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## TOPOLOGY ON EQUALITY ALGEBRAS

### Abstract

In this paper, by special upsets on equality algebras, we construct a topology on bounded equality algebras and investigate some of their topological properties, such as Hausdorff,  $T_0$ -space,  $T_1$ -space and disconnected. In addition, we express the relation between closed and compact sets in this topology. Moreover, by considering the binary operation  $\rightarrow$  and constructing a topology on the bounded equality algebra  $\mathbf{E}$ , we introduce the notion of semi-topological algebra and prove that any involutive equality algebra is a right semi-topological algebra and by some conditions it can be a semi- $\wedge$ -topological algebra. Also, we show that it is not necessarily a left semi-topological algebra. Finally, we investigate converse image, product and quotient topology on equality algebra and show that under what condition we can make finer topology.

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## 1. Introduction

Algebra and topology are indeed fundamental areas of mathematics that complement each other in many ways. Algebra deals with the study of mathematical operations and structures, while topology focuses on properties that are preserved under continuous transformations, such as bending and stretching. Together, these two fields provide a powerful toolkit for understanding and solving mathematical problems. They are used together in the field of mathematics known as algebraic topology. Algebraic topology is a branch of mathematics that uses algebraic tools to study topological spaces. Topological spaces are mathematical structures that capture the notion of closeness or continuity, while algebraic structures provide a way to analyze these spaces using algebraic techniques. By combining algebraic and topological methods, algebraic topology allows mathematicians to study properties of spaces that are not easily accessible through purely topological or algebraic means.

Topology on logical algebraic structures is a fascinating field that explores the relationship between the algebraic properties of structures like lattices, Boolean algebras, etc., and their topological properties. It involves studying concepts like open and closed sets, convergence, continuity, and compactness within the context of these structures.

The motivation for introducing equality algebras came from EQ-algebras which are defined by Novák in [17]. In EQ-algebras, compared to equality algebras, there is an additional operation  $\otimes$ , called product, which is very loosely related to the other operations. Therefore, there might not exist deep algebraic characterizations of EQ-algebras, and intention was to define a structure similar to EQ-algebras but without the product. This new logical algebra, the equality algebra, has two connectives, a meet operation and an equivalence, and a constant. Equality algebra is introduced by Jenei [8]. Since equality algebra can be a good alternative to possible algebraic

semantics for fuzzy type theory, the study of equality algebra is very valuable. Recently, many mathematicians have studied this algebraic structures in various fields. For instance, the relation between equality algebra with other algebraic structure, filters, ideals, annihilators and co-annihilators are investigated that continued in [11, 12, 13, 5, 6, 20, 15, 18, 19].

Lately a lot of mathematicians have been looking into the idea of topology on logical algebraic structures specifically  $BL$ -algebras. For instance Borzooei in [4] looked at how separation axioms are connected to (semi)topological quotient  $BL$ -algebras. They explored when a (semi)topological quotient  $BL$ -algebra can be a  $T_1$ -space or Hausdorff or regular or normal. In [1], the authors introduced the concept of quasi-filter neighborhoods in (semi)topological  $BL$ -algebras and discussed some of their characteristics through statements and proofs. Furthermore by applying the idea of quasi-filter they discovered certain situations in which a  $BL$ -algebra can be turned into a metrizable structure. To learn more about this topic you can check out the articles by [9, 10, 2, 3, 7, 21, 22, 23].

In this paper, we introduce a special subset of equality algebra which is upset and by using these upsets we construct a topology on bounded equality algebras and we investigate some of their topological properties, such as some types of topological space (Hausdorff,  $T_0$ -space and  $T_1$ -space) and disconnected. In addition, we express the relation between closed and compact sets in this topology. Moreover, by considering the binary operation  $\rightarrow$  and the constructed topology on the bounded equality algebra  $\mathbf{E}$ , we introduce the notion of a semi-topological algebra and we prove that any involutive equality algebra is a right semi-topological algebra and by some conditions it can be a semi- $\wedge$ -topological algebra. Also, we show that it is not necessarily a left semi-topological algebra. Finally, we investigate converse image, product and quotient topology on equality algebra and show that under what condition we can make a finer topology.

## 2. Preliminaries

In this section, we have some basic concepts about equality algebra that will be used in later sections. We only remind some definitions and results.

DEFINITION 2.1 ([8]). An algebraic structure  $(\mathbf{E}; \wedge, \sim, 1)$  of type  $(2, 2, 0)$  is called an *equality algebra* if it satisfies the following conditions, for all  $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbf{E}$ ,

(E1)  $(\mathbf{E}, \wedge, 1)$  is a commutative idempotent integral monoid,

(E2)  $\mathfrak{x} \sim \mathfrak{y} = \mathfrak{y} \sim \mathfrak{x}$ ,

(E3)  $\mathfrak{x} \sim \mathfrak{x} = 1$ ,

(E4)  $\mathfrak{x} \sim 1 = \mathfrak{x}$ ,

(E5)  $\mathfrak{x} \preceq \mathfrak{y} \preceq \mathfrak{z}$  implies  $\mathfrak{x} \sim \mathfrak{z} \preceq \mathfrak{y} \sim \mathfrak{z}$  and  $\mathfrak{x} \sim \mathfrak{z} \preceq \mathfrak{x} \sim \mathfrak{y}$ ,

(E6)  $\mathfrak{x} \sim \mathfrak{y} \preceq (\mathfrak{x} \wedge \mathfrak{z}) \sim (\mathfrak{y} \wedge \mathfrak{z})$ ,

(E7)  $\mathfrak{x} \sim \mathfrak{y} \preceq (\mathfrak{x} \sim \mathfrak{z}) \sim (\mathfrak{y} \sim \mathfrak{z})$ .

An equality algebra  $(\mathbf{E}, \preceq)$  with a binary relation defined by “ $\mathfrak{x} \preceq \mathfrak{y}$  if and only if  $\mathfrak{x} \wedge \mathfrak{y} = \mathfrak{x}$ ” is a poset. Also, we define the operation “ $\rightarrow$ ” on  $\mathbf{E}$  as:  $\mathfrak{x} \rightarrow \mathfrak{y} = \mathfrak{x} \sim (\mathfrak{x} \wedge \mathfrak{y})$ .

The set  $\emptyset \neq \mathcal{X} \subseteq \mathbf{E}$  is called a *subequality algebra* if it is closed under all operations on  $\mathbf{E}$ . It means that for any  $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$ ,  $\mathfrak{x} \wedge \mathfrak{y}, \mathfrak{x} \sim \mathfrak{y} \in \mathcal{X}$ .

An equality algebra  $(\mathbf{E}, \sim, \wedge, 0, 1)$  is called a *bounded equality algebra* if for every  $\mathfrak{x} \in \mathbf{E}$ ,  $0 \preceq \mathfrak{x}$ . In a bounded equality algebra  $\mathbf{E}$ , we define the negation operation “ $'$ ” by:  $\mathfrak{x}' = \mathfrak{x} \rightarrow 0 = \mathfrak{x} \sim 0$ , for all  $\mathfrak{x} \in \mathbf{E}$ . If for any  $\mathfrak{x} \in \mathbf{E}$ ,  $\mathfrak{x}'' = \mathfrak{x}$ , then  $\mathbf{E}$  is called an *involutive equality algebra*. (See [8])

**Note:** From now on, let  $(\mathbf{E}; \wedge, \sim, 1)$  or  $\mathbf{E}$  be an equality algebra, unless otherwise stated.

PROPOSITION 2.2 ([8, 24]). The following conditions hold, for any  $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathbf{E}$ :

(i)  $\mathfrak{x} \rightarrow \mathfrak{y} = 1$  if and only if  $\mathfrak{x} \preceq \mathfrak{y}$ ,

(ii)  $1 \rightarrow \mathfrak{x} = \mathfrak{x}$ ,  $\mathfrak{x} \rightarrow 1 = 1$ , and  $\mathfrak{x} \rightarrow \mathfrak{x} = 1$ ,

(iii)  $\mathfrak{x} \preceq \mathfrak{y} \rightarrow \mathfrak{x}$ ,

- (iv)  $\mathfrak{x} \preceq (\mathfrak{x} \rightarrow \eta) \rightarrow \eta$ ,
- (v)  $\mathfrak{x} \rightarrow (\eta \rightarrow \mathfrak{z}) = \eta \rightarrow (\mathfrak{x} \rightarrow \mathfrak{z})$ ,
- (vi)  $\mathfrak{x} \preceq \eta$  implies  $\eta \rightarrow \mathfrak{z} \preceq \mathfrak{x} \rightarrow \mathfrak{z}$  and  $\mathfrak{z} \rightarrow \mathfrak{x} \preceq \mathfrak{z} \rightarrow \eta$ ,
- (vii)  $\mathfrak{x} \rightarrow \eta = \mathfrak{x} \rightarrow (\mathfrak{x} \wedge \eta)$ ,
- (viii) If  $\mathbf{E}$  is a lattice, then  $(\mathfrak{x} \vee \eta) \rightarrow \mathfrak{z} = (\mathfrak{x} \rightarrow \mathfrak{z}) \wedge (\eta \rightarrow \mathfrak{z})$ .
- (ix)  $((\mathfrak{x} \rightarrow \eta) \rightarrow \eta) \rightarrow \eta = \mathfrak{x} \rightarrow \eta$ ,
- (x)  $\mathfrak{x} \sim \eta \preceq \mathfrak{x} \rightarrow \eta$ ,
- (xi)  $\mathfrak{x} \rightarrow \eta \preceq (\mathfrak{x} \wedge \mathfrak{z}) \rightarrow (\eta \wedge \mathfrak{z})$ ,
- (xii)  $\mathfrak{x} \rightarrow \eta \preceq (\mathfrak{z} \rightarrow \mathfrak{x}) \rightarrow (\mathfrak{z} \rightarrow \eta)$  and  $\mathfrak{x} \rightarrow \eta \preceq (\eta \rightarrow \mathfrak{z}) \rightarrow (\mathfrak{x} \rightarrow \mathfrak{z})$ .

DEFINITION 2.3.

- (i) A lattice equality algebra is an equality algebra which is lattice.
- (ii) If for any  $\mathfrak{x}, \eta \in \mathbf{E}$ , 1 is the unique upper bound of the set  $\{\mathfrak{x} \rightarrow \eta, \eta \rightarrow \mathfrak{x}\}$ , then  $\mathbf{E}$  is called *prelinear*. Also, any prelinear equality algebra is a distributive lattice. ([24, Theorem 3.8])

DEFINITION 2.4 ([8]). Let  $\mathbf{F}$  be a non-empty subset of  $\mathbf{E}$ . Then  $\mathbf{F}$  is called a *filter* of  $\mathbf{E}$  if for all  $\mathfrak{x}, \eta \in \mathbf{E}$ , we have

- (i)  $\mathfrak{x} \in \mathbf{F}$  and  $\mathfrak{x} \preceq \eta$  imply  $\eta \in \mathbf{F}$ ,
- (ii)  $\mathfrak{x} \in \mathbf{F}$  and  $\mathfrak{x} \sim \eta \in \mathbf{F}$  imply  $\eta \in \mathbf{F}$ .

The set of all filters of  $\mathbf{E}$  is denoted by  $\mathcal{F}(\mathbf{E})$ .

PROPOSITION 2.5 ([8]). Let  $\emptyset \neq \mathbf{F} \subseteq \mathbf{E}$ . Then  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$  if and only if, for all  $\mathfrak{x}, \eta \in \mathbf{E}$ ,  $1 \in \mathbf{F}$ , and if  $\mathfrak{x} \in \mathbf{F}$  and  $\mathfrak{x} \rightarrow \eta \in \mathbf{F}$ , then  $\eta \in \mathbf{F}$ .

Clearly, for any  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ ,  $1 \in \mathbf{F}$  and  $\mathbf{F}$  is called a *proper filter* of  $\mathbf{E}$  if  $\mathbf{F} \neq \mathbf{E}$ . Clearly, if  $\mathbf{E}$  is bounded, then  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$  is proper if and only if it is not containing 0.

Let  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ . Define the relation  $\theta_{\mathbf{F}}$  on  $\mathbf{E}$  by

$$(\mathfrak{x}, \mathfrak{y}) \in \theta_{\mathbf{F}} \quad \text{if and only if} \quad \{\mathfrak{x} \twoheadrightarrow \mathfrak{y}, \mathfrak{y} \twoheadrightarrow \mathfrak{x}\} \subseteq \mathbf{F}.$$

Let  $\frac{\mathbf{E}}{\mathbf{F}} = \{[\mathfrak{x}] \mid \mathfrak{x} \in \mathbf{E}\}$ , where  $[\mathfrak{x}] = \{\mathfrak{y} \in \mathbf{E} \mid (\mathfrak{x}, \mathfrak{y}) \in \theta_{\mathbf{F}}\}$ . Then define the binary relation  $\preceq_{\mathbf{F}}$  on  $\frac{\mathbf{E}}{\mathbf{F}}$  by:

$$[\mathfrak{x}] \preceq_{\mathbf{F}} [\mathfrak{y}] \quad \text{if and only if} \quad \mathfrak{x} \twoheadrightarrow \mathfrak{y} \in \mathbf{F},$$

which is an order relation on  $\frac{\mathbf{E}}{\mathbf{F}}$ . For any  $\mathfrak{x}, \mathfrak{y} \in \mathbf{E}$ , define

$$[\mathfrak{x}] \sim_{\mathbf{F}} [\mathfrak{y}] = [\mathfrak{x} \sim \mathfrak{y}] \quad \text{and} \quad [\mathfrak{x}] \wedge_{\mathbf{F}} [\mathfrak{y}] = [\mathfrak{x} \wedge \mathfrak{y}].$$

Then  $\left(\frac{\mathbf{E}}{\mathbf{F}}, \sim_{\mathbf{F}}, \wedge_{\mathbf{F}}, 1_{\frac{\mathbf{E}}{\mathbf{F}}}\right)$  is called a *quotient equality algebra* and denoted by  $\frac{\mathbf{E}}{\mathbf{F}}$ , where  $1_{\frac{\mathbf{E}}{\mathbf{F}}} = [1]_{\theta_{\mathbf{F}}} = \mathbf{F}$ .

DEFINITION 2.6 ([16]). Let  $\emptyset \neq \mathbf{A} \subseteq \mathbf{E}$ . The smallest filter of  $\mathbf{E}$  containing  $\mathbf{A}$  is called *the generated filter by  $\mathbf{A}$  in  $\mathbf{E}$*  which is denoted by  $\langle \mathbf{A} \rangle$ . Indeed,  $\langle \mathbf{A} \rangle = \bigcap_{\mathbf{A} \subseteq \mathbf{F} \in \mathcal{F}(\mathbf{E})} \mathbf{F}$ .

PROPOSITION 2.7 ([16]). Let  $\emptyset \neq \mathbf{A} \subseteq \mathbf{E}$ . Then

$$\langle \mathbf{A} \rangle = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{a}_1 \twoheadrightarrow (\mathfrak{a}_2 \twoheadrightarrow (\dots \twoheadrightarrow (\mathfrak{a}_n \twoheadrightarrow \mathfrak{x}) \dots))\} = 1, \\ \text{for some } n \in \mathbb{N} \text{ and } \mathfrak{a}_1, \dots, \mathfrak{a}_n \in \mathbf{A}\}.$$

In particular, for any element  $\mathfrak{a} \in \mathbf{E}$ , we have  $\langle \mathfrak{a} \rangle = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{a} \twoheadrightarrow^n \mathfrak{x} = 1, \text{ for some } n \in \mathbb{N}\}$ , where  $\mathfrak{x} \twoheadrightarrow^0 \mathfrak{y} = \mathfrak{y}$  and  $\mathfrak{x} \twoheadrightarrow^n \mathfrak{y} = \mathfrak{x} \twoheadrightarrow (\mathfrak{x} \twoheadrightarrow^{n-1} \mathfrak{y})$ . If  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$  and  $\mathfrak{a} \in \mathbf{E} \setminus \mathbf{F}$ , then

$$\langle \mathbf{F} \cup \{\mathfrak{a}\} \rangle = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{a} \twoheadrightarrow^n \mathfrak{x} \in \mathbf{F}, \text{ for some } n \in \mathbb{N}\}.$$

If  $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$ , then

$$\begin{aligned} \langle \mathbf{F} \cup \mathbf{G} \rangle &= \{ \mathfrak{x} \in \mathbf{E} \mid \mathfrak{g} \twoheadrightarrow \mathfrak{x} \in \mathbf{F}, \text{ for some } \mathfrak{g} \in \mathbf{G} \} \\ &= \{ \mathfrak{x} \in \mathbf{E} \mid \mathfrak{f} \twoheadrightarrow \mathfrak{x} \in \mathbf{G}, \text{ for some } \mathfrak{f} \in \mathbf{F} \}. \end{aligned}$$

### 3. Topology on equality algebras

In this section, we introduce a special subset of equality algebra which is upset and by using this upsets we construct a topology on bounded equality algebras and investigate their some types of topological space properties, such as Hausdorff,  $T_0$ -space,  $T_1$ -space and disconnected. (For studying more details in topology, we refer the reader to [14].)

**Note:** Let  $(\mathcal{X}; \preceq)$  be a poset. Then for any  $\mathcal{Y} \subseteq \mathcal{X}$ , we define

$$\uparrow \mathcal{Y} = \{ \mathfrak{x} \in \mathcal{X} \mid \exists \mathfrak{y} \in \mathcal{Y} \text{ s.t. } \mathfrak{y} \preceq \mathfrak{x} \}.$$

The set  $\mathcal{Y}$  is called *upset* if  $\uparrow \mathcal{Y} = \mathcal{Y}$ . The set of all upsets of  $\mathcal{X}$  is denoted by  $\Gamma(\mathcal{X})$ .

An upset  $\mathcal{Y}$  is called *finitely generated* if there exists  $n \in \mathbb{N}$  such that  $\mathcal{Y} = \uparrow \{ \mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_n \}$ , for some  $\mathfrak{y}_1, \mathfrak{y}_2, \dots, \mathfrak{y}_n \in \mathcal{Y}$ . Obviously,  $\mathcal{Y} \subseteq \uparrow \mathcal{Y}$ , for any  $\mathcal{Y} \subseteq \mathcal{X}$ .

**PROPOSITION 3.1.** Suppose  $(\mathcal{X}; \preceq)$  is a poset and  $\{ \mathcal{Y}_i \}_{i \in I} \subseteq P(\mathcal{X})$ . Then

- (i) If  $\mathcal{Y}_i \subseteq \mathcal{Y}_j$ , then  $\uparrow \mathcal{Y}_i \subseteq \uparrow \mathcal{Y}_j$ .
- (ii) If  $\{ \mathcal{Y}_i \}_{i \in I} \subseteq \Gamma(\mathcal{X})$ , then  $\bigcup_{i \in I} \mathcal{Y}_i \in \Gamma(\mathcal{X})$ .
- (iii) If  $\{ \mathcal{Y}_i \}_{i \in I} \subseteq \Gamma(\mathcal{X})$ , then  $\bigcap_{i \in I} \mathcal{Y}_i \in \Gamma(\mathcal{X})$ .
- (iv) If  $\mathcal{Y}_i \in \Gamma(\mathcal{X})$ , then  $\mathcal{Y}_i = \bigcup_{\mathfrak{x} \in \mathcal{Y}_i} \uparrow \mathfrak{x}$ .

**PROOF:** (i) The proof is clear.

(ii) Obviously,  $\bigcup_{i \in I} \mathcal{Y}_i \subseteq \uparrow \bigcup_{i \in I} \mathcal{Y}_i$ . Assume  $\mathfrak{x} \in \uparrow \bigcup_{i \in I} \mathcal{Y}_i$ . Then there exists  $\mathfrak{y} \in \bigcup_{i \in I} \mathcal{Y}_i$  such that  $\mathfrak{y} \preceq \mathfrak{x}$ . Thus, there exists  $i \in I$  such that  $\mathfrak{y} \in \mathcal{Y}_i$ . Since

$\mathcal{Y}_i$  is upset, we get  $\mathfrak{r} \in \uparrow \mathcal{Y}_i = \mathcal{Y}_i$ . Hence,  $\mathfrak{r} \in \bigcup_{i \in I} \mathcal{Y}_i$ . Therefore,  $\bigcup_{i \in I} \mathcal{Y}_i \in \Gamma(\mathcal{X})$ .

(iii) The proof is similar to the proof of (ii).

(iv) Let  $\eta \in \mathcal{Y}_i$ . Since  $\eta \preceq \eta$ , we have  $\eta \in \uparrow \eta \subseteq \bigcup_{\mathfrak{r} \in \mathcal{Y}_i} \uparrow \mathfrak{r}$ . Then  $\mathcal{Y}_i \subseteq \bigcup_{\mathfrak{r} \in \mathcal{Y}_i} \uparrow \mathfrak{r}$ .

Furthermore, if  $\mathfrak{t} \in \bigcup_{\mathfrak{r} \in \mathcal{Y}_i} \uparrow \mathfrak{r}$ , then there is  $\mathfrak{r} \in \mathcal{Y}_i$  such that  $\mathfrak{t} \in \uparrow \mathfrak{r}$ . Then  $\mathfrak{r} \preceq \mathfrak{t}$ ,

and so  $\mathfrak{t} \in \uparrow \mathcal{Y}_i$ . Since  $\mathcal{Y}_i \in \Gamma(\mathcal{X})$ , we have  $\mathfrak{t} \in \mathcal{Y}_i$ . Hence,  $\bigcup_{\mathfrak{r} \in \mathcal{Y}_i} \uparrow \mathfrak{r} \subseteq \mathcal{Y}_i$ .

Therefore,  $\mathcal{Y}_i = \bigcup_{\mathfrak{r} \in \mathcal{Y}_i} \uparrow \mathfrak{r}$ . □

**Note:** From now on,  $(\mathbf{E}, \wedge, \sim, 0, 1)$  or  $\mathbf{E}$  for short is a bounded equality algebra, unless otherwise stated.

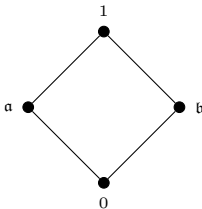
DEFINITION 3.2. Let  $\mathfrak{a} \in \mathbf{E}$  and  $\mathcal{X} \in \Gamma(\mathbf{E})$ . Then, we define

$$\mathcal{DN}_{\mathfrak{a}}(\mathcal{X}) := \{\mathfrak{r} \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ s.t } \mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}\},$$

where

$$\begin{aligned} \mathfrak{a} \xrightarrow{1} \mathfrak{r} &:= (\mathfrak{a} \rightarrow \mathfrak{r})'', \\ \mathfrak{a} \xrightarrow{2} \mathfrak{r} &:= (\mathfrak{a} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{r})'')'', \\ \mathfrak{a} \xrightarrow{n} \mathfrak{r} &:= \underbrace{(\mathfrak{a} \rightarrow \dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots)}_{n\text{-times}}''. \end{aligned}$$

*Example 3.3.* Let  $\mathbf{E} = \{0, \mathfrak{a}, \mathfrak{b}, 1\}$  be a poset with the following Hasse diagram. Define “ $\sim$ ” on  $\mathbf{E}$  as follows:



$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then  $(\mathbf{E}, \wedge, \vee, \sim, 0, 1)$  is a bounded lattice equality algebra and

$$\begin{aligned} \mathcal{DN}_a(\{1\}) &= \mathcal{DN}_a(\{\mathbf{a}, 1\}) \\ &= \{\mathbf{a}, 1\}, \\ \mathcal{DN}_a(\{\mathbf{b}, 1\}) &= \mathcal{DN}_a(\{\mathbf{a}, \mathbf{b}, 1\}) \\ &= \mathcal{DN}_a(\mathbf{E}) \\ &= \{0, \mathbf{a}, \mathbf{b}, 1\}. \end{aligned}$$

PROPOSITION 3.4. Let  $\mathbf{a}, \mathbf{b} \in \mathbf{E}$  and  $\emptyset \neq \mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{X})$ . Then

- (i)  $\mathcal{DN}_0(\mathcal{X}) = \mathbf{E}$  and  $\mathcal{DN}_1(\mathcal{X}) = \{\mathfrak{x} \in \mathbf{E} \mid \mathfrak{x}'' \in \mathcal{X}\}$ . Particularly, if  $\mathbf{E}$  is involutive, then  $\mathcal{DN}_1(\mathcal{X}) = \mathcal{X}$ .
- (ii)  $\mathcal{DN}_a(\emptyset) = \emptyset$  and  $\mathcal{DN}_a(\mathbf{E}) = \mathbf{E}$ .
- (iii)  $\mathbf{a} \in \mathcal{DN}_a(\mathcal{X})$ .
- (iv)  $\mathcal{DN}_a(\mathcal{X}) \in \Gamma(\mathbf{E})$ .
- (v) If  $\mathcal{X} \subseteq \mathcal{Y}$ , then  $\mathcal{DN}_a(\mathcal{X}) \subseteq \mathcal{DN}_a(\mathcal{Y})$ .
- (vi)  $\mathcal{X} \subseteq \mathcal{DN}_a(\mathcal{X})$ .
- (vii) If  $\mathbf{a} \preceq \mathbf{b}$ , then  $\mathcal{DN}_b(\mathcal{X}) \subseteq \mathcal{DN}_a(\mathcal{X})$ . Particularly,  $\mathcal{DN}_{b''}(\mathcal{X}) \subseteq \mathcal{DN}_b(\mathcal{X})$ .
- (viii) If  $\mathbf{E}$  is involutive, then  $\mathcal{DN}_a(\mathcal{X}) = \langle \mathcal{X} \cup \{\mathbf{a}\} \rangle$ . Particularly, if  $\mathbf{a} \in \mathcal{X}$  and  $\mathcal{X} \in \mathcal{F}(\mathbf{E})$ , then  $\mathcal{DN}_a(\mathcal{X}) = \mathcal{X}$ .

PROOF: The proofs of (i) and (ii) are clear.

(iii) Let  $\mathbf{a} \in \mathbf{E}$ . Then  $\mathbf{a} \xrightarrow{1} \mathbf{a} = (\mathbf{a} \rightarrow \mathbf{a})'' = 1'' = 1$ . Since  $\mathcal{X} \in \Gamma(\mathbf{E})$ , we have  $1 \in \mathcal{X}$ , and so  $\mathbf{a} \in \mathcal{DN}_a(\mathcal{X})$ .

(iv) Suppose  $\mathfrak{x} \in \mathcal{DN}_a(\mathcal{X})$  and  $\mathbf{b} \in \mathbf{E}$  such that  $\mathfrak{x} \preceq \mathbf{b}$ . Then there is  $n \in \mathbb{N}$  such that

$$\underbrace{(\mathbf{a} \rightarrow (\dots (\mathbf{a} \rightarrow \mathfrak{x})'' \dots))''}_{n\text{-times}} \in \mathcal{X}.$$

Since  $\mathfrak{r} \preceq \mathfrak{b}$ , by Proposition 2.2(vi),  $\mathfrak{a} \rightarrow \mathfrak{r} \preceq \mathfrak{a} \rightarrow \mathfrak{b}$ . Thus,  $(\mathfrak{a} \rightarrow \mathfrak{r})'' \preceq (\mathfrak{a} \rightarrow \mathfrak{b})''$ . By repeating this method, we have

$$\mathfrak{a} \xrightarrow{n} \mathfrak{r} = \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots))''}_{n\text{-times}} \preceq \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{b})'' \dots))''}_{n\text{-times}} = \mathfrak{a} \xrightarrow{n} \mathfrak{b}.$$

Since  $\mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}$  and  $\mathcal{X} \in \Gamma(\mathbf{E})$ , we conclude  $\mathfrak{a} \xrightarrow{n} \mathfrak{b} \in \mathcal{X}$ . Hence,  $\mathfrak{b} \in \mathcal{DN}_a(\mathcal{X})$ . Therefore,  $\mathcal{DN}_a(\mathcal{X}) \in \Gamma(\mathbf{E})$ .

(v) If  $\mathfrak{r} \in \mathcal{DN}_a(\mathcal{X})$ , then there is  $n \in \mathbb{N}$  such that  $\mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}$ . Since  $\mathcal{X} \subseteq \mathcal{Y}$ , we have  $\mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{Y}$ , and so  $\mathfrak{r} \in \mathcal{DN}_a(\mathcal{Y})$ . Hence,  $\mathcal{DN}_a(\mathcal{X}) \subseteq \mathcal{DN}_a(\mathcal{Y})$ .

(vi) Let  $\mathfrak{r} \in \mathcal{X}$ . By Proposition 2.2(iii) and (iv), we have

$$\mathfrak{r} \preceq \mathfrak{a} \rightarrow \mathfrak{r} \preceq (\mathfrak{a} \rightarrow \mathfrak{r})'' \preceq \dots \preceq \mathfrak{a} \xrightarrow{n} \mathfrak{r}.$$

Since  $\mathfrak{r} \in \mathcal{X}$  and  $\mathcal{X} \in \Gamma(\mathbf{E})$ , we get for any  $n \in \mathbb{N}$ ,  $\mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}$  and so  $\mathfrak{r} \in \mathcal{DN}_a(\mathcal{X})$ . Hence,  $\mathcal{X} \subseteq \mathcal{DN}_a(\mathcal{X})$ .

(vii) If  $\mathfrak{r} \in \mathcal{DN}_b(\mathcal{X})$ , then there is  $n \in \mathbb{N}$  such that  $\mathfrak{b} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}$ . Since  $\mathfrak{a} \preceq \mathfrak{b}$ , by Proposition 2.2(vi), we have  $\mathfrak{b} \rightarrow \mathfrak{r} \preceq \mathfrak{a} \rightarrow \mathfrak{r}$ , and so  $(\mathfrak{b} \rightarrow \mathfrak{r})'' \preceq (\mathfrak{a} \rightarrow \mathfrak{r})''$ . By Proposition 2.2(vi), we have  $\mathfrak{b} \rightarrow (\mathfrak{b} \rightarrow \mathfrak{r})'' \preceq \mathfrak{b} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{r})''$ . Since  $\mathfrak{a} \preceq \mathfrak{b}$ , by Proposition 2.2(vi), we get  $\mathfrak{b} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{r})'' \preceq \mathfrak{a} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{r})''$ . Thus,  $\mathfrak{b} \rightarrow (\mathfrak{b} \rightarrow \mathfrak{r})'' \preceq \mathfrak{a} \rightarrow (\mathfrak{a} \rightarrow \mathfrak{r})''$  and so  $\mathfrak{b} \xrightarrow{2} \mathfrak{r} \preceq \mathfrak{a} \xrightarrow{2} \mathfrak{r}$ . By repeating this method, we have  $\mathfrak{b} \xrightarrow{n} \mathfrak{r} \preceq \mathfrak{a} \xrightarrow{n} \mathfrak{r}$ . Since  $\mathfrak{b} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}$  and  $\mathcal{X} \in \Gamma(\mathbf{E})$ , we get  $\mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}$ . Hence,  $\mathfrak{r} \in \mathcal{DN}_a(\mathcal{X})$ . In addition, by Proposition 2.2(iv),  $\mathfrak{r} \preceq \mathfrak{r}''$ . Then the proof is clear.

(viii) Let  $\mathfrak{r} \in \mathcal{DN}_a(\mathcal{X})$ . Then

$$\mathfrak{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X} \iff \exists n \in \mathbb{N} \text{ such that } (\mathfrak{a} \rightarrow (\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots))'' \dots)'' \in \mathcal{X}.$$

Since  $\mathbf{E}$  is involutive, we get  $(\mathfrak{a} \rightarrow (\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r}) \dots))) \in \mathcal{X}$  and by Proposition 2.7, we have  $\mathfrak{r} \in \langle \mathcal{X} \cup \{\mathfrak{a}\} \rangle$ . The proof of other case is clear.  $\square$

LEMMA 3.5. *The sequence  $\{\mathfrak{r} \xrightarrow{n} \mathfrak{a}\}_{n=1}^\infty$  is an increasing sequence in  $\mathbf{E}$ , for any  $\mathfrak{r} \in \mathbf{E}$ .*

PROOF: By Proposition 2.2(iii),  $\mathbf{a} \preceq \mathfrak{x} \rightarrow \mathbf{a}$ . Then by using two times Proposition 2.2(vi),  $\mathbf{a}'' \preceq (\mathfrak{x} \rightarrow \mathbf{a})''$  and so  $\mathbf{a} \preceq (\mathfrak{x} \rightarrow \mathbf{a})''$ . Now, by Proposition 2.2(vi), we obtain  $\mathfrak{x} \rightarrow \mathbf{a} \preceq \mathfrak{x} \rightarrow (\mathfrak{x} \rightarrow \mathbf{a})''$ , and so  $(\mathfrak{x} \rightarrow \mathbf{a})'' \preceq (\mathfrak{x} \rightarrow (\mathfrak{x} \rightarrow \mathbf{a})'')$ . Then  $\mathfrak{x} \xrightarrow{1} \mathbf{a} \preceq \mathfrak{x} \xrightarrow{2} \mathbf{a}$ . Similarly, we have  $\mathfrak{x} \xrightarrow{2} \mathbf{a} \preceq \mathfrak{x} \xrightarrow{3} \mathbf{a}$ . Therefore,

$$\mathfrak{x} \xrightarrow{1} \mathbf{a} \preceq \mathfrak{x} \xrightarrow{2} \mathbf{a} \preceq \mathfrak{x} \xrightarrow{3} \mathbf{a} \preceq \dots \preceq \mathfrak{x} \xrightarrow{n} \mathbf{a} \preceq \dots \quad \square$$

PROPOSITION 3.6. If  $\{\mathcal{X}_\alpha\}_{\alpha \in I} \subseteq \Gamma(\mathbf{E})$ , then

(i)  $\mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right) = \bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ .

(ii)  $\mathcal{DN}_a \left( \bigcap_{\alpha \in I} \mathcal{X}_\alpha \right) \subseteq \bigcap_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ .

(iii) If  $I$  is finite, then  $\mathcal{DN}_a \left( \bigcap_{\alpha \in I} \mathcal{X}_\alpha \right) = \bigcap_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ .

PROOF: (i) By Propositions 3.1(ii), we have  $\bigcup_{\alpha \in I} \mathcal{X}_\alpha \in \Gamma(\mathbf{E})$ . If  $\mathfrak{x} \in \mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right)$ , then there is  $n \in \mathbb{N}$  such that  $\mathbf{a} \xrightarrow{n} \mathfrak{x} \in \bigcup_{\alpha \in I} \mathcal{X}_\alpha$ . Thus, there exists  $\eta \in I$  such that  $\mathbf{a} \xrightarrow{n} \mathfrak{x} \in \mathcal{X}_\eta$ , and so  $\mathfrak{x} \in \mathcal{DN}_a(\mathcal{X}_\eta)$ . Hence,  $\mathfrak{x} \in \bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ , and so  $\mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right) \subseteq \bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ . Now, since  $\mathcal{X}_\alpha \subseteq \bigcup_{\alpha \in I} \mathcal{X}_\alpha$  for any  $\alpha \in I$ , by Proposition 3.4(v) we get  $\mathcal{DN}_a(\mathcal{X}_\alpha) \subseteq \mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right)$ . So  $\bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha) \subseteq \mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right)$ . Hence,  $\mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right) = \bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ .

(ii) If  $\mathfrak{x} \in \bigcap_{\alpha} \mathcal{DN}_a(\mathcal{X}_\alpha)$ , then  $\mathfrak{x} \in \mathcal{DN}_a(\mathcal{X}_\alpha)$  for any  $\alpha \in I$ . Then there is

$n_\alpha \in \mathbb{N}$  for any  $\alpha \in I$  such that  $\mathbf{a} \xrightarrow{n_\alpha} \mathfrak{r} \in \mathcal{X}_\alpha$ . We put  $t = \max\{n_\alpha\}_{\alpha \in I}$ . By Lemma 3.5,  $\mathbf{a} \xrightarrow{n_\alpha} \mathfrak{r} \preceq \mathbf{a} \xrightarrow{t} \mathfrak{r}$  for any  $\alpha \in I$ . Since  $\mathcal{X}_\alpha \in \Gamma(\mathbf{E})$  for  $\alpha \in I$ , we obtain  $\mathbf{a} \xrightarrow{t} \mathfrak{r} \in \mathcal{X}_\alpha$  for any  $\alpha \in I$ . Hence,  $\mathbf{a} \xrightarrow{t} \mathfrak{r} \in \bigcap_{\alpha \in I} \mathcal{X}_\alpha$ , and so

$$\mathfrak{r} \in \mathcal{DN}_a \left( \bigcap_{\alpha \in I} \mathcal{X}_\alpha \right). \text{ Hence, } \mathcal{DN}_a \left( \bigcap_{\alpha \in I} \mathcal{X}_\alpha \right) \subseteq \bigcap_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha).$$

(iii) If  $|I| < \infty$ , then by Propositions 3.1(iii), we get  $\bigcap_{\alpha \in I} \mathcal{X}_\alpha \in \Gamma(\mathbf{E})$ . We

show  $\bigcap_{\alpha=1}^k \mathcal{DN}_a(\mathcal{X}_\alpha) = \mathcal{DN}_a \left( \bigcap_{\alpha=1}^k \mathcal{X}_\alpha \right)$ . If  $\mathfrak{r} \in \mathcal{DN}_a \left( \bigcap_{\alpha=1}^k \mathcal{X}_\alpha \right)$ , then there is

$n \in \mathbb{N}$  such that  $\mathbf{a} \xrightarrow{n} \mathfrak{r} \in \bigcap_{\alpha=1}^k \mathcal{X}_\alpha$ . Then  $\mathbf{a} \xrightarrow{n} \mathfrak{r} \in \mathcal{X}_\alpha$  for  $1 \preceq \alpha \preceq k$ . Thus,

$\mathfrak{r} \in \mathcal{DN}_a(\mathcal{X}_\alpha)$  for  $1 \preceq \alpha \preceq k$  and so  $\mathfrak{r} \in \bigcap_{\alpha=1}^k \mathcal{DN}_a(\mathcal{X}_\alpha)$ . Conversely, holds by

(ii). Therefore,  $\mathcal{DN}_a \left( \bigcap_{\alpha \in I} \mathcal{X}_\alpha \right) = \bigcap_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ . □

*Remark 3.7.* Let  $\mathbf{a} \in \mathbf{E}$ . Define

$$\tau_a := \{\mathcal{DN}_a(\mathcal{X}) \mid \mathcal{X} \in \Gamma(\mathbf{E})\}.$$

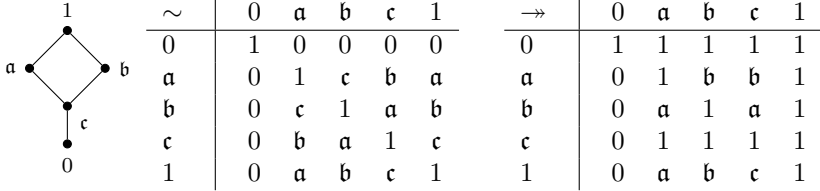
**THEOREM 3.8.** *The structure  $(\mathbf{E}, \tau_a)$  is a topological space.*

**PROOF:** By Proposition 3.4(ii),  $\mathcal{DN}_a(\emptyset) = \emptyset$  and  $\mathcal{DN}_a(\mathbf{E}) = \mathbf{E}$ . Then  $\emptyset, \mathbf{E} \in \tau_a$ . Furthermore, if  $\{\mathcal{DN}_a(\mathcal{X}_\alpha)\}_{\alpha \in I}$  is a family of  $\tau_a$ , then by Propositions 3.6(i), we have  $\mathcal{DN}_a \left( \bigcup_{\alpha \in I} \mathcal{X}_\alpha \right) = \bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ . Therefore,  $\bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha) \in$

$\tau_a$ . Now, if  $\{\mathcal{DN}_a(\mathcal{X}_\alpha)\}_{\alpha \in I}$  is a finite family of  $\tau_a$ , then by Propositions 3.6(iii), we get  $\mathcal{DN}_a \left( \bigcap_{\alpha \in I} \mathcal{X}_\alpha \right) = \bigcap_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha)$ . Hence,  $\bigcap_{\alpha \in I} \mathcal{DN}_a(\mathcal{X}_\alpha) \in \tau_a$ .

Therefore,  $\tau_a$  is a topology on  $\mathbf{E}$ . □

*Example 3.9.* Consider  $\mathbf{E} = \{0, \mathbf{a}, \mathbf{b}, \mathbf{c}, 1\}$  be a poset. Define “ $\sim$ ” on  $\mathbf{E}$  as the following table:



Then  $\mathbf{E} = (\mathbf{E}, \wedge, \sim, 1)$  is a bounded equality algebra. Clearly,

$$\Gamma(\mathbf{E}) = \{\emptyset, \{1\}, \{a, 1\}, \{b, 1\}, \{a, b, 1\}, \{a, b, c, 1\}, \mathbf{E}\}.$$

Then since  $\mathcal{DN}_b(\mathcal{X}) := \{\mathfrak{x} \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ s.t. } \mathfrak{b} \xrightarrow{n} \mathfrak{x} \in \mathcal{X}\}$ , we get that

$$\mathcal{DN}_b(\emptyset) = \emptyset, \mathcal{DN}_b(\mathbf{E}) = \mathbf{E}, \mathcal{DN}_b(\{1\}) = \mathcal{DN}_b(\{b, 1\}) = \{b, 1\},$$

$$\mathcal{DN}_b(\{a, 1\}) = \mathcal{DN}_b(\{a, b, c, 1\}) = \mathcal{DN}_b(\{a, b, 1\}) = \{a, b, c, 1\}.$$

Hence,  $\tau_b = \{\emptyset, \{b, 1\}, \{a, b, c, 1\}, \mathbf{E}\}$ .

*Remark 3.10.* By Proposition 3.4(ii),  $\mathcal{DN}_0(\emptyset) = \emptyset$ . Also, for any  $\emptyset \neq \mathcal{X} \in \Gamma(\mathbf{E})$ , we have  $\mathcal{DN}_0(\mathcal{X}) = \mathbf{E}$ . Hence,  $\tau_0 = \{\mathcal{DN}_0(\mathcal{X}) \mid \mathcal{X} \in \Gamma(\mathbf{E})\} = \{\emptyset, \mathbf{E}\}$ .

**THEOREM 3.11.** *Let  $|\mathbf{E}| \geq 2$ . If  $0 \neq a \in \mathbf{E}$ , then  $\tau_a$  is not a discrete topology on  $\mathbf{E}$ .*

**PROOF:** We know that if  $\tau_a$  is a discrete topology on  $\mathbf{E}$ , then  $\tau_a = \mathcal{P}(\mathbf{E})$ . Also, clearly,  $\tau_a \subseteq \mathcal{P}(\mathbf{E})$ . In addition, we have some subsets of  $\mathbf{E}$  that not containing 1. On the other hand, for any  $\emptyset \neq \mathcal{U} \in \tau_a$ ,  $\mathcal{U} \in \Gamma(\mathbf{E})$  and so  $1 \in \mathcal{U}$ . Hence,  $\mathcal{P}(\mathbf{E}) \not\subseteq \tau_a$ . Therefore,  $\tau_a$  is not a discrete topology on  $\mathbf{E}$ .  $\square$

**PROPOSITION 3.12.** Let  $a \in \mathbf{E}$ . If

$$\beta_a := \{\mathcal{DN}_a(\uparrow \mathfrak{x}) \mid \mathfrak{x} \in \mathbf{E}\} \cup \{\emptyset\},$$

then  $\beta_a$  is a base for  $\tau_a$ .

**PROOF:** Let  $\mathfrak{x} \in \mathbf{E}$ . Since  $\uparrow(\uparrow \mathfrak{x}) = \uparrow \mathfrak{x}$ , we have  $\mathcal{DN}_a(\uparrow \mathfrak{x}) \in \tau_a$ . Then  $\beta_a$  is a set of open subsets of  $\mathbf{E}$ . Furthermore, if  $\emptyset \neq \mathcal{U} \in \tau_a$ , then there is  $\mathcal{X} \in \Gamma(\mathbf{E})$  such that  $\mathcal{U} = \mathcal{DN}_a(\mathcal{X})$ . By Propositions 3.1(iv) and 3.6(i) we

have

$$\mathcal{U} = \mathcal{DN}_a(\mathcal{X}) = \mathcal{DN}_a\left(\bigcup_{\mathfrak{r} \in \mathcal{X}} \uparrow \mathfrak{r}\right) = \bigcup_{\mathfrak{r} \in \mathcal{X}} \mathcal{DN}_a(\uparrow \mathfrak{r}).$$

Therefore,  $\beta$  is a basis for  $\tau_a$ . □

**PROPOSITION 3.13.** If  $\mathfrak{a} \preceq \mathfrak{b}$ , then  $\tau_{\mathfrak{b}}$  is finer than  $\tau_a$ , for  $\mathfrak{a}, \mathfrak{b} \in \mathbf{E}$ .

**PROOF:** Let  $\mathfrak{t} \in \mathbf{E}$  and  $\mathfrak{t} \in \mathbf{B} \in \beta_a$ . Then  $\mathbf{B} = \mathcal{DN}_a(\uparrow \mathfrak{r})$  for some  $\mathfrak{r} \in \mathbf{E}$ . Since  $\mathfrak{t} \in \uparrow \mathfrak{t}$  and  $\uparrow \mathfrak{t} \subseteq \mathcal{DN}_b(\uparrow \mathfrak{t})$ , we have  $\mathfrak{t} \in \mathcal{DN}_b(\uparrow \mathfrak{t})$ . Also, since  $\mathfrak{a} \preceq \mathfrak{b}$ , by Proposition 3.4(vii)  $\mathcal{DN}_b(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_a(\uparrow \mathfrak{t})$ . Now, we show  $\mathcal{DN}_a(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_a(\uparrow \mathfrak{r})$ . Suppose  $\mathfrak{r} \in \mathcal{DN}_a(\uparrow \mathfrak{t})$ . Then there is  $m \in \mathbb{N}$  such that  $\mathfrak{a} \xrightarrow{m} \mathfrak{r} \in \uparrow \mathfrak{t}$ . Then

$$\mathfrak{t} \preceq \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots))''}_{m\text{-times}}.$$

Since  $\mathfrak{t} \in B$ , we have  $\mathfrak{t} \in \mathcal{DN}_a(\uparrow \mathfrak{r})$ . Then there is  $n \in \mathbb{N}$  such that  $\mathfrak{a} \xrightarrow{n} \mathfrak{t} \in \uparrow \mathfrak{r}$ . Thus,

$$\mathfrak{r} \preceq \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{t})'' \dots))''}_{n\text{-times}}.$$

Since  $\mathfrak{t} \preceq \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots))''}_{m\text{-times}}$ , by Proposition 2.2(vi), we have

$$\mathfrak{a} \rightarrow \mathfrak{t} \preceq \mathfrak{a} \rightarrow \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots))''}_{m \text{ times}}.$$

By assumption, and by Proposition 2.2(vi),

$$(\mathfrak{a} \rightarrow \mathfrak{t})'' \preceq \mathfrak{a} \rightarrow \underbrace{(\mathfrak{a} \rightarrow (\dots (\mathfrak{a} \rightarrow \mathfrak{r})'' \dots))''}_{m\text{-times}}.$$

So,  $\mathfrak{a} \xrightarrow{1} \mathfrak{t} \preceq \mathfrak{a} \xrightarrow{m+1} \mathfrak{r}$ . Similarly, we conclude  $\mathfrak{a} \xrightarrow{n} \mathfrak{t} \preceq \mathfrak{a} \xrightarrow{m+n} \mathfrak{r}$ , and so  $\mathfrak{a} \xrightarrow{m+n} \mathfrak{r} \in \uparrow \mathfrak{r}$ . Thus,  $\mathfrak{r} \in \mathcal{DN}_a(\uparrow \mathfrak{r})$ , and so  $\mathcal{DN}_a(\uparrow \mathfrak{t}) \subseteq \mathcal{DN}_a(\uparrow \mathfrak{r})$ . Since

$\mathcal{DN}_b(\uparrow t) \subseteq \mathcal{DN}_a(\uparrow t)$ , we have  $\mathcal{DN}_b(\uparrow t) \subseteq \mathcal{DN}_a(\uparrow t)$ . Finally, it is enough to set  $\mathbf{B}' = \mathcal{DN}_b(\uparrow t)$ . Then  $\mathbf{B}' \in \beta_b$  and  $t \in \mathbf{B}' \subseteq \mathbf{B}$ . Thus,  $\tau_b$  is finer than  $\tau_a$ .  $\square$

PROPOSITION 3.14. The singleton  $\{\mathbf{a}\}$  is dense in the topological space  $(\mathbf{E}, \tau_a)$ , where  $\mathbf{a} \in \mathbf{E}$ .

PROOF: We prove  $\overline{\{\mathbf{a}\}} = \mathbf{E}$ . Let  $\mathbf{B}$  be a closed subset in  $\mathbf{E}$  and  $\{\mathbf{a}\} \subseteq \mathbf{B}$ . Then  $\mathbf{B}^c$  is open, and so there is  $\mathcal{X} \in \Gamma(\mathbf{E})$  such that  $\mathbf{B}^c = \mathcal{DN}_a(\mathcal{X})$ . We know that, if  $\mathcal{DN}_a(\mathcal{X}) \neq \emptyset$ , then  $\mathbf{a} \in \mathcal{DN}_a(\mathcal{X})$ . Thus,  $\mathbf{a} \in \mathbf{B}^c$  a contradiction. Hence,  $\mathcal{DN}_a(\mathcal{X}) = \emptyset$ , and so  $\mathbf{B}^c = \emptyset$ . Therefore,  $\mathbf{B} = \mathbf{E}$  i.e.  $\overline{\{\mathbf{a}\}} = \bigcap_{\{\mathbf{a}\} \subseteq \mathbf{B} = \overline{\mathbf{B}}} \mathbf{B} = \mathbf{E}$ .  $\square$

PROPOSITION 3.15. The topological space  $(\mathcal{X}, \tau)$  is a  $T_0$  space if and only if  $\overline{\{\mathbf{x}\}} \neq \overline{\{\mathbf{y}\}}$  for any distinct pair  $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ .

PROOF: Let  $(\mathcal{X}, \tau)$  be  $T_0$  space and  $\mathbf{x} \neq \mathbf{y}$ . Consider  $\overline{\{\mathbf{x}\}} = \overline{\{\mathbf{y}\}}$  and  $\mathbf{x} \in \mathcal{U} \in \tau$ . Then  $\mathcal{U}^c$  is closed and  $\mathbf{x} \notin \mathcal{U}^c$ . If  $\mathbf{y} \in \mathcal{U}^c$ , then

$$\mathbf{x} \in \overline{\{\mathbf{x}\}} = \overline{\{\mathbf{y}\}} = \bigcap_{\eta \in \mathcal{C}} \mathcal{C} \subseteq \mathcal{U}^c \quad \text{where } \mathcal{C} \text{ is closed subset of } \mathcal{X}.$$

Then  $\mathbf{x} \in \mathcal{U}^c$ , and so  $\mathbf{x} \notin \mathcal{U}$ , a contradiction. Hence,  $\mathbf{y} \notin \mathcal{U}^c$  and  $\mathbf{y} \in \mathcal{U}$ . Therefore,  $\overline{\{\mathbf{x}\}} \neq \overline{\{\mathbf{y}\}}$ . Conversely, consider  $\overline{\{\mathbf{x}\}} \neq \overline{\{\mathbf{y}\}}$  for any  $\mathbf{x}, \mathbf{y} \in \mathcal{X}$  such that  $\mathbf{x} \neq \mathbf{y}$ . Then

$$\mathbf{y} \in \overline{\{\mathbf{y}\}} \neq \overline{\{\mathbf{x}\}} = \bigcap_{\mathbf{x} \in \beta = \overline{\beta}} \beta.$$

Thus,  $\mathbf{y} \notin \bigcap_{\mathbf{x} \in \beta = \overline{\beta}} \beta$ . So, there exists a closed subset  $C$  such that  $\mathbf{x} \in C$  and  $\mathbf{y} \notin C$ . Now, set  $V := C^c$ . Then,  $V \in \tau$  such that  $\mathbf{y} \in V$  and  $\mathbf{x} \notin V$ . Hence,  $(\mathcal{X}, \tau)$  is  $T_0$  space.  $\square$

THEOREM 3.16. For any  $\mathbf{a} \in \mathbf{E}$ ,

- (i)  $(\mathbf{E}, \tau_a)$  is not Hausdorff.
- (ii)  $(\mathbf{E}, \tau_a)$  is not  $T_0$ .
- (iii)  $(\mathbf{E}, \tau_a)$  is not  $T_1$ .

(iv) *The topological space  $(\mathbf{E}, \tau_{\mathbf{a}})$  is connected.*

(v) *The topological space  $(\mathbf{E}, \tau_{\mathbf{a}})$  is irreducible.*

PROOF: (i) The proof is clear.

(ii) Let  $\mathbf{a} \in \mathbf{E}$ . By Proposition 3.14,  $\{\mathbf{a}\}$  is dense. Thus,  $\overline{\{\mathbf{a}\}} = \overline{\{1\}} = \mathbf{E}$  and by Proposition 3.15,  $(\mathbf{E}, \tau_{\mathbf{a}})$  is not a  $T_0$  space.

(iii) By (ii), it is clear.

(iv) For any  $\mathcal{U} \in \tau_{\mathbf{a}}$ , we have  $1 \in \mathcal{U}$  and so  $(\mathbf{E}, \tau_{\mathbf{a}})$  is a connected space.

(v) Suppose  $\mathbf{E} = \mathbf{B}_1 \cup \mathbf{B}_2$  such that  $\mathbf{B}_1$  and  $\mathbf{B}_2$  are proper closed subsets. Since  $\mathbf{B}_1^c \in \tau$ , there exists  $\mathcal{X}_1 \in \Gamma(\mathbf{E})$  such that  $\mathbf{B}_1^c = \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_1)$ . Similarly, there is  $\mathcal{X}_2 \in \Gamma(\mathbf{E})$  such that  $\mathbf{B}_2^c = \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_2)$ . In addition, from  $\mathbf{a} \in \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_1)$  and  $\mathbf{a} \in \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_2)$  we obtain,  $\mathbf{a} \notin \mathbf{B}_1$  and  $\mathbf{a} \notin \mathbf{B}_2$ . Hence,  $\mathbf{a} \notin \mathbf{B}_1 \cup \mathbf{B}_2$ , and so  $\mathbf{a} \notin \mathbf{E}$ , a contradiction. Therefore,  $\mathbf{E}$  is an irreducible topological space.  $\square$

**THEOREM 3.17.** *If  $\emptyset \neq \mathcal{X} \subseteq \mathbf{E}$  is finitely generated, then  $\mathcal{X}$  is compact in  $(\mathbf{E}, \tau_{\mathbf{a}})$ , for any  $\mathbf{a} \in \mathbf{E}$ .*

PROOF: Suppose  $\mathcal{X} = \uparrow \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}$ , for some  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n \in \mathbf{E}$ . In addition, let  $\mathcal{C} = \{\mathcal{DN}_{\mathbf{a}}(\mathcal{X}_i)\}_{i \in I}$  be an open cover for  $\mathcal{X}$ . Then  $\mathcal{X} = \bigcup_{i \in I} \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_i)$ . For every  $1 \preccurlyeq j \preccurlyeq n$ , there is  $i_j \in I$  such that  $\mathbf{r}_j \in \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_j})$ .

Then  $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\} \subseteq \bigcup_n \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_1}) \cup \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_2}) \cup \dots \cup \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_n})$ . Thus,

$\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\} \subseteq \bigcup_{j=1}^n \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_j})$ . If  $\boldsymbol{\eta} \in \mathcal{X} = \uparrow \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}$ , then there exists

$1 \preccurlyeq \mathbf{t} \preccurlyeq n$  such that  $\mathbf{r}_{\mathbf{t}} \preccurlyeq \boldsymbol{\eta}$ . Since  $\mathbf{r}_{\mathbf{t}} \in \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_{\mathbf{t}}})$ , we get  $\boldsymbol{\eta} \in \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_j})$ .

Thus,  $\mathcal{X} \subseteq \bigcup_{j=1}^n \mathcal{DN}_{\mathbf{a}}(\mathcal{X}_{i_j})$ , and so  $\mathcal{C}$  has a finite subcover for  $\mathcal{X}$ . Therefore,

$\mathcal{X}$  is compact.  $\square$

**THEOREM 3.18.** *If  $\mathcal{X} \subseteq \mathbf{E}$  is closed, then  $\mathcal{X}$  is compact in the topological space  $(\mathbf{E}, \tau_{\mathbf{a}})$ , where  $\mathbf{a} \in \mathbf{E}$ .*

PROOF: Let  $\mathcal{X} \subseteq \mathbf{E}$  is closed. Then  $\mathcal{X}^c$  is open. In this case, either  $\mathcal{X}^c = \mathbf{E}$  or  $\mathcal{X}^c \neq \mathbf{E}$ . If  $\mathcal{X}^c = \mathbf{E}$ , then  $\mathcal{X} = \emptyset$ , and so  $\emptyset$  is compact. If  $\mathcal{X}^c \neq \mathbf{E}$ , since  $\mathcal{X}^c$  is upset, we get  $0 \notin \mathcal{X}^c$ , and so  $0 \in \mathcal{X}$ . Let  $\mathcal{C} = \{\mathcal{DN}_{\mathbf{a}}(\mathcal{V}_{\alpha})\}_{\alpha \in I}$  be an

open cover for  $\mathcal{X}$ . Then  $0 \in \bigcup_{\alpha \in I} \mathcal{DN}_a(\mathcal{Y}_\alpha)$ . Thus, there is  $\alpha \in I$  such that  $0 \in \mathcal{DN}_a(\mathcal{Y}_\alpha)$ . Since  $\mathcal{DN}_a(\mathcal{Y}_\alpha)$  is upset  $\mathcal{DN}_a(\mathcal{Y}_\alpha) = \mathbf{E}$ . Hence,  $\{\mathbf{E}\}$  is a finite sub-cover for  $\mathcal{X}$ . Therefore,  $\mathcal{X}$  is compact.  $\square$

The following example shows that the converse of the above theorem is not true.

*Example 3.19.* (i) According to Example 3.9,  $\mathcal{X} = \{\mathbf{b}, 1\}$  is finite and so it is compact but  $\mathcal{X}^c = \{0, \mathbf{a}, \mathbf{c}\} \notin \tau_b$ . Hence,  $\mathcal{X}$  is not closed.

(ii) According to Example 3.3,  $\tau_a = \{\emptyset, \{\mathbf{a}, 1\}, \mathbf{E}\}$ . Clearly,  $\mathcal{X} = \{\mathbf{a}, 1\}$  is finite and so it is compact but  $\mathcal{X}^c = \{0, \mathbf{b}\} \notin \tau_a$ . Hence,  $\mathcal{X}$  is not closed.

**THEOREM 3.20.** *If  $\mathbf{E}$  is an involutive equality algebra, then  $(\mathbf{E}, \rightarrow, \tau_a)$  is a right semi-topological algebra, where  $\mathbf{a} \in \mathbf{E}$*

**PROOF:** We have to show that  $\rightarrow_b: \mathbf{E} \rightarrow \mathbf{E}$  defined by  $\mathfrak{x} \mapsto (\mathbf{b} \rightarrow \mathfrak{x})$  is continuous. It means for any  $\mathbf{B} \in \beta_a, (\rightarrow_b)^{-1}(\mathbf{B}) \in \tau_a$ . Let  $\mathfrak{x} \in \mathbf{E}$  such that  $\mathbf{B} := \mathcal{DN}_a(\uparrow \mathfrak{x}) \in \beta_a$ . We prove  $\rightarrow_b^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x})) \in \tau_a$ . Thus, we have

$$\begin{aligned} (\rightarrow_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x})) &= \{\eta \in \mathbf{E} \mid \rightarrow_b(\eta) \in \mathcal{DN}_a(\uparrow \mathfrak{x})\} \\ &= \{\eta \in \mathbf{E} \mid (\mathbf{b} \rightarrow \eta) \in \mathcal{DN}_a(\uparrow \mathfrak{x})\} \\ &= \{\eta \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ such that } \mathbf{a} \xrightarrow{n} (\mathbf{b} \rightarrow \eta) \in \uparrow \mathfrak{x}\} \\ &= \{\eta \in \mathbf{E} \mid \exists n \in \mathbb{N} \text{ such that } \mathfrak{x} \preceq \mathbf{a} \xrightarrow{n} (\mathbf{b} \rightarrow \eta)\}. \end{aligned} \tag{3.1}$$

First, take  $\mathcal{X} := (\rightarrow_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x}))$  and we show that  $\mathcal{X} \in \Gamma(\mathbf{E})$ . Let  $\eta \preceq \mathfrak{z}$  and  $\eta \in \mathcal{X}$ . Thus by Proposition 2.2(vi) and since  $\mathbf{E}$  is involutive, we have  $\mathbf{b} \rightarrow \eta \preceq \mathbf{b} \rightarrow \mathfrak{z}$ . Then  $\mathbf{a} \rightarrow (\mathbf{b} \rightarrow \eta) \preceq \mathbf{a} \rightarrow (\mathbf{b} \rightarrow \mathfrak{z})$ , and so

$$(\mathbf{a} \rightarrow (\mathbf{b} \rightarrow \eta))'' \preceq (\mathbf{a} \rightarrow (\mathbf{b} \rightarrow \mathfrak{z}))''.$$

By repeating this method, for any  $n \in \mathbb{N}$  we have  $\mathbf{a} \xrightarrow{n} (\mathbf{b} \rightarrow \eta) \preceq \mathbf{a} \xrightarrow{n} (\mathbf{b} \rightarrow \mathfrak{z})$ . Since  $\mathfrak{x} \preceq \mathbf{a} \xrightarrow{n} (\mathbf{b} \rightarrow \eta)$  we get  $\mathfrak{x} \preceq \mathbf{a} \xrightarrow{n} (\mathbf{b} \rightarrow \mathfrak{z})$ . Thus,  $\mathfrak{z} \in \mathcal{X}$ , and so  $\mathcal{X} \in \Gamma(\mathbf{E})$ . We claim that  $\mathcal{DN}_a(\mathcal{X}) = \mathcal{X}$ . By Proposition 3.4(vi), we have  $\mathcal{X} \subseteq \mathcal{DN}_a(\mathcal{X})$ . For other side of inclusion, consider  $\eta \in \mathcal{DN}_a(\mathcal{X})$ . Then there exists  $m \in \mathbb{N}$  such that  $\mathbf{a} \xrightarrow{m} \eta \in \mathcal{X} = (\rightarrow_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x}))$ . Thus there

is  $k \in \mathbb{N}$  such that  $\mathfrak{x} \preceq \mathfrak{a} \xrightarrow{k} (\mathfrak{b} \rightarrow (\mathfrak{a} \xrightarrow{m} \eta))$ . By Proposition 2.2(v) and since  $\mathbf{E}$  is involutive we get,

$$\mathfrak{x} \preceq \mathfrak{b} \rightarrow (\mathfrak{a} \xrightarrow{k} (\mathfrak{a} \xrightarrow{m} \eta)) = \mathfrak{b} \rightarrow (\mathfrak{a} \xrightarrow{m+k} \eta) = \mathfrak{a} \xrightarrow{m+k} (\mathfrak{b} \rightarrow \eta).$$

By (3.1),  $\eta \in \mathcal{X}$  and so  $\mathcal{DN}_a(\mathcal{X}) \subseteq \mathcal{X}$ . Hence,  $\mathcal{X} = \mathcal{DN}_a(\mathcal{X}) \in \tau_a$ . Therefore,  $(\mathbf{E}, \rightarrow, \tau_a)$  is a right semi-topological algebra.  $\square$

*Remark 3.21.* Clearly,  $\rightarrow_b: \mathbf{E} \rightarrow \mathbf{E}$  defined by  $\mathfrak{x} \mapsto (\mathfrak{x} \rightarrow \mathfrak{b})$  is not continuous, since by Proposition 2.2(vi),  $(\rightarrow_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x}))$  is not upset necessarily.

**Open Problem:** Is there any example of a finite equality algebra such as  $\mathbf{E}$  which  $\tau_a$  is not a left semi-topology on  $\mathbf{E}$ ?

**THEOREM 3.22.** *If  $\mathbf{E}$  satisfies the following condition*

$$\mathfrak{x} \rightarrow (\eta \wedge \mathfrak{z}) = (\mathfrak{x} \rightarrow \eta) \wedge (\mathfrak{x} \rightarrow \mathfrak{z}), \quad \text{for any } \mathfrak{x}, \eta, \mathfrak{z} \in \mathbf{E}$$

*then  $(\mathbf{E}, \wedge, \tau_a)$  is a semi-topological algebra, where  $\mathfrak{a} \in \mathbf{E}$ .*

**PROOF:** We have to show that  $\wedge_b: \mathbf{E} \rightarrow \mathbf{E}$  defined by  $\mathfrak{x} \mapsto (\mathfrak{b} \wedge \mathfrak{x})$  is continuous. For this, we show that the inverse image of every element of  $\beta_a$  is an open subset in  $\tau_a$ , i.e., for any  $\mathbf{B} \in \beta_a$ ,  $(\wedge_b)^{-1}(\mathbf{B}) \in \tau_a$ . Let  $\mathfrak{x} \in \mathbf{E}$  such that  $\mathbf{B} := \mathcal{DN}_a(\uparrow \mathfrak{x}) \in \beta_a$ . We prove  $\wedge_b^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x})) \in \tau_a$ . Thus, we have

$$\begin{aligned} (\wedge_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x})) &= \{\eta \in \mathbf{E} \mid \wedge_b(\eta) \in \mathcal{DN}_a(\uparrow \mathfrak{x})\} \\ &= \{\eta \in \mathbf{E} \mid (\mathfrak{b} \wedge \eta) \in \mathcal{DN}_a(\uparrow \mathfrak{x})\} \\ &= \{\eta \in \mathbf{E} \mid \text{there exists } n \in \mathbb{N} \text{ such that } \mathfrak{a} \xrightarrow{n} (\mathfrak{b} \wedge \eta) \in \uparrow \mathfrak{x}\} \\ &= \{\eta \in \mathbf{E} \mid \text{there exists } n \in \mathbb{N} \text{ such that } \mathfrak{x} \preceq \mathfrak{a} \xrightarrow{n} (\mathfrak{b} \wedge \eta)\}. \end{aligned} \tag{3.2}$$

First, take  $\mathcal{X} := (\wedge_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{x}))$  and we show that  $\mathcal{X} \in \Gamma(\mathbf{E})$ . Let  $\eta \preceq \mathfrak{z}$  and  $\eta \in \mathcal{X}$ . Thus  $\mathfrak{b} \wedge \eta \preceq \mathfrak{b} \wedge \mathfrak{z}$  and so  $\mathfrak{a} \rightarrow (\mathfrak{b} \wedge \eta) \preceq \mathfrak{a} \rightarrow (\mathfrak{b} \wedge \mathfrak{z})$ . Hence  $(\mathfrak{a} \rightarrow (\mathfrak{b} \wedge \eta))'' \preceq (\mathfrak{a} \rightarrow (\mathfrak{b} \wedge \mathfrak{z}))''$ . By repeating this method, for any  $n \in \mathbb{N}$  we have  $\mathfrak{a} \xrightarrow{n} (\mathfrak{b} \wedge \eta) \preceq \mathfrak{a} \xrightarrow{n} (\mathfrak{b} \wedge \mathfrak{z})$ . Since  $\mathfrak{x} \preceq \mathfrak{a} \xrightarrow{n} (\mathfrak{b} \wedge \eta)$  we get  $\mathfrak{x} \preceq \mathfrak{a} \xrightarrow{n} (\mathfrak{b} \wedge \mathfrak{z})$ . Thus,  $\mathfrak{z} \in \mathcal{X}$ , and so  $\mathcal{X} \in \Gamma(\mathbf{E})$ . We claim that

$\mathcal{DN}_a(\mathcal{X}) = \mathcal{X}$ . By Proposition 3.4(vi), we have  $\mathcal{X} \subseteq \mathcal{DN}_a(\mathcal{X})$ . For other side of inclusion, consider  $\eta \in \mathcal{DN}_a(\mathcal{X})$ . Then there exists  $m \in \mathbb{N}$  such that  $\mathbf{a} \xrightarrow{m} \eta \in \mathcal{X} = (\lambda_b)^{-1}(\mathcal{DN}_a(\uparrow \mathfrak{r}))$ . Thus there is  $k \in \mathbb{N}$  such that

$$\mathfrak{r} \preceq \mathbf{a} \xrightarrow{k} (\mathbf{b} \wedge (\mathbf{a} \xrightarrow{m} \eta)).$$

By assumption we have

$$\begin{aligned} \mathfrak{r} \preceq \mathbf{a} \xrightarrow{k} (\mathbf{b} \wedge (\mathbf{a} \xrightarrow{m} \eta)) &= (\mathbf{a} \xrightarrow{k} \mathbf{b}) \wedge (\mathbf{a} \xrightarrow{k} (\mathbf{a} \xrightarrow{m} \eta)) \\ &\preceq (\mathbf{a} \xrightarrow{k+m} \mathbf{b}) \wedge (\mathbf{a} \xrightarrow{k+m} \eta) \\ &= \mathbf{a} \xrightarrow{k+m} (\mathbf{b} \wedge \eta) \end{aligned}$$

By (3.2),  $\eta \in \mathcal{X}$  and so  $\mathcal{DN}_a(\mathcal{X}) \subseteq \mathcal{X}$ . Hence,  $\mathcal{X} = \mathcal{DN}_a(\mathcal{X}) \in \tau_a$ . Therefore,  $(\mathbf{E}, \wedge, \tau_a)$  is a semi-topological algebra.  $\square$

*Example 3.23.* An equality algebra as in Example 3.3, satisfies in conditions of Theorem 3.22.

**THEOREM 3.24.** Consider  $\mathbf{E}_1$  and  $\mathbf{E}_2$  be two bounded equality algebras and  $\mathbf{F} \in \mathcal{F}(\mathbf{E}_2)$ . If  $f : \mathbf{E}_1 \rightarrow \mathbf{E}_2$  is an equality homomorphism, then

- (i)  $f^{-1}(\mathcal{DN}_{f(a)}(\mathbf{F})) = \mathcal{DN}_a(f^{-1}(\mathbf{F}))$ .
- (ii)  $\mathcal{DN}_a(\ker(f)) = f^{-1}(\mathcal{DN}_{f(a)}(\{1\}))$ .
- (iii)  $f$  is continuous map from  $(\mathbf{E}_1, \tau_a)$  to  $(\mathbf{E}_2, \tau_{f(a)})$ .

**PROOF:** (i) Since  $f$  is an equality homomorphism, then  $f^{-1}(F) \in \mathcal{F}(\mathbf{E}_1)$  and

$$\begin{aligned}
f^{-1}(\mathcal{DN}_{f(\mathbf{a})}(\mathbf{F})) &= \{\mathbf{x} \in \mathbf{E}_1 \mid f(\mathbf{x}) \in \mathcal{DN}_{f(\mathbf{a})}(\mathbf{F})\} \\
&= \{\mathbf{x} \in \mathbf{E}_1 \mid \exists n \in \mathbb{N}, \text{ s.t. } f(\mathbf{a}) \xrightarrow{n} f(\mathbf{x}) \in \mathbf{F}\} \\
&= \{\mathbf{x} \in \mathbf{E}_1 \mid \exists n \in \mathbb{N}, \text{ s.t. } f(\mathbf{a} \xrightarrow{n} \mathbf{x}) \in \mathbf{F}\} \\
&= \{\mathbf{x} \in \mathbf{E}_1 \mid \exists n \in \mathbb{N}, \text{ s.t. } \mathbf{a} \xrightarrow{n} \mathbf{x} \in f^{-1}(\mathbf{F})\} \\
&= \mathcal{DN}_{\mathbf{a}}(f^{-1}(\mathbf{F})).
\end{aligned}$$

(ii) For  $\mathbf{x} \in \mathbf{E}_1$ , we have

$$\begin{aligned}
\mathbf{x} \in \mathcal{DN}_{\mathbf{a}}(\ker(f)) &\Leftrightarrow \mathbf{a} \xrightarrow{n} \mathbf{x} \in \ker(f), \text{ for some } n \in \mathbb{N} \\
&\Leftrightarrow f(\mathbf{a} \xrightarrow{n} \mathbf{x}) = 1, \text{ for some } n \in \mathbb{N} \\
&\Leftrightarrow f(\mathbf{a}) \xrightarrow{n} f(\mathbf{x}) = 1, \text{ for some } n \in \mathbb{N} \\
&\Leftrightarrow f(\mathbf{x}) \in \mathcal{DN}_{f(\mathbf{a})}(\{1\}) \\
&\Leftrightarrow \mathbf{x} \in f^{-1}(\mathcal{DN}_{f(\mathbf{a})}(\{1\})).
\end{aligned}$$

(iii) By (i),  $f^{-1}(\mathcal{DN}_{f(\mathbf{a})}(\mathbf{F})) = \mathcal{DN}_{\mathbf{a}}(f^{-1}(\mathbf{F})) \in \tau_{\mathbf{a}}$ . Thus,  $f$  is continuous map from  $(\mathbf{E}_1, \tau_{\mathbf{a}})$  to  $(\mathbf{E}_2, \tau_{f(\mathbf{a})})$ .  $\square$

Let  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$  and  $\pi : \mathbf{E} \rightarrow \frac{\mathbf{E}}{\mathbf{F}}$  where  $\pi(\mathbf{x}) = [\mathbf{x}]$ . Clearly,  $(\frac{\mathbf{E}}{\mathbf{F}}, \tau_{\mathbf{a}})$  is a topological space, where the set  $\tau_{[\mathbf{a}]} = \left\{ \mathcal{DN}_{[\mathbf{a}]} \left( \frac{\mathcal{X}}{\mathbf{F}} \right) \mid \mathbf{F} \subseteq \mathcal{X} \in \Gamma(\mathbf{E}) \right\}$  is a topology on  $\frac{\mathbf{E}}{\mathbf{F}}$  and the set  $\left\{ \mathcal{DN}_{[\mathbf{a}]}(\uparrow[\mathbf{x}]) \mid [\mathbf{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \right\}$  is a basis for the topology  $\tau_{[\mathbf{a}]}$  on  $\frac{\mathbf{E}}{\mathbf{F}}$ . In addition, the set  $\hat{\tau}_{\mathbf{a}} = \{[\mathcal{DN}_{\mathbf{a}}(\mathcal{X})] \mid \mathbf{F} \subseteq \mathcal{X} \in \Gamma(\mathbf{E})\}$  is a quotient topology on  $\frac{\mathbf{E}}{\mathbf{F}}$  and  $(\frac{\mathbf{E}}{\mathbf{F}}, \hat{\tau}_{\mathbf{a}})$  is the quotient topology space. The set  $\{[\mathcal{DN}_{\mathbf{a}}(\uparrow \mathbf{x})] \mid \mathbf{x} \in \mathbf{E}\}$  is a basis for the quotient topology  $\hat{\tau}_{\mathbf{a}}$ .

PROPOSITION 3.25. Let  $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$  such that  $\mathbf{F} \subseteq \mathbf{G}$ . Then  $\mathcal{DN}_{[\mathbf{a}]} \left( \frac{\mathbf{G}}{\mathbf{F}} \right) = \frac{\mathcal{DN}_{\mathbf{a}}(\mathbf{G})}{\mathbf{F}}$ .

PROOF: For any  $\mathfrak{x} \in \mathbf{E}$ , we have

$$\begin{aligned} \mathcal{DN}_{[\mathfrak{a}]} \left( \frac{\mathbf{G}}{\mathbf{F}} \right) &= \left\{ [\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N}, \text{ s.t. } [\mathfrak{a}] \xrightarrow{n} [\mathfrak{x}] \in \frac{\mathbf{G}}{\mathbf{F}} \right\} \\ &= \left\{ [\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N}, \text{ s.t. } [\mathfrak{a} \xrightarrow{n} \mathfrak{x}] \in \frac{\mathbf{G}}{\mathbf{F}} \right\} \\ &= \left\{ [\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N}, \text{ s.t. } \mathfrak{a} \xrightarrow{n} \mathfrak{x} \in \mathbf{G} \right\} \\ &= \left\{ [\mathfrak{x}] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \mathfrak{x} \in \mathcal{DN}_{\mathfrak{a}}(\mathbf{G}) \right\} \\ &= \frac{\mathcal{DN}_{\mathfrak{a}}(\mathbf{G})}{\mathbf{F}}. \end{aligned}$$

□

THEOREM 3.26. Let  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ . Then

(i)  $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] \subseteq \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$ , for any  $\mathfrak{x} \in \mathbf{E}$ .

(ii) The quotient topology  $\widehat{\tau}_{\mathfrak{a}}$  is finer than the topology  $\tau_{[\mathfrak{a}]}$ .

PROOF: (i) Consider  $[\mathfrak{y}] \in [\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})]$ . Then there exists  $\mathfrak{z} \in \mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})$  such that  $[\mathfrak{y}] = [\mathfrak{z}]$ . Thus there is  $n \in \mathbb{N}$  such that  $\mathfrak{x} \xrightarrow{n} (\mathfrak{a} \xrightarrow{n} \mathfrak{z}) = 1$ , and so  $[\mathfrak{x}] \xrightarrow{n} ([\mathfrak{a}] \xrightarrow{n} [\mathfrak{z}]) = [1]$ . Since  $[\mathfrak{y}] = [\mathfrak{z}]$  we get  $[\mathfrak{x}] \xrightarrow{n} ([\mathfrak{a}] \xrightarrow{n} [\mathfrak{y}]) = [1]$ . Hence,  $[\mathfrak{y}] \in \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$ .

(ii) By (i) the proof is clear. □

THEOREM 3.27. Consider  $\mathbf{E}$  be involutive and  $\mathbf{F} \in \mathcal{F}(\mathbf{E})$ . Then

(i)  $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] = \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$ , for any  $\mathfrak{x} \in \mathbf{E}$ .

(ii) The topology  $\widehat{\tau}_{\mathfrak{a}}$  and the topology  $\tau_{[\mathfrak{a}]}$  are same on  $\frac{\mathbf{E}}{\mathbf{F}}$ .

PROOF: (i) By Theorem 3.26(i), for any  $\mathfrak{x} \in \mathbf{E}$ , we have  $[\mathcal{DN}_{\mathfrak{a}}(\uparrow \mathfrak{x})] \subseteq \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$ . Conversely, suppose  $[\mathfrak{y}] \in \mathcal{DN}_{[\mathfrak{a}]}(\uparrow [\mathfrak{x}])$ . Since

$$\begin{aligned} \mathcal{DN}_{[a]}(\uparrow [x]) &= \left\{ [y] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N}, \text{ s.t. } [x] \preceq [a] \xrightarrow{n} [y] \right\} \\ &= \left\{ [y] \in \frac{\mathbf{E}}{\mathbf{F}} \mid \exists n \in \mathbb{N}, \text{ s.t. } \mathfrak{x} \rightarrow (\mathfrak{a} \xrightarrow{n} \eta) \in \mathbf{F} \right\} \end{aligned}$$

Since  $\mathfrak{x} \rightarrow (\mathfrak{a} \xrightarrow{n} \eta) \in \mathbf{F}$ , there is  $\mathfrak{t} \in \mathbf{F}$  such that  $1 = \mathfrak{t} \rightarrow (\mathfrak{x} \rightarrow (\mathfrak{a} \xrightarrow{n} \eta))$  and by Proposition 2.2(v), we have  $1 = \mathfrak{x} \rightarrow (\mathfrak{a} \xrightarrow{n} (\mathfrak{t} \rightarrow \eta))$ . Thus,  $\mathfrak{t} \rightarrow \eta \in \mathcal{DN}_a(\uparrow \mathfrak{x})$ . Since  $\mathfrak{t} \in \mathbf{F}$  we get  $[\mathfrak{t}] = [1]$ , and so  $[\mathfrak{t} \rightarrow \eta] = [\mathfrak{t}] \rightarrow [\eta] = [\eta]$ . Hence,  $[\eta] \in [\mathcal{DN}_a(\uparrow \mathfrak{x})]$ , and so  $\mathcal{DN}_{[a]}(\uparrow [x]) \subseteq [\mathcal{DN}_a(\uparrow \mathfrak{x})]$ . Therefore,  $[\mathcal{DN}_a(\uparrow \mathfrak{x})] = \mathcal{DN}_{[a]}(\uparrow [x])$ .

(ii) By (i) the proof is clear.  $\square$

Let  $\mathbf{E}_1$  and  $\mathbf{E}_2$  be two equality algebras. Then  $\mathbf{G} \in \mathcal{F}(\mathbf{E}_1 \times \mathbf{E}_2)$  if and only if there exist  $\mathbf{F}_1 \in \mathcal{F}(\mathbf{E}_1)$  and  $\mathbf{F}_2 \in \mathcal{F}(\mathbf{E}_2)$  such that  $\mathbf{G} = \mathbf{F}_1 \times \mathbf{F}_2$ .

PROPOSITION 3.28. Consider  $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$  and  $\mathfrak{a} \in \mathbf{E}$ . Then

$$\mathcal{DN}_a(\mathbf{F}) \times \mathcal{DN}_a(\mathbf{G}) = \mathcal{DN}_a(\mathbf{F} \times \mathbf{G}).$$

PROOF: Let  $\mathfrak{a} \in \mathbf{E}$ . Then by Lemma 3.5, we have

$$\begin{aligned} \mathcal{DN}_a(\mathbf{F}) \times \mathcal{DN}_a(\mathbf{G}) &= \{(\mathfrak{x}, \eta) \in \mathbf{E} \times \mathbf{E} \mid \mathfrak{x} \in \mathcal{DN}_a(\mathbf{F}), \eta \in \mathcal{DN}_a(\mathbf{G})\} \\ &= \{(\mathfrak{x}, \eta) \in \mathbf{E} \times \mathbf{E} \mid \exists n, m \in \mathbb{N} \text{ s.t. } \mathfrak{a} \xrightarrow{n} \mathfrak{x} \in \mathbf{F}, \mathfrak{a} \xrightarrow{m} \eta \in \mathbf{G}\} \\ &= \{(\mathfrak{x}, \eta) \in \mathbf{E} \times \mathbf{E} \mid \exists t \in \mathbb{N} \text{ s.t. } t \geq \max\{n, m\} \\ &\quad \text{s.t. } \mathfrak{a} \xrightarrow{t} \mathfrak{x} \in \mathbf{F}, \mathfrak{a} \xrightarrow{t} \eta \in \mathbf{G}\} \\ &= \{(\mathfrak{x}, \eta) \in \mathbf{E} \times \mathbf{E} \mid \exists t \in \mathbb{N} \text{ s.t. } t \geq \max\{n, m\} \\ &\quad \text{s.t. } (\mathfrak{a}, \mathfrak{a}) \xrightarrow{t} (\mathfrak{x}, \eta) \in \mathbf{F} \times \mathbf{G}\} \\ &= \mathcal{DN}_a(\mathbf{F} \times \mathbf{G}). \end{aligned} \quad \square$$

THEOREM 3.29. Consider  $\mathbf{F}, \mathbf{G} \in \mathcal{F}(\mathbf{E})$ . Then

$$\psi : \frac{\mathbf{E} \times \mathbf{E}}{\mathcal{DN}_a(\mathbf{F} \times \mathbf{G})} \rightarrow \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{F})} \times \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{G})}$$

is a homeomorphism, for any  $\mathfrak{a} \in \mathbf{E}$ .

PROOF: Define the map  $\varphi : \mathbf{E} \times \mathbf{E} \rightarrow \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{F})} \times \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{G})}$  such that  $\varphi(\mathfrak{x}, \mathfrak{y}) = ([\mathfrak{x}]_{\mathcal{DN}_a(\mathbf{F})}, [\mathfrak{y}]_{\mathcal{DN}_a(\mathbf{G})})$ . Obviously, the map  $\varphi$  is onto and  $(\mathfrak{x}, \mathfrak{y}) \in \ker \varphi$  if and only if  $[\mathfrak{x}]_{\mathcal{DN}_a(\mathbf{F})} = [1]_{\mathcal{DN}_a(\mathbf{F})}$  and  $[\mathfrak{y}]_{\mathcal{DN}_a(\mathbf{G})} = [1]_{\mathcal{DN}_a(\mathbf{G})}$ . Hence, Proposition 3.28,  $\mathcal{DN}_a(\mathbf{F}) \times \mathcal{DN}_a(\mathbf{G}) = \mathcal{DN}_a(\mathbf{F} \times \mathbf{G})$  and so  $\ker(\varphi) = \mathcal{DN}_a(\mathbf{F}) \times \mathcal{DN}_a(\mathbf{G})$ .

In addition, suppose

$$\psi : \frac{\mathbf{E} \times \mathbf{E}}{\mathcal{DN}_a(\mathbf{F} \times \mathbf{G})} \longrightarrow \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{F})} \times \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{G})}$$

defined by  $\psi([\mathfrak{x}, \mathfrak{y}]_{\mathcal{DN}_a(\mathbf{F} \times \mathbf{G})}) = ([\mathfrak{x}]_{\mathcal{DN}_a(\mathbf{F})}, [\mathfrak{y}]_{\mathcal{DN}_a(\mathbf{G})})$ . Thus by the first isomorphism theorem  $\psi$  is an isomorphism. Suppose that  $\mathcal{U}$  is an open subset of  $\frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{F})} \times \frac{\mathbf{E}}{\mathcal{DN}_a(\mathbf{G})}$ . Then there exist open subset  $V, W \in \tau_a$  such that  $\mathcal{U} = \frac{V}{\mathcal{DN}_a(\mathbf{F})} \times \frac{W}{\mathcal{DN}_a(\mathbf{G})}$ . Clearly,  $\psi^{-1}(\mathcal{U}) = \frac{V \times W}{\mathcal{DN}_a(\mathbf{F} \times \mathbf{G})}$  is an open subset of  $\frac{\mathbf{E} \times \mathbf{E}}{\mathcal{DN}_a(\mathbf{F} \times \mathbf{G})}$ . Hence,  $\psi$  is a continuous map. Therefore,  $\psi$  is a homeomorphism. □

### 4. Conclusion

In this paper, a special subset of equality algebra which is upset is introduced and by using this upsets a topology on a bounded equality algebras is constructed and some of their topological properties, such as some types of topological space (Hausdorff,  $T_0$ -space and  $T_1$ -space ) and connectedness are investigated. In addition, the relation between closed and compact sets in this topology is expressed. Moreover, by considering the binary operation  $\rightarrow$  and the constructed topology on the bounded equality algebra  $\mathbf{E}$ , the notion of a semi-topological algebra is introduced and it is proved that any involutive equality algebra is a right semi-topological algebra and a semi- $\lambda$ -topological algebra but not a left semi-topological algebra. Finally, converse image, product and quotient topology on equality algebra are studied and showed that under what condition finer topology can be made.

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