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ON GENERALIZATION OF MODULAR LATTICES

Abstract

We introduce the concepts of dually balanced lattices and $\mathcal{M}$-lattices and provide some basic properties of these classes of lattices. Both classes can be viewed as generalizations of the well-known class of modular lattices. In particular, we obtain analogues of the Kurosh-Ore theorem for dually balanced lattices and the Jordan-Hölder theorem for $\mathcal{M}$-lattices. Furthermore, we investigate the behaviour of several invariants, including the hollow dimension and the Kurosh-Ore dimension in dually balanced lattices, as well as the maximal dimension in $\mathcal{M}$-lattices.

Keywords: modular lattice, hollow dimension, Kurosh-Ore dimension.

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1. Introduction

The only lattices considered in this paper are finite. In this section we present some notions and their properties concerning lattices which will be used in next sections. For a given lattice L its partial order is denoted by $\leq$ and the join and meet operations are denoted by $\vee$ and $\wedge$, respectively. Moreover, we write 0 and 1 for the least and the greatest elements of L, respectively. For given $a, b \in L$, $a \leqslant b$, we denote by $[a, b]$ the interval $\{x \in L : \ a \leqslant x \leqslant b\}$ in $\mathcal{L}$. Undefined notions and fundamental results on lattices can be found in [2, 6].

The concept of the uniform (Goldie) dimension of a module was introduced by Goldie in [5], while Varadarajan defined the hollow dimension (dual Goldie dimension) of modules in [16]. In [7, 8], Grzeszczuk and Puczyłowski naturally extended this concept to modular lattices, showing that the uniform dimension of a module M is equal to the uniform dimension of the lattice $L(M)$ of submodules of M, and the uniform dimension of the lattice dual to $L(M)$ is equal to the hollow dimension of M. Thus, the uniform dimension of a modular lattice can be regarded as a generalization of both the uniform and hollow dimensions of modules.

In [9], Krempa and Terlikowska-Osłowska extended the notion of the uniform dimension to balanced lattices. A lattice L is called *balanced* if, for all $x, y, z \in L$, the following condition holds:

$$x \wedge y = (x \vee y) \wedge z = 0 \implies (y \vee z) \wedge x = (z \vee x) \wedge y = 0. \tag{1.1}$$

Unlike modular lattices, balanced lattices are not self-dual. As a result, the class of lattices whose dual lattices are balanced differs significantly from the class of balanced lattices. To study the hollow dimension in lattices, the concept of a dually balanced lattice was introduced in [1]. A lattice L is called dually balanced if its dual lattice is balanced. According to (1.1), this means that for all $x, y, z \in L$, the following condition holds:

$$x \vee y = (x \wedge y) \vee z = 1 \implies (y \wedge z) \vee x = (z \wedge x) \vee y = 1. \tag{1.2}$$

In [11], it was shown that the uniform dimension of modular lattices is related to the well-known Kurosh-Ore dimension of lattices and to the length of lattices. A natural question arising from this is whether the uniform and hollow dimensions exhibit similar properties in dually balanced lattices as in the modular case.

In group theory, for a finite group G and a family M of its maximal subgroups, M is called irredundant if the intersection of all its elements is not equal to the intersection of any proper subfamily of M. The maximal dimension of G is the maximal size of an irredundant family of maximal subgroups of G. It turns out that for a group whose subgroup lattice $L(G)$ is dually balanced, the maximal dimension of G is equal to the hollow dimension of $L(G)$. A natural direction for further study is extending this observation to lattices. For this purpose, we introduce the concept of an $\mathcal{M}$-lattice, where for any dual atoms $x, y, z \in L$, condition (1.2) holds.

It should be mentioned that the term balanced lattice was also used by Reuter in [12] to describe a different class of lattices, unrelated to the definition introduced by Krempa and Terlikowska-Osłowska in [9].

This paper is organized as follows. In Section 2, we introduce basic concepts and notation. In Section 3, we study the radical of a lattice, defined as the intersection of all its maximal elements. In Proposition 3.1, we prove that the interval $[0, rad(L)]$ consists of all non-generators of L as an ideal. In Section 4, following [1], we introduce the concept of a dually balanced lattice and describe properties of the hollow dimension in such lattices. We proved in Theorem 4.7 that the hollow dimension can be regarded as the rank of a matroid define on the set of uniform elements. In Section 5, motivated by [11, 17], we extend the Kurosh-Ore dimension to dually balanced lattices. In Theorem 5.7, we prove that for a dually balanced lattice, the uniform dimension is equal to the Kurosh-Ore dimension, and both are equal to the largest size of a minimal generating set of the ideal L. In Section 6, motivated by [3, 4, 15], we introduce the concept of $\mathcal{M}$-lattices, considering only maximal elements of a lattice L. Next we extend the notion of the hollow dimension to $\mathcal{M}$-lattices and studying their basic properties. In Section 7, we present some invariants of lattices with a trivial radical, related to intersections of maximal elements. In Theorem 7.5, we obtain an analog of the Jordan-Hölder theorem: every maximal chain in the subposet of a lattice L, consisting of 1 and all intersections of maximal elements in L, has the same length.

2. Preliminaries

If X is a subset of a lattice L then $(X]$ denotes the ideal of L generated by the set X, that is the smallest ideal of L containing X. We say that X is a *generating set* for $(X]$. A generating set X of L (as an ideal) is said to be *minimal* if no proper subset of X generates an ideal L.

LEMMA 2.1 ([6]). *Let L be a lattice and X be a non-empty subset of L. If $x \in (X]$ then there exist an integer $n \geq 1$ and $x_1, \ldots, x_n \in X$ such that $x \leqslant x_1 \vee \ldots \vee x_n$. In particular, $(\{a\}] = (a] = \{x \in L : x \leqslant a\}$.*

We say that $a \in L$ is a *non-generating element* if $(S \cup \{a\}] = L$ implies $(S] = L$ for each subset $S \subseteq L$.

We say that

- [6] $x \in L$ is a *dual atom* of L if $x < 1$ and there is no element $z \in L$ such that $x < z < 1$,
- [6] $a \in L$ is *irreducible* in L if $a = x \vee y$ implies $a = x$ or $a = y$,
- [7] $a \in L \setminus \{1\}$ is *small in L* if $a \vee x \neq 1$ for every $1 \neq x \in L$,
- [7] $h \in L \setminus \{1\}$ is *hollow in L* if every element from $[h, 1)$ is small in $[h, 1]$.

It is obvious that $0 \in L$ is always small and any dual atom in L is a hollow element. Moreover, observe that $a \in L$ is irreducible in L if and only if 0 is hollow in $[0, a]$. We shall denote by $C(L)$, $S(L)$, $H(L)$, $I(L)$ the set of all dual atoms, small elements, hollow elements and irreducible elements of L respectively.

A lattice L is *modular* if for every $x, y, z \in L$

$$x \leqslant z \implies x \vee (y \wedge z) = (x \vee y) \wedge z. \tag{2.1}$$

It is known (see [2]) that L is a modular lattice if and only if L does not contain a sublattice isomorphic to the pentagon (see the lattice N_5 in Figure 1). In a similar way we can define a balanced lattice. For this purpose we say that a sublattice $K \subseteq L$ is *a 0-sublattice* if it is containing the least element of L.

THEOREM 2.2 ([9]). *A finite lattice L is balanced if and only if L does not contain neither 0-sublattice isomorphic to L_6 nor 0-sublattice isomorphic to L_7 (see Figure 1).*

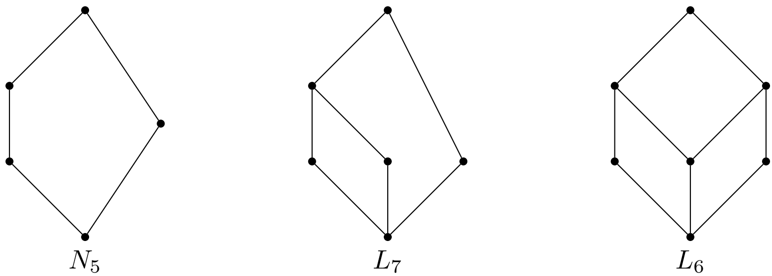

Figure 1: Lattices N_5, L_6, and L_7

We will say that $b \in L$ is a *complement of a in L* if $a \vee b = 1$ and $a \wedge b = 0$. As a consequence, we will say that L is *complemented* if every its element has a complement. Analogously, dually to [9], we will say that b is an *S-complement of a in L* if $a \vee b = 1$ and $a \wedge b$ is small in L. As a consequence, we will say that L is *S-complemented* if every its element has an S-complement. As $0 \in L$ is a small element we obtain

PROPOSITION 2.3. Every complemented lattice is S-complemented.

3. Radical of lattices

In this Section, following [13, 14], we introduce the notion of radical of a lattice, that is the meet of all dual atoms, and study its basic properties. If we consider the radical of lattices of substructures of the same algebraic structures one obtains well known objects like the Jacobson radical or the Frattini subgroup.

Let L be an arbitrary lattice. The *radical* $rad(L)$ of a lattice L is defined as the meet of all dual atoms of L.

Proposition 3.1. The ideal of a lattice L generated by $rad(L)$ is the set of all non-generating elements of L.

Proof: Assume that $a \in L$ is a non-generating element of L and $m \in C(L)$. If $m \vee a = 1$ then, by Lemma 2.1, $(m, a] = L$. Furthermore $(m, a] = (m] = L$, which is in contradiction with the choice of m. It follows that $a \leqslant m$ for all $m \in C(L)$. Hence $a \leqslant rad(L)$. Conversely, assume that $b \leqslant rad(L)$. Let S be a subset of L such that $(S, b] = L$. If $b \notin (S]$, then $\bigvee S \vee b = 1$. This implies that there exists $m \in C(L)$ such that $\bigvee S \leqslant m$ and $b \nleqslant m$, a contradiction with the choice of b. Hence $b \in (S]$ and $(S, b] = (S] = L$. It follows that b is a non-generating element of L. □

Corollary 3.2. Let L be a lattice and $X \subseteq L$. Then X generates L as an ideal if and only if its image in $[rad(L), 1]$ generates $[rad(L), 1]$ as an ideal.

Proposition 3.3. If L is a lattice then the radical of L is equal to the join of all small elements of L.

Proof: Let $x \in L \setminus \{1\}$ and $m \in C(L)$, such that $x \leqslant m$. Then $rad(L) \vee x \leqslant rad(L) \vee m = m \neq 1$. Hence $rad(L)$ is a small element of L and so $rad(L) \leqslant \bigvee S(L)$.

Assume that $s \in S(L)$ and there exists $m \in C(L)$ such that $s \nleqslant m$. Then $s \vee m = 1$, a contradiction. Hence for every $m \in C(L)$, $s \leqslant m$. It follows that $s \leqslant \bigwedge C(L) = rad(L)$. Since s is arbitrary, then $\bigvee S(L) \leqslant rad(L)$. It implies that $rad(L) = \bigvee S(L)$. □

The above proposition implies that $rad(L)$ is a unique small element of $[rad(L), 1]$. Hence we obtain

Proposition 3.4. If L is an S-complemented lattice, then $[rad(L), 1]$ is complemented.

4. Dually balanced lattice and the hollow dimension

The aim of this section is to present some properties of dually balanced lattices and their hollow dimension. Most of the results in this section can be obtained by dualizing the corresponding results from [9] of balanced lattices and the uniform dimension. Therefore, we omit their proofs.

Let $X = \{x_1, \ldots, x_n\} \subset L \setminus \{1\}$. Then we will say that X is :

- *strongly coindependent* if for every disjoint subsets Y, $Z \subset X$ we have

$$\left(\bigwedge Y\right) \vee \left(\bigwedge Z\right) = 1.$$

- *coindependent* if for every $1 \leq i \leq n$ we have

$$x_i \vee \left(\bigwedge_{j \neq i} x_j\right) = 1;$$

- *sequentially coindependent (s-coindependent)* if for every $1 < i \leq n$ we have

$$x_i \vee \left(\bigwedge_{j < i} x_j\right) = 1.$$

THEOREM 4.1. *Let L be a dually balanced lattice and let $X \subset L$ be its finite subset. Then the following conditions are equivalent:*

1. *X is strongly coindependent;*
2. *X is coindependent;*
3. *X is sequentially coindependent.*

A subset $B \subset L$ is called a *cobase* of L if every element of B is hollow and B is a maximal coindependent subset of L.

Proposition 4.2. Let L be a lattice. If $B \subset L$ is a cobase of L, then $\bigwedge B$ is a small element.

Proof: Assume $B \subset L$ is a cobase of L. Since B is maximal coindependent, $\bigwedge B \vee x < 1$ for all $x \in L \setminus \{1\}$. Hence $\bigwedge B$ is a small element. □

Proposition 4.3. Every modular lattice is dually balanced.

Proof: Assume $I = \{1, \ldots, n\}$ and $X = \{x_1, \ldots, x_n\}$ is an s-coindependent subset of L. Then for all $i \in I \setminus \{1\}$, $x_i \vee \left(\bigwedge_{j<i} x_j\right) = 1$. Observe that, by the modular law, we get for all $i \in I$

$$\left(\bigwedge_{j\neq i} x_j\right) \vee x_i = \left(\bigwedge_{j\neq i} x_j\right) \vee x_i \vee \left(\bigwedge_{j\neq n} x_j\right) =$$

$$= \left(\bigwedge_{j\neq i,n} x_j \wedge x_n\right) \vee \left(\bigwedge_{j\neq n} x_j\right) \vee x_i = \left[\bigwedge_{j\neq i,n} x_j \wedge \left(x_n \vee \left(\bigwedge_{j\neq n} x_j\right)\right)\right] \vee x_i =$$

$$= \left(\bigwedge_{j\neq i,n} x_j \wedge 1\right) \vee x_i = \left(\bigwedge_{j\neq i,n} x_j\right) \vee x_i.$$

It follows by induction on n that

$$\left(\bigwedge_{j\neq i,n} x_j\right) \vee x_i = 1,$$

which completes the proof. □

We say that a sublattice $K \subseteq L$ is *a* 1-*sublattice* of L if it is containing the greatest element of L.

Theorem 4.4. *[9] A finite lattice L is dually balanced if and only if L does not contain neither* 1*-sublattice isomorphic to* D_1 *nor* 1*-sublattice isomorphic to* D_2 *(see Figure 2).*

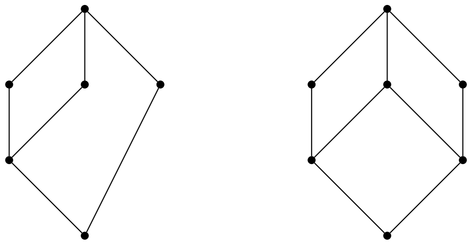

Figure 2: Lattices D_1 and D_2

A sublattice of a dually balanced lattice does not need to be dually balanced. However, by Theorem 4.4 we get

Corollary 4.5. If L is a dually balanced lattice and $a \in L$, then $[a, 1]$ is dually balanced.

Proposition 4.6. Lattices L_1, L_2 are dually balanced if and only if $L_1 \times L_2$ has the same property.

In [11], it was proved that a matroid can be defined on the set of uniform elements of a modular lattice, where the independent sets of this matroid are precisely the independent sets of uniform elements. By the dual version of [9, Lemma 3.4] it implies the following result.

Theorem 4.7. *Let L be a dually balanced lattice and $\mathcal{I}$ be a set of all coindependent subsets of $H(L)$. Then the pair $(H(L), \mathcal{I})$ forms a matroid. In particular, cobases of L are bases of the matroid $(H(L), \mathcal{I})$.*

Thus, for a dually balanced lattice L if B is a cobase of L then the cardinality of B is called the *hollow dimension of* L and is denoted by $h(L)$. Hence, by the above theorem, $h(L)$ is well defined. Moreover, $h(L)$ equals the rank of $(H(L), \mathcal{I})$.

Proposition 4.8.

1. If L_1, L_2 are dually balanced lattices then $h(L_1 \times L_2) = h(L_1) + h(L_2)$.
2. If L is a dually balanced lattice and $a \in L$ then $h([a, 1]) \leq h(L)$.

Proposition 4.9. If L is a dually balanced lattice, then L is S-complemented.

Proof: Assume that L is dually balanced and $a_1 \in L$. Theorem 4.7 implies that any coindependent set of L can be extend to a cobase of L. Hence there exist elements $a_2, \ldots, a_n \in L$ such that $\{a_1, \ldots, a_n\}$ is a cobase of L. Thus $a_1 \vee (a_2 \wedge \ldots \wedge a_n) = 1$ and, by Proposition 4.2, $a_1 \wedge \ldots \wedge a_n$ is small in L. It follows that L is S-complemented. □

5. The Kurosh-Ore dimension

A subset X of a lattice L is called $\vee$-*irredundant* if for every element $x \in X$,

$$x \nleq \bigvee (X \setminus \{x\}).$$

In other words, no element of X is expressible as the join of the remaining elements of X. If an element $a \in L$ can be written as $a = x_1 \vee x_2 \vee \ldots \vee x_m$, where $x_1, x_2, \ldots, x_m \in I(L)$, then $x_1 \vee x_2 \vee \ldots \vee x_m$ is called a $\vee$-*decomposition* of a. A $\vee$-decomposition is said to be $\vee$-*irredundant* if the set $\{x_1, \ldots, x_m\}$ is $\vee$-irredundant.

In view of Lemma 2.1, we obtain the following.

Proposition 5.1. Let L be a lattice and $X \subseteq L$. Then X is a minimal generating set of L if and only if X is a $\vee$-irredundant and $\bigvee X = 1$.

Proof: Assume that $X \subset L$ is a $\vee$-irredundant set and $\bigvee X = 1$. Hence, by Lemma 2.1, $1 \in (X]$ and so $(X] = L$. Since $\bigvee Y < 1$ for every $Y \subset X$, it follows that $(Y] \subset L$. Thus X is a minimal generating set of L.

Now let $X \subset L$ be a minimal generating set of L. Then $(X] = L$ and $(X \setminus \{x\}] \subset L$ for all $x \in X$. Hence, by Lemma 2.1, $\bigvee X = 1$ and $\bigvee (X \setminus \{x\}) < 1$. It follows that X is $\vee$-irredundant. □

Let L be a lattice. The *Kurosh-Ore dimension* $k_\vee(L)$ of L is defined as the maximal size of a $\vee$-irredundant $\vee$-decomposition of 1 in L.

By the above definition and Proposition 5.1 we may conclude:

THEOREM 5.2. *If L is a lattice then $k_\vee(L)$ is equal to the largest size of a minimal generating set of L. In particular, $k_\vee(L) = k_\vee([rad(L), 1])$.*

THEOREM 5.3 ([2, Kurosh-Ore Theorem]).
Let L be a modular lattice. If

$$1 = x_1 \vee \ldots \vee x_n = y_l \vee \ldots \vee y_m$$

are $\vee$-irredundant $\vee$-decompositions of 1 then $n = m$.

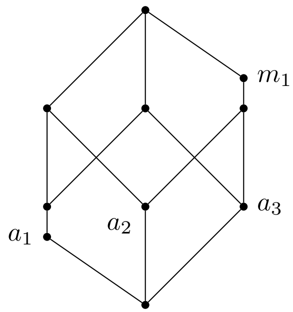

Figure 3: Lattice L that is dually balanced but not modular

Example 5.4. Let L be the lattice presented in Figure 3. The lattice L is dually balanced, but not modular. The sets $\{a_1, a_2, a_3\}$ and $\{a_1, m_1\}$ are $\vee$-irredundant $\vee$-decompositions of 1 in L. Thus the Kurosh-Ore theorem does not hold in L and $k_\vee(L) = 3$. On the other hand the set $\{a_1 \vee a_2, a_1 \vee a_3, a_2 \vee a_3\}$ is a cobase of L. It is easy to verify that $h(L) = 3$.

THEOREM 5.5. *Let L be a dually balanced lattice. If there exists a $\vee$-irredundant $\vee$-decomposition of 1 of size n then there exists an n-element coindependent set of hollow elements in L. In particular, $k_\vee(L) \leq h(L)$.*

PROOF: Assume that $\{x_1, \ldots, x_n\} \subseteq I(L)$, $x_1 \vee \ldots \vee x_n = 1$ and $x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n < 1$ for $1 \leq i \leq n$. Put

$$\check{x}_i = x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n$$

for $1 \leq i \leq n$. If $i \neq j$, then $\check{x}_i > x_j$. Thus

$$\check{x}_j \vee (\bigwedge_{i \neq j} \check{x}_i) \geqslant \check{x}_j \vee (\bigwedge_{i \neq j} x_j) = \check{x}_j \vee x_j = 1,$$

and therefore $\check{x}_j \vee (\bigwedge_{i \neq j} \check{x}_i) = 1$. Hence $\{\check{x}_1, \ldots, \check{x}_n\}$ is a coindependent set of L and it follows that $k_\vee(L) \leq h(L)$. □

THEOREM 5.6. *If L is a dually balanced lattice and $h(L) = n$ then there exists a $\vee$-irredundant $\vee$-decomposition of* 1 *of size n.*

PROOF: Suppose that $h(L) = n$ and $\{x_1, \ldots, x_n\} \subset L \setminus \{1\}$ be a cobase of L. Put

$$\hat{x}_i = x_1 \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_n$$

for $1 \leq i \leq n$. Notice that $x_2 \vee \bar{x}_2 = 1$. We show that $\hat{x}_1 \vee \ldots \vee \hat{x}_n = 1$ and $\hat{x}_1 \vee \ldots \vee \hat{x}_{i-1} \vee \hat{x}_{i+1} \vee \ldots \vee \hat{x}_n < 1$, for $1 \leq i \leq n$. For this purpose we use induction on n.

Assume $a = x_n$, $b = \hat{x}_2 \vee \ldots \vee \hat{x}_n$ i $c = x_2 \wedge \ldots \wedge x_{n-1}$. Then, by induction assumption, we obtain

$$a \vee b = x_n \vee \hat{x}_2 \vee \ldots \vee \hat{x}_n = x_n \vee \hat{x}_n = 1,$$

$$(a \wedge b) \vee c = (x_n \wedge (\bar{x}_2 \vee \ldots \vee \hat{x}_n)) \vee (x_2 \wedge \ldots \wedge x_{n-1}) \geqslant$$

$$(x_n \wedge \hat{x}_2) \vee \ldots \vee (x_n \wedge \hat{x}_n) \vee (x_2 \wedge \ldots \wedge x_{n-1}) =$$

$$\hat{x}_2 \vee \ldots \vee \hat{x}_{n-1} \vee (x_2 \wedge \ldots \wedge x_{n-1}) \geqslant \hat{x}_2 \vee \ldots \vee \hat{x}_{n-1} \vee \hat{x}_1 = 1.$$

Since L is dually balanced,

$$1 = (a \wedge c) \vee b = (x_2 \wedge \ldots \wedge x_{n-1} \wedge x_n) \vee \hat{x}_2 \vee \ldots \vee \hat{x}_n = \hat{x}_1 \vee \ldots \vee \hat{x}_n$$

Suppose that $\hat{x}_1 \vee \ldots \vee \hat{x}_{i-1} \vee \hat{x}_{i+1} \vee \ldots \vee \hat{x}_n = 1$ for some $1 \leq i \leq n$.

Since $\hat{x}_j \leqslant x_i$ for $i \neq j$, we get $\hat{x}_1 \vee \ldots \vee \hat{x}_{i-1} \vee \hat{x}_{i+1} \vee \ldots \vee \hat{x}_n = 1 \leqslant x_i$, what contradict our assumption. Hence for all $1 \leq i \leq n$

$$\hat{x}_1 \vee \ldots \vee \hat{x}_{i-1} \vee \hat{x}_{i+1} \vee \ldots \vee \hat{x}_n < 1.$$

Thus $\{\hat{x}_1, \ldots, \hat{x}_n\}$ is a $\vee$-irredundant set.

We claim that $X = \{\hat{x}_1, \ldots, \hat{x}_n\} \subseteq I(L)$. To prove our claim fix an element $y_1 \in L$, $y_1 \leqslant \hat{x}_1$ such that $\bigvee(X \setminus \{\hat{x}_1\}) \vee y_1 = 1$ and minimal with that property. Assume that y_1 is not irreducible. Then there exist elements $z_1, z_2 < y_1$ such that $y_1 = z_1 \vee z_2$. Since $\bigvee(X \setminus \{\hat{x}_1\}) \vee z_1 \vee z_2 = 1$ and X is a minimal generating set, the set $(X \setminus \{\hat{x}_1\}) \cup \{z_1, z_2\}$ is not $\vee$-irredundant. As for any $j \neq 1$, $\bigvee(X \setminus \{\hat{x}_1, \hat{x}_j\}) \vee z_1 \vee z_2 < \bigvee(X \setminus \{\hat{x}_j\}) < 1$, we obtain $\bigvee(X \setminus \{\hat{x}_1\}) \vee z_j = 1$ for $j = 1$ or $j = 2$, leading to a contradiction with the choice of y_1. This shows that y_1 is irreducible. Obviously, the $\vee$-decomposition $\bigvee((X \setminus \{\hat{x}_1\}) \vee y_1) = 1$ remains $\vee$-irredundant. Similarly, we may replace the elements $\hat{x}_2, \ldots, \hat{x}_n$ by some $y_2, \ldots y_n \in L$ which are $\vee$-irreducible and yield a $\vee$-irredundant $\vee$-decomposition $y_1 \vee \ldots \vee y_n = 1$. □

As a consequence of Proposition 5.1 and Theorems 5.5, 5.6, we obtain the following result.

THEOREM 5.7. *If L is a dually balanced lattice then $h(L) = k_\vee(L)$ is equal to the largest size of the minimal generating set of L.*

6. $\mathcal{M}$-lattice

In this section, motivated by the results of [3, 4, 10, 15] concerning on the maximal dimension of finite groups, we study the class of M-lattices. Furthermore, we adapt the concept of the hollow dimension to this class of lattices.

PROPOSITION 6.1. Let L be a lattice. If $B = \{x_1, \ldots, x_n\} \subseteq L$ is a coindependent set of L then there exist a coindependent set of dual atoms $M = \{m_1, \ldots, m_n\} \subseteq L$ with $x_i \leqslant m_i$ for every $1 \leq i \leq n$.

PROOF: As $x_i \leqslant m_i$ for every $1 \leq i \leq n$, we get

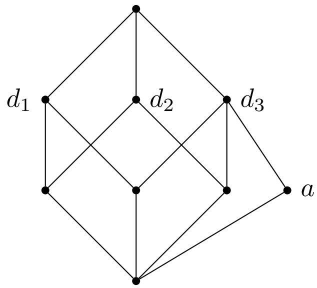

Figure 4: Lattice L that is not dually balanced

$$1 = \left(\bigwedge_{j\neq i} x_j\right) \vee x_i \leqslant \left(\bigwedge_{j\neq i} m_j\right) \vee m_i .$$

Hence M is a coindependent set of L. □

Example 6.2. Let L be a lattice presented in Figure 4. Observe that

$$d_1 \vee d_2 = 1, \quad (d_1 \wedge d_2) \vee a = 1 \quad \text{but}$$

$$(d_2 \wedge a) \vee d_1 = d_1, \quad (a \wedge d_1) \vee d_2 = d_2.$$

It follows that L is not dually balanced. In particular, sets $\{d_1, d_2, d_3\}$ and $\{d_1, a\}$ are cobases of L. Since $\{d_1, d_2, d_3\}$ is the set of all dual atoms of L, it is easy to check that every s-coindependent set of dual atoms of L is coindependent. Moreover, there exists only one cobase $\{d_1, d_2, d_3\}$ that consists only of dual atoms.

A lattice L is called an $\mathcal{M}$*-lattice*, if every sequentially coindependent set of dual atoms of L is coindependent. The example of an $\mathcal{M}$-lattice is a lattice presented in Figure 4.

It is not difficult to see that if L is a lattice and $M = \{m_1, \ldots, m_k\} \subseteq C(L)$ then M is

- s-coindependent if and only if

$$m_1 \wedge m_2 \wedge \ldots \wedge m_n < \ldots < m_1 \wedge m_2 < m_1 \tag{6.1}$$

- coindependent if and only if for every $1 \leq i \leq k$

$$\bigwedge_j m_j < \bigwedge_{i \neq j} m_j. \tag{6.2}$$

Further a subset $B \subset L \setminus \{0\}$ will be called an *m-base of* L if B is a maximal coindependent subset of dual atoms of L and $\bigwedge B = rad(L)$.

Theorem 6.3. *If L is an $\mathcal{M}$-lattice then every maximal coindependent set of dual atoms of L is an m-base of L.*

Proof: Assume that $D \subseteq C(L)$ is a maximal coindependent set of L. The inequality $rad(L) \leqslant \bigwedge D$ is obvious.

We claim that $\bigwedge D \leqslant rad(L)$. If $d \in C(L)$, then $d \leqslant \bigwedge D \vee d \leqslant 1$. Assume $\bigwedge D \vee d = 1$ for some $d \in C(L)$. In this case $D \cup \{d\}$ is an s-coindependent set of L. Since L is an $\mathcal{M}$-lattice, $D \cup \{d\}$ is coindependent, which is in contradiction with the maximality of D. Therefore $(\bigwedge D) \vee d = d$, for every $d \in C(L)$. It implies that $\bigwedge D \leqslant \bigwedge C(L) = rad(L)$. □

Theorem 6.4. *Let L be an $\mathcal{M}$-lattice and B_1, B_2 be m-bases of L. Then for every $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$, such that $\{B_1 \setminus \{x\}\} \cup \{y\}$ is an m-base of L.*

Proof: Let $B_1 = \{b_1, \ldots, b_n\}$ and $B_2 = \{d_1, \ldots, d_k\}$. Clearly $\bigwedge(B_1 \setminus \{b_i\}) > rad(L)$, for all $1 \leq i \leq n$. Suppose that $\bigwedge(B_1 \setminus \{b_i\}) \vee d_j = d_j$, for all $1 \leq j \leq k$. Hence $\bigwedge(B_1 \setminus \{b_i\}) \leqslant d_j$, for all $1 \leq j \leq k$. This implies that $\bigwedge(B_1 \setminus \{b_i\}) \leqslant \bigwedge B_2 = rad(L)$, a contradiction. It follows that there exists $1 \leq j \leq k$ such that $\bigwedge(B_1 \setminus \{b_i\}) \vee d_j = 1$. Put $B = (B_1 \setminus \{b_i\}) \cup \{d_j\}$. We have that B is s-coindependent. Since L is an $\mathcal{M}$-lattice, B is coindependent.

Next we claim that $\bigwedge B = rad(L)$. To prove our claim assume, on the contrary, that $\bigwedge B > rad(L)$. Then there exists $m \in C(L)$ such that $\bigwedge B \vee m = 1$. Hence $B \cup \{m\}$ is s-coindependent. Because L is an $\mathcal{M}$-lattice,

$B \cup \{m\}$ is coindependent, which is in contradiction with the maximality of B_1. Therefore $\bigwedge B = rad(L)$ and B is an m-base. Hence the result follows. □

By the above theorem and the definitions of a matroid in terms of bases (see [18]), we obtain the following result.

Theorem 6.5. *Let L be an $\mathcal{M}$-lattice and $\mathcal{I}$ be a family of all coindependent subsets of $C(L)$. Then the pair $(C(L), \mathcal{I})$ forms a matroid. In particular, m-bases of L are bases of $(C(L), \mathcal{I})$.*

The *maximal dimension* of a lattice L (denoted by $Maxdim(L)$) is the maximal size of a coindependent set of dual atoms of L and the *minimal dimension* of a lattice L (denoted by $Mindim(L)$) is the minimal size of a maximal coindependent set of dual atoms of L. Theorem 6.5 implies the following result.

Corollary 6.6. If L is an $\mathcal{M}$-lattice then

$$Mindim(L) = Maxdim(L).$$

In particular, $Maxdim(L)$ equals the rank of the matroid $(C(L), \mathcal{I})$.

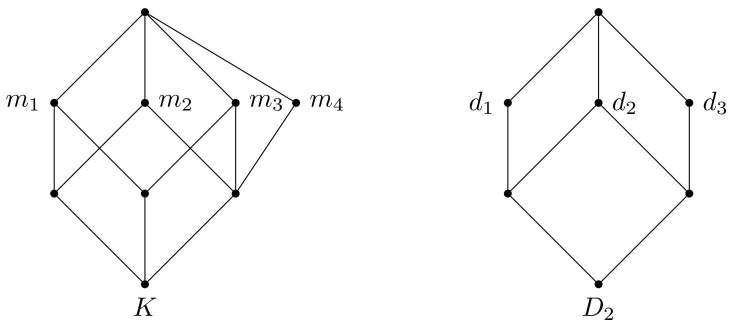

Figure 5: Lattices K and D_2 that are not $\mathcal{M}$-lattices

Example 6.7. The converse theorem to Theorem 6.5 need not hold. The lattice K (see Figure 5) is not an $\mathcal{M}$-lattice. The sets $\{m_1, m_2, m_3\}$ and $\{m_1, m_4\}$ are m-bases of K. So $Mindim(K) = 2$, $Maxdim(K) = 3$. The lattice D_2 (see Figure 5) also is not an $\mathcal{M}$-lattice. All maximal coindependent sets of D_2 are equal to 2, but there exists the maximal coindependent set $\{d_1, d_2\}$ such that $d_1 \wedge d_2 > rad(D_2)$.

Using the definitions (6.1) and (6.1) we are able to formulate the following properties.

LEMMA 6.8. *Let L be a lattice.*

1. *If $x \in L$ and $X \subseteq C(L)$ is coindependent (s-coindependent) in $[x, 1]$, then X is coindependent (s-coindependent) L.*
2. *$X \subseteq C(L)$ is coindependent (s-coindependent) in $[rad(L), 1]$ if and only if X is coindependent (s-coindependent) in L.*

PROOF: First observation is obvious. The second depends on the fact that the radical of lattice is the meet of all duals atoms of L. □

THEOREM 6.9. *If L is an $\mathcal{M}$-lattice then for every $x \in L$ the interval $[x, 1]$ of L is an $\mathcal{M}$-lattice. Moreover, $Maxdim([x, 1]) \le Maxdim(L)$.*

PROOF: The first part of the Theorem is obvious. We claim that $Maxdim([x, 1]) \le Maxdim(L)$. To prove this let $Maxdim([x, 1]) = k$. Then there exists an m-base A of $[x, 1]$ such that $|A| = k$. Then $A \subseteq C(L)$ and A is a coindependent set of L. Since L is an $\mathcal{M}$-lattice, A may be extended to an m-base of L, by Theorem 6.5. Hence $Maxdim([x, 1]) \le Maxdim(L)$. □

COROLLARY 6.10. Let L be a lattice. L is an $\mathcal{M}$-lattice if and only if $[rad(L), 1]$ is an $\mathcal{M}$-lattice. In particular, $Maxdim([rad(L), 1]) = Maxdim(L)$.

PROOF: By Lemma 6.8 the first part of the theorem is clear. Assume that L is an $\mathcal{M}$-lattice and $Maxdim(L) = k$. Then there exists an m-base $B = \{b_1, \ldots, b_k\}$ of L. Hence, by Lemma 6.8, B is a coindependent set in $[rad(L), 1]$. Since $[rad(L), 1]$ is $\mathcal{M}$-lattices, B may be extended to an m-base of $[rad(L), 1]$, by Theorem 6.5. Therefore $Maxdim([rad(L), 1]) \ge$

$Maxdim(L)$. The converse inequality follows from Theorem 6.9 and so $Maxdim([rad(L), 1]) = Maxdim(L)$. □

THEOREM 6.11. *Let L_1 and L_2 be lattices. L_1 and L_2 are $\mathcal{M}$-lattices if and only if $L_1 \times L_2$ is an $\mathcal{M}$-lattice. In particular, if L_1 and L_2 are $\mathcal{M}$-lattices, then*

$$Maxdim(L_1 \times L_2) = Maxdim(L_1) + Maxdim(L_2).$$

PROOF: Let $X_1 = \{x_1, \ldots, x_k\} \subseteq M(L_1)$ and $X_2 = \{y_1, \ldots, y_l\} \subseteq M(L_2)$. Then $X = \{(x_1, 1), \ldots, (x_k, 1), (1, y_1), \ldots, (1, y_l)\} \subseteq M(L_1 \times L_2)$. Observe that X is coindependent (s-coindependent) if and only if X_1, X_2 are coindependent (s-coindependent). It follows that $L_1 \times L_2$ is an $\mathcal{M}$-lattice if and only if L_1, L_2 are $\mathcal{M}$-lattices. Note also that $x \in L_1 \times L_2$ is a dual atom of $L_1 \times L_2$ if and only if $x = (x_1, 1)$ or $x = (1, x_2)$, where x_1 is a dual atom of L_1 and x_2 is a dual atom of L_2. Therefore $Maxdim(L_1 \times L_2) = Maxdim(L_1) + Maxdim(L_2)$. □

PROPOSITION 6.12. Let L be a dually balanced lattice. Then

$$Maxdim(L) = h(L).$$

PROOF: We know that every dually balanced lattice is an $\mathcal{M}$-lattice. The inequality $Maxdim(L) \leq h(L)$ is obvious. Let $h(L) = h$. Then there exists a coindependent set in L of size h. By Proposition 6.1, there exist a coindependent set X of dual atoms in L of size h. Since L is $\mathcal{M}$-lattice, the set X may be extended to an m-base. Hence $Maxdim(L) \geq h(L)$. □

7. Dimensions of $\mathcal{M}$-lattice

A chain between two elements a and b is called a *maximal* if

$$a = a_0 < a_1 < \ldots < a_n = b$$

and $[a_{i-1}, a_i] = \{a_{i-1}, a_i\}$ for every $1 \leq i \leq n$. The number n is called the *length* of the maximal chain.

For a non-trivial lattice L, the *intersection number* of L, denoted $\alpha(L)$, is the minimum number of dual atoms whose intersection equals $rad(L)$.

PROPOSITION 7.1. Let L be a lattice. Then

$$Mindim(L) \leq \alpha(L) \leq Maxdim(L).$$

PROOF: Assume that $\alpha(L) = k$. Then there exists $M = \{m_1, m_2, \ldots, m_k\} \subseteq C(L)$ such that $m_1 \wedge m_2 \wedge \ldots \wedge m_k = rad(L)$. By the minimality of M we obtain that $rad(L) = \bigwedge_j m_j < \bigwedge_{j \neq i} m_j$ for each $1 \leq i \leq k$. Hence M is an m-base of L. It follows that $Mindim(L) \leq \alpha(L) \leq Maxdim(L)$. □

We say that an element m of a lattice L is a *maximal intersection* in L if there exists the set $\{m_1, \ldots, m_k\} \subseteq C(L)$ with $m = m_1 \wedge \ldots \wedge m_k$. We denote by $Max(L)$ the subposet of L consisting of 1 and all the maximal intersections in L. $Max(L)$ does not need to be a sublattice of L.

THEOREM 7.2 ([2],The Jordan-Holder Theorem). *If L is a modular lattice, then any two maximal chains of L are of the same length.*

In our case, in place of arbitrary maximal chains of L, we restrict our attention to the maximal chains in the poset $Max(L)$. The minimal and maximal length of a maximal chain in $Max(L)$ we call $MinInt(L)$ and $MaxInt(L)$ respectively.

LEMMA 7.3. *Let L be a lattice. If*

$$c_k < c_{k-1} < \ldots < c_1 < c_0 = 1$$

is a maximal chain in the poset $Max(L)$, then there exists a maximal s-coindependent set $\{m_1, \ldots, m_k\} \subseteq C(L)$ such that $c_i = m_1 \wedge \ldots \wedge m_i$ and $c_k = rad(L)$.

PROOF: Suppose that

$$C : c_k < c_{t-1} < \ldots < c_1 < c_0 = 1$$

is a maximal chain in $Max(L)$. Hence c_1 is a dual atom of L. Put $c_1 = m_1$. Assume by induction that $c_i = m_1 \wedge \ldots \wedge m_i$, where $m_1, \ldots, m_i \in C(L)$. Since $c_{i+1} \in Max(L)$, there exist $d_1, \ldots, d_l \in C(L)$, such that $c_{i+1} = d_1 \wedge \ldots \wedge d_l$. Then $c_{i+1} \leqslant c_i \wedge d_j \leqslant c_i$ for each $1 \leq j \leq l$. If $c_i = c_i \wedge d_j$ for all $1 \leq j \leq l$, then $c_i \leqslant d_1 \wedge \ldots \wedge d_l = c_{i+1}$, a contradiction. Thus there exists j such that $c_i \wedge d_j = c_{i+1}$. Put $m_{i+1} = d_j$. It follows that $c_{i+1} = m_1 \wedge \ldots \wedge m_i \wedge m_{i+1}$.

Assume that $c_k \neq rad(L)$. In this case there exists $m_{k+1} \in C(L)$ such that $c_k \nleqslant m_{k+1}$. Then $c_k \wedge m_{k+1} < c_k < \ldots < c_0 = 1$ is a chain of length $k+1$. This contradicts the maximality of C. Hence we get $c_k = rad(L)$ and $c_i = m_1 \wedge m_2 \wedge \ldots \wedge m_i$. It follows, by (6.1), that $M = \{m_1, \ldots, m_k\}$ is a maximal s-coindependent set of L. □

THEOREM 7.4. *If L is a lattice, then*

1. $Mindim(L) \leq MinInt(L)$;
2. $Maxdim(L) \leq MaxInt(L)$.

PROOF: Assume that $MinInt(L) = k$ and

$$C : rad(L) = c_k < c_{t-1} < \ldots < c_1 < c_0 = 1$$

is a maximal chain in $Max(L)$. By Lemma 7.3, we have an s-coindependent set $M = \{m_1, \ldots, m_k\}$. If M is not coindependent, then there exist $1 \leq i \leq k$ such that $\bigwedge_{j \neq i} m_j = \bigwedge_j m_j$. By induction we get that $M \setminus \{m_i\}$ is coindependent. Thus $Mindim(L) \leq MinInt(L)$.

Let $Maxdim(L) = k$ and suppose that $\{m_1, \ldots, m_k\}$ is an m-base of L. Then $rad(L) = m_1 \wedge m_2 \wedge \ldots \wedge m_k < \ldots < m_1 \wedge m_2 < m_1 < 1$ is a chain in $Max(L)$ of length k. This yields $Maxdim(L) \leq MaxInt(L)$. □

THEOREM 7.5. *If L is an $\mathcal{M}$-lattice, then*

$$\alpha(L) = Mindim(L) = MinInt(L) = Maxdim(L) = MaxInt(L).$$

PROOF: Assume that $MaxInt(L) = k$ and C is a maximal chain of the length k in $Max(L)$. By Lemma 7.3, there exists an s-coindependent set M of size k. Because L is an $\mathcal{M}$-lattice, M is coindependent. Thus

$MaxInt(L) \leq Maxdim(L)$ what implies that $MaxInt(L) = Maxdim(L)$. Moreover, in an $\mathcal{M}$-lattice, by Corollary 6.6, Proposition 7.1 and Theorem 7.4, we have $Maxdim(L) = \alpha(L) = Mindim(L) \leq MinInt(L) \leq MaxInt(L)$ and the result follows. □

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