqslant t$, making t is an upper bound for the sequence $\{t \xrightarrow{n} y\}_{n \in \mathbb{N}}$. Applying Proposition 3.8(ii), it follows that $t \in \mathcal{J}(L)$. □

In the following example, we show that the condition (C) in the definition of Glivenko algebra in Corollary 3.10, is necessary.

Example 3.11. Assume $(L = \{0, a, b, 1\}; \leqslant)$ is a poset with the following Hasse diagram. Consider the operation $\to$ on L as follows.

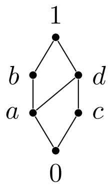
$\to$	0	a	b	1
0	1	1	1	1
a	0	1	0	1
b	0	0	1	1
1	0	a	b	1



Then $(L; \to, 1)$ is an L-algebra. Since $a \nleqslant b \to a = 0$, we conclude that L is not a CKL-algebra. On the other hand, L is a simple L-algebra. Hence, $\mathcal{J}(L) = \{1\}$. Moreover, $\mathcal{D}(L) = \{a, b, 1\}$. Hence, $\mathcal{D}(L) \nsubseteq \mathcal{J}(L)$.

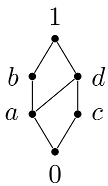
Definition 3.12. A Glivenko algebra is called *proper* if all of its principal ideals, except for $\langle 0 \rangle$, are proper ideals.

Example 3.13. Assume that $(L = \{0, a, b, c, d, 1\}; \leq)$ is a poset with the following Hasse diagram. Consider the operation $\to$ on L as defined below.



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	c	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	0	b	b	c	1	1
1	0	a	b	c	d	1

Then $(L;\rightarrow,1)$ is a CKL-algebra. Moreover, $\mathscr{I}_d(L)=\{\langle 1\rangle=\{1\},\langle b\rangle=\{1,b\},\langle d\rangle=\{1,d\},\langle c\rangle=\{1,c,d\},\langle a\rangle=\{a,b,d,1\},\langle 0\rangle=L\}$. Therefore, L is a proper Glivenko algebra.
(ii) Let $(L=\{0,a,b,c,d,1\};\leq)$ be a poset with the following Hasse diagram. Define the operation $\rightarrow$ on L as follows.



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(L;\rightarrow,1)$ is a CKL-algebra. Since $\langle a\rangle=L$, we conclude that L is not a proper Glivenko algebra.

We are now ready to state the main result of this section, which is the characterization of the Jacobson radical elements in the proper Glivenko algebras.

Theorem 3.14. *If L is a proper Glivenko algebra, then $\mathcal{J}(L)=\mathcal{D}(L)$.*

Proof: By Corollary 3.10, since L is a Glivenko algebra, it follows that $\mathcal{D}(L)\subseteq\mathcal{J}(L)$. Let $t\in\mathcal{J}(L)$ and $s\in L\setminus\{0\}$ such that $t\rightarrow 0=s$. If $\langle s\rangle\neq L$, then by Proposition 2.10, there exists $T\in\mathscr{M}ax(L)$ such that $\langle s\rangle\subseteq T$. Therefore, $s\in T$. Since $t\in\bigcap_{M\in\mathscr{M}ax(L)}M$, we have $t\in T$. By (I_2), we conclude $0\in T$ and this is a contradiction. So, $\langle t\rightarrow 0\rangle=L$. By the hypothesis, all of the principal ideals of L (except for $\langle 0\rangle$) are proper. On the other hand, $\langle s \rangle \notin p\mathscr{I}_d(L)$. Therefore, $s = 0$. Thus, $\mathcal{J}(L) \subseteq \{t \in L \mid t \to 0 = 0\} \subseteq \mathcal{D}(L)$ and $\mathcal{J}(L) = \mathcal{D}(L)$. $\square$

Question. What are the elements of $\mathcal{J}(L)$ in Glivenko algebras?

After examining a large number of examples, we arrived at this conjecture.

Open problem. We conjecture that if L is a Glivenko algebra, then

$$\mathcal{J}(L) = \mathcal{D}(L).$$

4. Radical and Jacobson radical in finite Brouwerian semi-lattices

This section aims to identify the elements of the radical of an ideal in finite Brouwerian semi-lattices. To achieve this, first we introduce the concept of I_x, for every element x and ideal I of L, and then we investigate its properties. We also characterize the Jacobson radical in bounded Brouwerian semi-lattices.

Notation. Let L be an L-algebra, $I \in \mathscr{I}_d(L)$ and $x \in L$. Considering

$$I_x := \{y \in L \mid x \to y \in I\} \quad \text{and} \quad I_L := \{I_x \mid x \in L\}$$

Clearly, $x \in I_x$ and $I_1 = I$.

Example 4.1. Consider Example 3.13. If $I = \{1, d\}$ and $J = \{b, 1\}$, then $I_b = \{a, b, d, 1\} = J_d$ and $J_b = \langle b \rangle = \{b, 1\}$.

Definition 4.2. [17] An L-algebra L is called a *Hilbert algebra* if its binary operation $\to$ is left distributive on itself, that is for any $x, y, z \in L$:

$$x \to (y \to z) = (x \to y) \to (x \to z),$$

Every Hilbert algebra is a CKL-algebra (See [14]).

Proposition 4.3. If L is a Hilbert algebra, $I \in \mathscr{I}_d(L)$ and $a \in L$, then $I_a \in \mathscr{I}_d(L)$.

Proof: Since any Hilbert algebra is a CKL-algebra, it is enough to show that (I_1) and (I_2) hold. As $a \to 1 = 1 \in I$, we conclude $1 \in I_a$. Thus, (I_1) holds. Now, let $x, x \to y \in I_a$. Then $a \to x \in I$ and $a \to (x \to y) \in I$. Since L is a Hilbert algebra, we have $(a \to x) \to (a \to y) \in I$. Hence, $a \to y \in I$ and so, $y \in I_a$. Therefore, (I_2) holds and $I_a \in \mathscr{I}_d(L)$. $\square$

The Hilbert condition in Proposition 4.3 is fundamental. The following example demonstrates that if L is not a Hilbert algebra, then I_a may not necessarily be an ideal.

Example 4.4. Let $L = \{1 = \eta_0, \eta_1, \eta_2, \eta_3, \dots\}$ be a chain such that $\cdots < \eta_3 < \eta_2 < \eta_1 < 1$ and for every $m, n \in \mathbb{N}$,

$$\eta_m \to \eta_n = \begin{cases} 1 & n \leqslant m \\ \eta_{n-m} & \text{otherwise.} \end{cases}$$

Then L is an L-algebra. Moreover,

$$1 = \eta_2 \to \eta_2 = \eta_2 \to (\eta_1 \to \eta_3) \neq (\eta_2 \to \eta_1) \to (\eta_2 \to \eta_3) = 1 \to \eta_1 = \eta_1.$$

So, L is not a Hilbert algebra. Consider the row corresponding to η_1 in the binary operation $\to$.

$\to$	$\cdots$	η_4	η_3	η_2	η_1	1
$\cdots$						
η_4	$\cdots$	1	1	1	1	1
η_3	$\cdots$	η_1	1	1	1	1
η_2	$\cdots$	η_2	η_1	1	1	1
η_1	$\cdots$	η_3	η_2	η_1	1	1
1	$\cdots$	η_4	η_3	η_2	η_1	1

If $I = \{1\}$, then $I_{\eta_1} = \{1, \eta_1\}$. Since $\eta_1 \in I_{\eta_1}$, $\eta_1 \to \eta_2 = \eta_1 \in I_{\eta_1}$ and $\eta_2 \notin I\eta_1$, we conclude I_{η_1} is not an ideal of L.

In the following, we see that the concept of I_x allows us to define a closure operator on $\mathscr{I}_d(L)$.

Remark 4.5. If L is a KL-algebra, $I \in \mathscr{I}_d(L)$ and $x \in L$, then, $I \subseteq I_x$. To see this, take $y \in I$. As $y \leqslant x \to y$, we have $x \to y \in I$. It implies $y \in I_x$.

PROPOSITION 4.6. If L is a Hilbert algebra, $I, J \in \mathscr{I}_d(L)$ and $x \in L$, then

(i) $I \subseteq I_x$.

(ii) $(I_x)_x = I_x$.

(iii) $I \subseteq J$ implies $I_x \subseteq J_x$.

PROOF: (i) Since L is a CKL-algebra and consequently a KL-algebra, applying Remark 4.5, the proof is obvious.
(ii) Let $y \in (I_x)_x$. Then $x \to y \in I_x$. Consequently, $x \to (x \to y) \in I$. Since L is a Hilbert algebra, we have

$$x \to y = 1 \to (x \to y) = (x \to x) \to (x \to y) = x \to (x \to y) \in I.$$

Thus, $y \in I_x$, which implies $(I_x)_x \subseteq I_x$. Additionally, from (i), we know $I_x \subseteq (I_x)_x$. Therefore, $(I_x)_x = I_x$.
(iii) Suppose $I \subseteq J$ and $y \in I_x$. So, $x \to y \in I \subseteq J$ and $y \in J_x$. Hence, $I_x \subseteq J_x$. □

COROLLARY 4.7. If L is a Hilbert algebra, then for any $x \in L$, $-_x : \mathscr{I}_d(L) \to \mathscr{I}_d(L)$ by $I \mapsto I_x$, for any $I \in \mathscr{I}_d(L)$, is a closure operator.

PROPOSITION 4.8. Assume L is a Hilbert algebra, $I \in \mathscr{I}_d(L)$ and $x \in L$. Then I is closed with respect to the closure operator $-_x$ if and only if $x \in I$.

PROOF: Let $x \in I$. We prove that $I_x = I$. By Proposition 4.6, $I \subseteq I_x$. Now, if $y \in I_x$ then $x \to y \in I$, leading to $y \in I$ by (I_2). Thus, $I_x \subseteq I$, which implies $I_x = I$. For the converse, suppose that $I_x = I$. Since $x \in I_x$, we conclude $x \in I$. □

PROPOSITION 4.9. Suppose that L is a Hilbert algebra, $I, J \in \mathscr{I}_d(L)$, and $x, y \in L$. Then

(i) $I \subseteq \langle x \rangle$ if and only if $I_x = \langle x \rangle$.

(ii) $(I \cap J)_x = I_x \cap J_x$.

(iii) If $x \leqslant y$, then $I_y \subseteq I_x$.

PROOF: (i) Let $I \subseteq \langle x \rangle$. As $x \in I_x$ and $\langle x \rangle$ is the smallest ideal of L containing x, we have $\langle x \rangle \subseteq I_x$. Since $I \subseteq \langle x \rangle$, using Propositions 4.6(iii)

and 4.8, we get $I_x \subseteq \langle x \rangle_x = \langle x \rangle$. Hence, $I_x = \langle x \rangle$. For the converse, assume $I_x = \langle x \rangle$ and $y \in I$. Since $I \subseteq I_x$, we have $y \in I_x$. Therefore, $I \subseteq \langle x \rangle$.
(ii) It is obvious.
(iii) Suppose that $t \in I_y$. Then $y \to t \in I$. As $x \leqslant y$, by Proposition 2.2, we conclude $y \to t \leqslant x \to t$. Therefore, $t \in I_x$, implying $I_y \subseteq I_x$. □

In an L-algebra L, for any two elements a and b, we denote the greatest lower bound (if it exists) by $a \wedge b$. Now, we introduce the concept of Brouwerian semi-lattice and then we characterize the elements of $\mathcal{R}ad(I)$, for the ideals of finite Brouwerian semi-lattices.

Definition 4.10. An L-algebra L is called a Brouwerian semi-lattice, if for every $a, b, c \in L$, $a \wedge b$ exists and

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

Example 4.11. (i) Consider the chain $(L = \{0, a, 1\}; \leq)$. Define the binary operation $\to$ on L as follows.

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then $(L; \to, 1)$ is a Brouwerian semi-lattice. But the L-algebra introduced in Example 3.11, is not a Brouwerian semi-lattice. This is because $a \wedge b = 0 \leqslant a$ while $a \nleqslant 0 = b \to a$.
(ii) Let $L = [0, 1]$. Define the operation $\to$ on L as follows

$$x \to y = \begin{cases} 1 & x \leqslant y \\ y & \text{otherwise.} \end{cases}$$

Then $(L; \to, 1)$ is an L-algebra. Since it is a chain, $x \wedge y \in L$ for any $x, y \in L$. Also, it is straightforward to see that L is an infinite Brouwerian semi-lattice.
(iii) Now, we define the operation $\to$ on L as follows

$$x \to y = \begin{cases} 1 & x \leqslant y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

With this operation, L becomes an infinite L-algebra that is not a Brouwerian semi-lattice. For example, we have $0.31 \leqslant 0.32 \to 0.3$, while $0.31 \nleqslant 0.3$.

From [21] and [17], we have the following remark.

Remark 4.12. (i) Every Brouwerian semi-lattice is a KL-algebra. To show this, let L be a Brouwerian semi-lattice and $a, b \in L$. Since $a \wedge b \leqslant a$, we have $a \leqslant b \to a$.
(ii) Every Brouwerian semi-lattice is a Hilbert algebra. To see this, let L be a Brouwerian semi-lattice. Applying [11, Page 105-Rule 1.6], for any $a, b, c \in L$, we have

$$a \to (b \to c) = (a \wedge b) \to c = (a \to b) \to (a \to c).$$

As a result of Proposition 4.3 and Remark 4.12, we have the following corollary.

Corollary 4.13. If L is a Brouwerian semi-lattice, $I \in \mathscr{I}_d(L)$ and $a \in L$, then $I_a \in \mathscr{I}_d(L)$.

Inspired by the notion of an atom in Boolean algebras ([13]), we define the concept of an atom for L-algebras.

Definition 4.14. An element a of a bounded L-algebra L is said to be an atom if $0 < a$ and there does not exist an element $c \in L$ such that $0 < c < a$. We denote the set of all atoms of L by $\mathcal{A}tom(L)$.

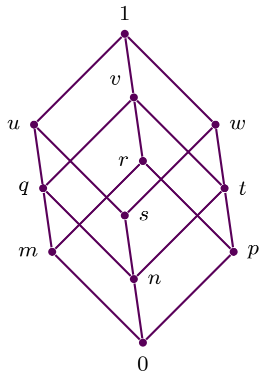
Example 4.15. Consider the chain $(L = \{0, a, 1\}; \leq)$ in Example 4.11(i). Then a is an atom of L.

Proposition 4.16. Let L be a bounded Brouwerian semi-lattice and $a \in \mathcal{A}tom(L)$. Then $\langle a \rangle \in \mathscr{M}ax(L)$.

Proof: Assume that $J \in p\mathscr{I}_d(L)$ and $\langle a \rangle \subsetneq J$. Then there exists $0 \neq b \in J$ such that $b \notin \langle a \rangle$. Since L is a Brouwerian semi-lattice, we have $b \leqslant a \to (a \wedge b)$. Also, as $b \in J$, we get $a \to (a \wedge b) \in J$. Since $a \in \langle a \rangle \subseteq J$, by (I_2), we have $a \wedge b \in J$. On the other hand, $a \wedge b \leqslant a$ and a is an atom of L. Since $b \notin \langle a \rangle$, thus $a \wedge b \neq a$. So $a \wedge b = 0$ and $0 \in J$. Hence, $J = L$ and this is a contradiction. Therefore, $\langle a \rangle = J$ and $\langle a \rangle \in \mathscr{M}ax(L)$. $\square$

In the following example, we show that the condition of Proposition 4.16, is necessary.

Example 4.17. Let $(L = \{0, m, n, p, q, r, s, t, u, v, w, 1\}; \leq)$ be a poset with the following Hasse diagram.



Define the operation $\rightarrow$ on L as follows.

$\rightarrow$	0	m	n	p	q	s	r	t	u	v	w	1
0	1	1	1	1	1	1	1	1	1	1	1	1
m	s	1	s	s	1	s	1	s	1	1	s	1
n	r	r	1	r	1	1	r	1	1	1	1	1
p	s	s	s	1	s	s	1	1	s	1	1	1
q	0	r	s	0	1	s	r	s	1	1	s	1
s	r	r	v	r	v	1	r	v	1	v	1	1
r	s	u	s	w	u	s	1	w	u	1	w	1
t	0	0	s	r	s	s	r	1	s	1	1	1
u	0	r	n	0	v	s	r	n	1	v	s	1
v	0	m	s	p	u	s	r	w	u	1	w	1
w	0	0	n	r	n	s	r	v	s	v	1	1
1	0	m	n	p	q	s	r	t	u	v	w	1

Then $(L; \rightarrow, 1)$ is an L-algebra. On the other hand, since $m \wedge p = 0 \leqslant n$, but $m \nleqslant p \rightarrow n = s$, it follows that L is not a Brouwerian semi-lattice. Notice that $m \in \mathcal{A}tom(L)$, but the ideal generated by it is L and is not a maximal ideal.

THEOREM 4.18. *Suppose that L is a finite Brouwerian semi-lattice and $I, J \in \mathscr{I}_d(L)$. Then*

$$J \in I_L \text{ if and only if } I \subseteq J.$$

PROOF: If L is a finite Brouwerian semi-lattice and $J \in I_L$ then, by Remarks 4.5 and 4.12, it follows that $I \subseteq J$. Conversely, take $I \subseteq J$. If $I = J$, then $J = I_1$. Let $I \subsetneq J$. Obviously, $J \setminus I \neq \emptyset$. Then $J \setminus I$ has the minimum element m. Because otherwise, if there exists an element $t \in J \setminus I$ such that $m \nleqslant t$, then by the proof of Proposition 4.16, we have $m \wedge t \in J$. Since J is finite, this process stops at one point. So, $J \setminus I$ has the minimum element m. If $s \in J$ then $m \leqslant s$. Thus, $m \rightarrow s = 1 \in I$ and $s \in I_m$. Hence, $J \subseteq I_m$. For the converse, if $s \in I_m$ then $m \rightarrow s \in I \subseteq J$ and since $m \in J$, it follows from (I_2) that $s \in J$. Therefore, $I_m \subseteq J$, leading to $I_m = J$. Hence, $J \in I_L$. □

COROLLARY 4.19. If L is a finite Brouwerian semi-lattice, then $I_L = \uparrow I$.

COROLLARY 4.20. If $I \in \mathscr{M}ax(L)$ and $a \in L \setminus I$, then $I_a = L$.

PROOF: We know that $I \subseteq I_a$. Since I is a maximal ideal in L, we have either $I_a = I$ or $I_a = L$. However, since $a \in I_a$ and $a \notin I$, it follows that $I_a \neq I$. Therefore, $I_a = L$. □

COROLLARY 4.21. Let L be a finite Brouwerian semi-lattice, $I \in \mathscr{I}_d(L)$ and $a \in \mathcal{A}tom(L)$ such that $I \subseteq \uparrow a$. Then $I_a \in \mathscr{M}ax(L)$.

PROOF: By Proposition 4.16, we have $\langle a \rangle \in \mathscr{M}ax(L)$. Since $\langle a \rangle \subseteq I_a$, we conclude that $I_a = \langle a \rangle$ or $I_a = L$. If $I_a = L$, then $0 \in I_a$. So, $a \rightarrow 0 \in I$, and by the assumption, we have $a \rightarrow 0 \in \langle a \rangle$. By (I_2), we get $0 \in \langle a \rangle$, which is a contradiction. Hence, $I_a = \langle a \rangle$. □

Now, we are ready to present the main results of this section. Our main results concern finding the elements of radical of an ideal and Jacobson radical in certain finite L-algebras.

THEOREM 4.22. *Let L be a finite Brouwerian semi-lattice and $I \in \mathscr{I}_d(L)$. Furthermore, let*

$$\Lambda = \bigcap_{x \in I} \downarrow x \setminus \{0\} \ , \ \Delta = \{m \in \Lambda \mid m \textit{ is an atom}\}.$$

Then

$$\mathcal{R}ad(I) = \{t \in L \mid m \to t \in I, \textit{ for any } m \in \Delta\}.$$

PROOF: Suppose $m \in \Delta$. Using Proposition 4.9(ii), we have $I \subseteq I_m$. If $m \in \Delta$, then m is a lower bound of I and m is an atom. So, $I \subseteq \uparrow m$ and by Corollary 4.21, we conclude $I_m \in \mathscr{M}ax(L)$. So,

$$\bigcap_{\substack{M \in \mathscr{M}ax(L) \\ I \subseteq M}} M \subseteq \bigcap_{m \in \Delta} I_m.$$

On the other hand, if $I \subseteq M \in \mathscr{M}ax(L)$, then by Theorem 4.18, there exists $m \in L$ such that $M = I_m$. Thus,

$$\bigcap_{m \in \Delta} I_m \subseteq \bigcap_{\substack{M \in \mathscr{M}ax(L) \\ I \subseteq M}} M.$$

Hence,

$$\mathcal{R}ad(I) = \bigcap_{m \in \Delta} I_m = \{t \in L \mid m \to t \in I, \text{ for any } m \in \Delta\}.$$

□

THEOREM 4.23. *If L is a bounded Brouwerian semi-lattice, then $\mathcal{J}(L) = \mathcal{D}(L)$.*

PROOF: By Corollary 3.10, we have $\mathcal{D}(L) \subseteq \mathcal{J}(L)$. If $x \in L$ is not dense, then $x \to 0 > 0$. Hence, $\langle x \to 0 \rangle = \uparrow (x \to 0) \neq L$. By Proposition 2.10, there is a maximal ideal M with $x \to 0 \in M$. Thus $x \notin M$(otherwise $0 \in M$), which yields $x \notin \mathcal{J}(L)$. □

5. Conclusion

In this paper, inspired by the concepts of the radical of an ideal and the Jacobson radical in Ring Theory, we have defined the radical of an ideal and the Jacobson radical in L-algebras. Our primary motivation has been to identify the elements of these radicals. To achieve this, we introduced the notion of I_a by considering an ideal I of an L-algebra L and an element $a \in L$. Additionally, we introduced specific types of L-algebras, such as proper Glivenko algebras, Brouwerian semi-lattices, and Hilbert algebras.

In this study, we successfully identified the elements of the radical of an ideal in finite Brouwerian semi-lattices and the elements of the Jacobson radical in proper Glivenko algebras and bounded Brouwerian semi-lattices. We aim to extend these results to characterize the elements of the radical of an ideal and the Jacobson radical in all L-algebras. Furthermore, we posed an open problem, whose resolution-whether through proof or counterexample-would provide a compelling direction for future research.

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