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SOME NEW KINDS OF BCK-FILTERS IN BOUNDED BCK-ALGEBRAS

Abstract

This paper investigates and characterizes some new types of BCK-filters in bounded BCK-algebras. We introduce the notions of EQI-filters, implicative BCK-filters, and commutative BCK-filters. We clarify the relationships among these filters and prove that every implicative BCK-filter is a commutative BCK-filter and every commutative BCK-filter is an EQI-filter, although the converses do not hold in general. We also study these filters in several important subclasses of BCK-algebras. For instance, in involutory BCK-algebras, the notions of EQI-filters, implicative BCK-filters, and commutative BCK-filters coincide, whereas they are distinct in general.

Keywords: BCK-filter, implicative BCK-filter, commutative BCK-filter, EQI-filter, EQI-algebras.

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1. Introduction

A BCK-algebra is an algebraic structure introduced by Y. Imai and K. Iséki [5] in 1966. It originated from set-theoretic considerations and is closely related to classical and non-classical logics. Since logical systems play a crucial role in computer science and artificial intelligence, research in this area is highly significant.

The class of bounded BCK-algebras includes important subclasses such as bounded commutative BCK-algebras, bounded implicative BCK-algebras, and involutory BCK-algebras. These structures contain various types of BCK-lattices. For example, S. Khosravi Shoar et al. [11] introduced the concept of PC-lattices in bounded BCK-algebras and proved that in PC-lattices satisfying condition (S), commutative BCK-algebras and involutory BCK-algebras coincide.

Filter theory, together with ideal theory, plays a vital role in studying the structure of BCK/BCI-algebras. J. Meng [8] introduced the concept of BCK-filters. W. A. Dudek and Y. B. Jun [2] studied poor and crazy filters in bounded BCK-algebras. In this paper, we introduce EQI-filters, implicative BCK-filters, and commutative BCK-filters in bounded BCK-algebras and investigate the relationships among them. We also examine these filters in several important subclasses of BCK-algebras.

2. Preliminaries

Definition 2.1 ([6, 3]). Let X be a set with a binary operation $*$ and a constant 0. Then $(X; *, 0)$ is called a BCK-algebra if it satisfies the following axioms for all $x, y, z \in X$:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$.

(BCK-2) $(x * (x * y)) * y = 0$.

(BCK-3) $x * x = 0$.

(BCK-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

(BCK-5) $0 * x = 0$.

A partial ordering $\leq$ on X is defined by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra X, for all $x, y, z \in X$ the following statements hold:

(I) $(x * y) * z = (x * z) * y$.

(II) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

(III) $x * (x * (x * y)) = (x * y)$.

(IV) $(z * x) * (y * x) \leq z * y$ and $(x * y) * (x * z) \leq z * y$.

Definition 2.2 ([1, 5, 12, 9]). Let X be a BCK-algebra. Then for all $x, y \in X$:

(i) X is called an *implicative* BCK-algebra if $x * (y * x) = x$.

(ii) X is called a *commutative* BCK-algebra if $x * (x * y) = y * (y * x)$.

(iii) X is called *bounded*, if there exists the greatest element of X, which is denoted by 1 (In this case for any $x \in X$, $1 * x$ is denoted by Nx).

(iv) If X is bounded, then $x \in X$ is called *involution* of X, if $NNx = x$ where $NNx = N(Nx)$.

(v) X is called an *involutory* BCK-algebra, if for all $x \in X$, $NNx = x$.

Definition 2.3 ([6, 7]). Let I be a nonempty subset of BCK-algebra X. Then for all $x, y, z \in X$:

(i) I is called an *ideal* of X if $0 \in I$, $y \in I$ and $x * y \in I$ imply $x \in I$.

(ii) I is called an *implicative ideal* of X if $0 \in I$, $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$.

Definition 2.4 ([8, 2]). Let F be a nonempty subset of bounded BCK-algebra X. Then for all $x, y, z \in X$:

(i) F is called a *BCK-filter* of X if $1 \in F$ and $N(Nx * Ny) \in F$ and $y \in F$ imply $x \in F$.

(ii) F is called a *crazy filter* of X if $1 \in F$, $N(y * x) \in F$ and $y \in F$ imply $x \in F$.

Theorem 2.5 ([8, 2]). *Let F be a nonempty subset of bounded BCK-algebra X. Then for all $x, y \in X$:*

(i) If F is a BCK-filter, $x \leq y$ and $x \in F$, then $y \in F$.

(ii) If F is a crazy filter, $x \leq y$ and $x \in F$, then $y \in F$.

Proposition 2.6 ([6, 9]). Let X be a bounded BCK-algebra with the greatest element 1. Then for all $x, y \in X$:

(i) $N1 = 0$ and $N0 = 1$.

(ii) $NNx \leq x$.

(iii) $Nx * Ny \leq y * x$.

(iv) $y \leq x$ implies $Nx \leq Ny$.

(v) $Nx * y = Ny * x$.

(vi) $NNNx = Nx$.

Lemma 2.7 ([4]). *Let X be a bounded BCK-algebra. Then for $x, y \in X$:*

(i) If the greatest lower bound $x \wedge y$ of x and y exists, then the least upper bound $Nx \vee Ny$ of Nx and Ny exists, and $Nx \vee Ny = N(x \wedge y)$.

(ii) If X is involutory and the least upper bound $x \vee y$ exists, then the greatest lower bound $Nx \wedge Ny$ exists, and $Nx \wedge Ny = N(x \vee y)$.

Definition 2.8 ([3]). Let X be a BCK-algebra. Then $(X; *, 0)$ is called a BCK-lattice, if $(X, \leq)$ is a lattice, where $\leq$ is the partial BCK-order on X.

Proposition 2.9 ([4]). Let X be a BCK-algebra. Then for all $x, y, z \in X$:

(i) if X is a BCK-lattice, then $x * (y \wedge z) = (x * y) \vee (x * z)$.

(ii) if X is an involutory BCK-lattice, then $(x \vee y) * z = (x * z) \vee (y * z)$.

Theorem 2.10 ([2]). *Every BCK-filter is a crazy filter (but the converse does not hold in general).*

Definition 2.11 ([11]). Let X be a BCK-lattice. Then X is called a PC-lattice if for all $x, y, z \in X$:

$$(z * x) * (y * x) = z * (x \vee y).$$

Definition 2.12 ([10]). Let X be a bounded BCK-algebra. Then X is called an EQI-algebra if for all $x \in X$:

$$N(x * NNx) = 1.$$

Theorem 2.13 ([10, 11, 9]). *Let X be a bounded BCK-algebra. Then*

(i) Every commutative BCK-algebra is an involutory BCK-algebra.

(ii) A bounded implicative BCK-algebra is a PC-lattice, but the converse does not hold in general.

(iii) Every involutory BCK-algebra and PC-lattice is an EQI-algebra, but the converse does not hold in general.

Definition 2.14 ([10]). Let I be an ideal of X. Then I is called an *EQI-ideal* of X if $NN(x * NNx) \in I$ for all $x \in X$.

3. EQI-filters in bounded BCK-algebras

Lemma 3.1. *Let X be a bounded BCK-algebra. Then for all $x, y, v \in X$:*

(i) $NN(NNx * y) = NNx * y$.

(ii) $NN(Nx * y) = Nx * y$.

(iii) $NN(Nx * v) * N(Nx * v) = ((Nx * x) * v) * v$.

Proof: Let $x, y, v \in X$.
(i) By Proposition 2.6 (ii), $(NN(NNx * y)) * (NNx * y) = 0$.
Moreover, we have

$$
\begin{aligned}
(NNx * y) * NN(NNx * y) &\overset{\text{(I)}}{=} (NNx * NN(NNx * y)) * y \\
&\overset{\text{(v)}}{=} (NNN(NNx * y) * Nx) * y \\
&\overset{\text{Prop. 2.6(vi)}}{=} (N(NNx * y) * Nx) * y \\
&\overset{\text{Prop. 2.6(v)}}{=} (NNx * (NNx * y)) * y \\
&\overset{\text{(v)}}{=} (NNx * y) * (NNx * y) \\
&\overset{\text{(BCK-3)}}{=} 0.
\end{aligned}
$$

Hence, by BCK-4, $NN(NNx * y) = NNx * y$.
(ii) By (i) and Proposition 2.6 (vi) we have

$$NN(Nx * y) = NN(NN(Nx) * y) = NN(Nx * y) = Nx * y. \tag{3.1}$$

(iii)

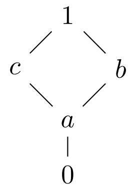
$$
\begin{aligned}
NN(Nx * v) * N(Nx * v) &\overset{\text{(ii)}}{=} (Nx * v) * N(Nx * v) \\
&\overset{\text{(I)}}{=} (Nx * N(Nx * v)) * v \\
&\overset{\text{(I)}}{=} (NN(Nx * v) * x) * v \\
&\overset{\text{(ii)}}{=} ((Nx * v) * x) * v \\
&\overset{\text{(I)}}{=} ((Nx * x) * v) * v.
\end{aligned}
$$
□

Definition 3.2. A BCK-filter F of X is called an *EQI-filter* if for all $x \in X$, $N(x * NNx) \in F$.

The following example shows that EQI-filters exist and that not every BCK-filter is an EQI-filter in general.

Example 3.3. Let $X = \{0, a, b, c, 1\}$ be a set, and the operation $*$ and the partial ordering $\leq$ on X be defined as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	a	a	0	0
1	1	b	a	b	0



(3.2)

Then $(X; *, 0)$ is a bounded BCK-algebra. It is easy to check that $\{1\}$ and $\{1, b\}$ are BCK-filters, $\{1, b\}$ is an EQI-filter, but $\{1\}$ is not an EQI-filter, because $N(c * NNc) = 1 * (c * (1 * (1 * c))) = 1 * (c * (1 * b)) = 1 * (c * a) = 1 * a = b \notin \{1\}$.

LEMMA 3.4. *Let X be a bounded BCK-algebra, F and E be BCK-filters with $F \subseteq E$. If F is an EQI-filter, then so is E.*

PROOF: The proof follows directly from the definition of an EQI-filter. □

THEOREM 3.5. *Let X be a bounded BCK-algebra. Then the following statements are equivalent:*

(i) $\{1\}$ is an EQI-filter.

(ii) Every BCK-filter is an EQI-filter.

(iii) X is an EQI-algebra.

PROOF: The theorem holds by Lemma 3.4, and Definition 2.12. □

If we define $NF = \{Nx \mid x \in F\}$, then we have the following theorem.

THEOREM 3.6. *Let F be a nonempty subset of X. Then F is an EQI-filter if and only if NF is an EQI-ideal.*

PROOF: Let F be a nonempty subset of X. Then by Definition 2.14 and 3.2, for all $x \in X$ we have

$$\begin{aligned} F \text{ is an } \textit{EQI-filter} &\Leftrightarrow N(x * NNx) \in F \\ &\Leftrightarrow NN(x * NNx) \in NF \\ &\Leftrightarrow NF \text{ is an } \textit{EQI-ideal}. \end{aligned}$$

□

Theorem 3.7. *In any EQI-algebra, the concepts of BCK-filters, EQI-filters and crazy filters coincide.*

Proof: Suppose that X is an EQI-algebra. Then by Theorem 3.5, the concepts of BCK-filters and EQI-filters coincide. By Theorem 2.10, it is sufficient to prove that any crazy filter is a BCK-filter. Hence, we must prove that if $y \in F$ and $N(Nx * Ny) \in F$, then $x \in F$. By Lemma 3.1, we have

$$\begin{aligned}
N(Nx * Ny) * N(y * x) &\overset{\text{Lem. } 3.1\text{(ii)}}{=} NN[N(Nx * Ny) * N(y * x)] \\
&\overset{\text{(I)}}{=} NN[NN(y * x) * (Nx * Ny)] \\
&\overset{\text{(II) and Prop. } 2.6\text{(ii)}}{\leq} NN[(y * x) * (Nx * Ny)] \\
&\overset{\text{(I)}}{=} NN[(y * x) * (NNy * x)] \\
&\overset{\text{(IV)}}{\leq} NN(y * NNy) \\
&\overset{\text{(Def. } 2.12\text{)}}{=} 0.
\end{aligned}$$

Therefore by Theorem 2.5(ii), we have $N(y * x) \in F$, which implies that $x \in F$. □

Corollary 3.8. Let X be a bounded BCK-algebra. Then in PC-lattices, involutory BCK-algebras, commutative BCK-algebras and implicative BCK-algebras, the concepts of BCK-filters, EQI-filters and crazy filters coincide.

Proof: The corollary holds by Theorem 2.13, and Theorem 3.5. □

Theorem 3.9. *Let X be a bounded BCK-algebra and F a nonempty subset of X. Then in PC-lattices, involutory BCK-algebras, commutative BCK-algebras and implicative BCK-algebras, F is a BCK-filter if and only if NF is an ideal of X.*

Proof: The theorem holds by Corollary 3.8, Theorem 3.6, Theorem 2.13 and Theorem 3.5. □

4. Implicative BCK-filters in bounded BCK-algebras

DEFINITION 4.1. A nonempty subset F of X is said to be an *implicative BCK-filter* if $1 \in F$ and for all $x, y, z \in X$: $N((Nx * (Ny * Nx)) * Nz) \in F$ and $z \in F$ imply $x \in F$.

THEOREM 4.2. *Any implicative BCK-filter of X is a BCK-filter, but the converse does not hold in general.*

PROOF: Let F be an implicative BCK-filter of X, and suppose $N(Nx * Ny) \in F$ and $y \in F$. Then

$$\begin{aligned} N((Nx * (N1 * Nx)) * Ny) &= N((Nx * (0 * Nx)) * Ny) \\ &= N((Nx * 0) * Ny) = N(Nx * Ny) \in F. \end{aligned}$$

Hence by Definition 4.1, we have $x \in F$. □

The following example shows that the converse of Theorem 4.2 does not hold in general.

Example 4.3. Let $X = \{0, a, b, 1\}$ be a set, and define the operation $*$ and the partial ordering $\leq$ on X as follows:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

$$\begin{array}{c} 1 \\ | \\ b \\ | \\ a \\ | \\ 0 \end{array} \qquad (4.1)$$

Then $(X; *, 0)$ is a bounded BCK-algebra. It is easy to check that $\{1\}$ and $\{1, b\}$ are BCK-filters. $\{1, b\}$ is an implicative BCK-filter, but $\{1\}$ is not an implicative BCK-filter because $N((Nb * (N0 * Nb)) * N1) = N((a * (1 * a)) * 0) = N((a * b)) = N0 = 1 \in \{1\}$ and $1 \in \{1\}$, yet $b \notin \{1\}$.

LEMMA 4.4. *Let F be a BCK-filter of X. Then the following statements are equivalent for all $x, y \in X$:*

(i) F is an implicative BCK-filter.

(ii) $N(Nx * (Ny * Nx)) \in F$ *implies* $x \in F$.

(iii) $N(Nx * x) \in F$ *implies* $x \in F$.

Proof: $(i) \Rightarrow (ii)$: Let F be an implicative BCK-filter and $N(Nx*(Ny*Nx)) \in F$. We have $1 \in F$ and

$$\begin{aligned} N((Nx * (Ny * Nx) * N1)) &= N((Nx * (Ny * Nx) * 0)) \\ &= N(Nx * (Ny * Nx)) \in F. \end{aligned}$$

So by Definition 4.1, we get $x \in F$.

$(ii) \Rightarrow (iii)$: Assume $N(Nx * x) \in F$. Then by Proposition 2.6(vi) and (I),

$$\begin{aligned} N(Nx * (N0 * Nx)) &= N(Nx * (1 * Nx)) \\ &= N(Nx * NNx) \\ &= N(NNNx * x) \\ &= N(Nx * x) \in F. \end{aligned}$$

Therefore by (ii) we have $x \in F$.

$(iii) \Rightarrow (ii)$: Suppose $N(Nx * (Ny * Nx)) \in F$. Given that $Ny * Nx \leq 1 * Nx = NNx$, by (II) we have $Nx * NNx \leq Nx * (Ny * Nx)$, so $N(Nx * (Ny * Nx)) \leq N(Nx * NNx)$. Consequently $N(Nx * NNx) \in F$. Moreover, by (I) and Proposition 2.6(vi), $N(Nx * x) = N(NNNx * x) = N(Nx * NNx) \in F$, so by (iii) we obtain $x \in F$.

$(ii) \Rightarrow (i)$: This follows directly from the definition of a BCK-filter. □

Theorem 4.5. *Let F be a BCK-filter of X. Then the following are equivalent for all $x, y, z \in X$:*

(i) F *is an implicative BCK-filter.*

(ii) $N((Nx * Ny) * Ny) \in F$ *implies* $N(Nx * Ny) \in F$.

(iii) $N((Nx * Ny) * Nz) \in F$ *implies* $N((Nx * Nz) * (Ny * Nz)) \in F$.

Proof: $(i) \Rightarrow (ii)$: Suppose F is an implicative BCK-filter and $N((Nx * Ny) * Ny) \in F$. We have

$$\begin{aligned}(Nx * Ny) * Ny &\overset{\text{Prop. 2.6(ii) and (II)}}{\geq} NN(Nx * Ny) * Ny \\ &\overset{\text{(IV)}}{\geq} (Nx * Ny) * (Nx * NN(Nx * Ny)) \\ &\overset{\text{Prop. 2.6(ii) and (II)}}{\geq} NN(Nx * Ny) * (Nx * NN(Nx * Ny)).\end{aligned}$$

Hence $N((Nx * Ny) * Ny) \leq N(NN(Nx * Ny) * (Nx * NN(Nx * Ny)))$. Given that F is a BCK-filter, $N(NN(Nx*Ny)*(Nx*NN(Nx*Ny))) \in F$. Consequently by Lemma 4.4 (ii), we have $N(Nx * Ny) \in F$.

$(ii) \Rightarrow (i)$: Let $N(Nx * x) \in F$. Then by (I),

$$N((NNx * (Nx * x)) * Nx) = N((NNx * Nx) * (Nx * x)).$$

Hence

$$\begin{aligned}N((N(Nx * NNx) * NNx) * NNx) &\overset{\text{(I)}}{=} N((NNNx*(Nx*NNx))*NNx) \\ &\overset{\text{Prop. 2.6(iv)}}{=} N((Nx * (Nx * NNx)) * NNx) \\ &\overset{\text{(I)}}{=} N((Nx * NNx) * (Nx * NNx)) \\ &= N(0) \\ &= 1 \in F.\end{aligned}$$

Thus by (ii) we have $N((N(Nx * NNx) * NNx)) \in F$. On the other hand,

$$\begin{aligned}N(Nx * NN(Nx * x)) &\overset{\text{Prop. 2.6(iv)}}{=} N(NNNx * NN(Nx * x)) \\ &\overset{\text{(I)}}{=} N(NNN(Nx * x) * NNx) \\ &\overset{\text{Prop. 2.6(iv)}}{=} N(N(Nx * x) * NNx) \\ &\overset{\text{Prop. 2.6(iv)}}{=} N(N(NNNx * x) * NNx) \\ &\overset{\text{(I)}}{=} N(N(Nx * NNx) * NNx).\end{aligned}$$

Hence $N(Nx * NN(Nx * x)) \in F$. Given that F is a BCK-filter, we get $x \in F$. Therefore by Lemma 4.4, F is an implicative BCK-filter.

$(ii) \Rightarrow (iii)$: Suppose $N((Nx * Ny) * Nz) \in F$. Then

$$\begin{aligned}
(((Nx * (Ny * Nz)) * Nz) * Nz) * ((Nx * Ny) * Nz) & \\
&\overset{\text{(IV)}}{\leq} ((Nx * (Ny * Nz)) * Nz) * (Nx * Ny) \\
&\overset{\text{(I)}}{\leq} ((Nx * (Ny * Nz)) * (Nx * Ny)) * Nz \\
&\overset{\text{(IV),(II)}}{\leq} (Ny * (Ny * Nz)) * Nz \\
&\overset{\text{(I)}}{=} (Ny * Nz) * (Ny * Nz) \\
&= 0.
\end{aligned}$$

It follows that $N((Nx * Ny) * Nz) \leq N(((Nx * (Ny * Nz)) * Nz) * Nz)$. Hence by Theorem 2.5, $N(((Nx*(Ny*Nz))*Nz)*Nz) \in F$. Consequently by (ii),

$$N((Nx * Nz) * (Ny * Nz)) = N((Nx * (Ny * Nz)) * Nz) \in F.$$

$(iii) \Rightarrow (ii)$: Put $z = y$; the statement clearly holds. □

THEOREM 4.6. *Let F and E be BCK-filters of X with $F \subseteq E$. If F is an implicative BCK-filter, then so is E.*

PROOF: Suppose F is an implicative BCK-filter, E is a BCK-filter, $F \subseteq E$, and $N((Nx * Ny) * Ny) \in E$. Set $U = (Nx * Ny) * Ny$. Then

$$\begin{aligned}
(NN(Nx * U) * Ny) * Ny &\overset{\text{Lem. 3.1(ii)}}{=} ((Nx * U) * Ny) * Ny \\
&\overset{\text{(I)}}{=} ((Nx * Ny) * U) * Ny \\
&\overset{\text{(I)}}{=} ((Nx * Ny) * Ny) * U \\
&= 0.
\end{aligned}$$

Therefore $N((NN(Nx * U) * Ny) * Ny) = N0 = 1 \in F$. Now by Theorem 4.5 (ii), we have $N(NN(Nx * U) * Ny) \in F$. By Lemma 3.1, and Theorem 2.6 (vi), $N((Nx * Ny) * U) \in F$. On the other hand, by Proposition 2.6 (iii),

$$NN(Nx * Ny) * NNU \leq (Nx * Ny) * U.$$

Thus $N((Nx * Ny) * U) \leq N(NN(Nx * Ny) * NNU)$. Given that F is a BCK-filter, $N(NN(Nx * Ny) * NNU) \in F \subseteq E$. Because $NU \in E$ and E is a BCK-filter, we obtain $N(Nx * Ny) \in E$. By Theorem 4.5, E is an implicative BCK-filter. $\square$

THEOREM 4.7. *Let F be a BCK-filter of X. Then F is an implicative BCK-filter of X if and only if for every $a \in X$, the set*

$$B^F(a) = \{x \in X : N(Nx * Na) \in F\}$$

is the smallest BCK-filter containing F and a.

PROOF: Suppose that for every $a \in X$, $B^F(a)$ is a BCK-filter of X and $N((Nx*Ny)*Ny) \in F$. By Theorem 4.5, it suffices to show $N(Nx*Ny) \in F$. By Lemma 3.1, $N(NN(Nx * Ny) * Ny) = N((Nx * Ny) * Ny) \in F$, so $N(Nx * Ny) \in B^F(y)$. Given that $y \in B^F(y)$ and $B^F(y)$ is a BCK-filter, we have $x \in B^F(y)$. Therefore $N(Nx * Ny) \in F$.

Conversely, assume F is an implicative BCK-filter. If $N(Nx * Ny) \in B^F(y)$ and $x \in B^F(y)$, then by definition of $B^F(y)$ and Lemma 3.1, we obtain $N((Nx * Ny) * Ny) = N(NN(Nx * Ny) * Ny) \in F$. Hence by Theorem 4.5, $N(Nx * Ny) \in F$, which shows $x \in B^F(y)$. Thus $B^F(a)$ is a BCK-filter. Clearly $a \in B^F(a)$. Moreover, for any $x \in F$, using Lemma 3.1(ii), we have $N(NN(Nx * Na) * Nx) = N((Nx * Na) * Nx) = N((Nx * Nx) * Na) = N0 = 1 \in F$. Given that F is a BCK-filter, $N(Nx*Na) \in F$, so $x \in B^F(a)$. Hence $F \subseteq B^F(a)$. If E is any BCK-filter containing F and a, then for $x \in B^F(a)$ we have $N(Nx*Na) \in F \subseteq E$, and because $a \in E$ and E is a BCK-filter, we get $x \in E$. Therefore $B^F(a) \subseteq E$. $\square$

THEOREM 4.8. *Let X be an involutory BCK-algebra. Then*

(i) $\{1\}$ is an implicative BCK-filter.

(ii) Every BCK-filter is an implicative BCK-filter.

(iii) For every $a \in X$, $B(a) = B^{\{1\}}(a)$ is an implicative BCK-filter.

(iv) X is an implicative BCK-algebra.

Proof: $(i) \Leftrightarrow (ii)$ follows directly from Theorem 4.6.

$(ii) \Rightarrow (iii)$: $\{1\}$ is a BCK-filter, so by (ii) it is an implicative BCK-filter. By Theorem 4.7, $B(a) = B^{\{1\}}(a)$ is a BCK-filter, and then by (ii) it is an implicative BCK-filter.

$(iii) \Rightarrow (iv)$: Given that X is involutory, $NNx = x$ for all $x \in X$. For $x, y \in X$, set $s = Nx$, $t = Ny$, so $x = Ns$, $y = Nt$. Put $u = N(x * (y * x))$. Then $N(Ns * (Nt * Ns)) = N(x * (y * x)) = u \in B(u)$, and $B(u)$ is an implicative BCK-filter. By Lemma 4.4(ii), $s \in B(u)$, hence $N(Ns * Nu) \in \{1\}$. Thus $N(x * (x * (y * x))) = N(x * NN(x * (y * x))) = N(Ns * Nu) = 1$, so $NN(x * (x * (y * x))) = x * (x * (y * x)) = 0$. Therefore $x \leq x * (y * x)$. Clearly $x * (y * x) \leq x$, so $x * (y * x) = x$. By Definition 2.2, X is an implicative BCK-algebra.

$(iv) \Rightarrow (i)$: Let $N(Nx * x) \in \{1\}$. By (iv), $x * Nx = x * (1 * x) = x$. Hence $N(Nx * x) = N(Nx * (x * Nx)) = N(Nx) = NNx = x \in \{1\}$, which implies $x = 1 \in \{1\}$. By Lemma 4.4, $\{1\}$ is an implicative BCK-filter. □

Theorem 4.9. *In EQI-algebras, involutory BCK-algebras and PC-lattices, F is an implicative BCK-filter if and only if NF is an implicative ideal.*

Proof: Given that $\{1\}$ is a BCK-filter, by Theorem 3.9, we have $N\{1\} = \{0\}$, which is an ideal. The result then follows from Theorem 4.8 and Definition 2.3. □

5. Commutative BCK-filters in bounded BCK-algebras

Definition 5.1. A BCK-filter F of X is called a *commutative BCK-filter* if for all $x, y \in X$, $N(Nx * Ny) \in F$ implies $N(Nx * (Ny * (Ny * Nx))) \in F$.

Theorem 5.2. *Let F be an implicative BCK-filter of X. Then F is a commutative BCK-filter, but the converse does not hold in general.*

Proof: Suppose F is an implicative BCK-filter, $N(Nx * Ny) \in F$, and let $v = Ny * (Ny * Nx)$. Then

$$\begin{aligned}
& N(Nx * Ny) * N(NN(Nx * v) * N(Nx * v)) \\
\overset{\text{Prop. 2.6(iii)}}{\leq} \quad & (NN(Nx * v) * N(Nx * v)) * (Nx * Ny) \\
\overset{\text{Lem. 3.1(iii)}}{=} \quad & (((Nx * x) * v) * v) * (Nx * Ny) \\
\overset{\text{(I)}}{=} \quad & (((Nx * x) * (Nx * Ny)) * v) * v \\
\overset{\text{(IV)}}{\leq} \quad & ((Ny * x) * v) * v \\
= \quad & ((Ny * x) * (Ny * (Ny * Nx))) * v \\
\overset{\text{(IV)}}{\leq} \quad & ((Ny * Nx) * x) * v \\
\overset{\text{(I)}}{=} \quad & ((NNx * y) * x) * v \\
\overset{\text{(I)}}{=} \quad & ((NNx * x) * y) * v \\
= \quad & 0 * v \\
= \quad & 0.
\end{aligned}$$

Therefore $N(Nx * Ny) \leq N(NN(Nx * v) * N(Nx * v))$. Given that $N(Nx * Ny) \in F$, we have $N(NN(Nx * v) * N(Nx * v)) \in F$, and by Lemma 4.4, we obtain $N(Nx * v) = N(Nx * (Ny * (Ny * Nx))) \in F$. Hence F is a commutative BCK-filter. □

The following example shows that the converse does not hold in general.

Example 5.3. Let $X = \{0, a, b, c, 1\}$ be a set, and define the operation $*$ and the partial ordering $\leq$ on X as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	b	0	0
1	1	c	b	a	0

$$\begin{array}{c} 1 \\ | \\ c \\ | \\ b \\ | \\ a \\ | \\ 0 \end{array} \tag{5.1}$$

Then $(X;*,0)$ is a bounded BCK-algebra. It is easy to check that $\{1\}$ and $\{1,c\}$ are BCK-filters. $\{1,c\}$ is a commutative BCK-filter, but it is not an implicative BCK-filter because $N(Nb*b) = N(b*b) = N(0) = 1 \in \{1,c\}$ while $b \notin \{1,c\}$.

Theorem 5.4. *Let X be an EQI-algebra. Then every commutative BCK-filter is an EQI-filter.*

Proof: Let F be a commutative BCK-filter of the EQI-algebra X. Given that every commutative BCK-filter is a BCK-filter, Theorem 3.5 implies that F is an EQI-filter. □

The following example shows that the converse of Theorem 5.4 does not hold in general.

In Example 5.3, it is easy to check that $\{1\}$ is an EQI-filter, but it is not a commutative BCK-filter because $N(Nc*Na) = N(a*c) = N0 = 1 \in \{1\}$, yet $N(Nc*(Na*(Na*Nc))) = N(a*(c*(c*a))) = N(a*(c*c)) = N(a*0) = N(a) = c \notin \{1\}$.

Theorem 5.5. *Let X be an involutory BCK-algebra and a PC-lattice. Then the concepts of crazy filters, BCK-filters, EQI-filters and commutative BCK-filters coincide.*

Proof: Assume X is an involutory BCK-algebra and a PC-lattice. By Corollary 3.8, the concepts of crazy filters, BCK-filters and EQI-filters coincide. Thus it suffices to prove that every BCK-filter is a commutative BCK-filter. Let F be a BCK-filter and $N(Nx*Ny) \in F$. Then

$$
\begin{aligned}
N(Nx*(Ny*(Ny*Nx))) &\overset{(\mathrm{I})}{=} N(Nx*(Ny*(NNx*y))) \\
&\overset{\text{Def. } 2.2(\text{iii})}{=} N(Nx*((1*y)*(NNx*y))) \\
&\overset{\text{Def. } 2.11}{=} N(Nx*(1*(NNx\vee y))) \\
&\overset{\text{Def. } 2.2(\text{iii})}{=} N(Nx*(N(NNx\vee y))) \\
&\overset{\text{Lem. } 2.7(\text{ii})}{=} N(Nx*(NNNx\wedge Ny)) \\
&\overset{\text{Prop. } 2.6(\text{vi})}{=} N(Nx*(Nx\wedge Ny)) \\
&\overset{\text{Prop. } 2.9(\text{i})}{=} N((Nx*Nx)\vee(Nx*Ny)) \\
&= N(Nx*Ny)\in F.
\end{aligned}
$$

Therefore, by Definition 5.1, F is a commutative BCK-filter. □

LEMMA 5.6. *Let F and E be BCK-filters of X with $F \subseteq E$. If F is a commutative BCK-filter, then so is E.*

PROOF: Assume that F is a commutative BCK-filter, E is a BCK-filter, $F \subseteq E$, and $u = N(Nx * Ny) \in E$. By (I) and (BCK-3) we have

$$N((Nx*(Nx*Ny))*Ny) = N((Nx*Ny)*(Nx*Ny)) = N(0) = 1 \in F.$$

Hence by the definition of a commutative BCK-filter,

$$N[(Nx*(Nx*Ny))*(Ny*(Ny*(Nx*(Nx*Ny))))] = v \in F \subseteq E.$$

Therefore

$$
\begin{aligned}
&N\big[Nx*(Ny*(Ny*(Nx*(Nx*Ny))))\big]*NN(Nx*Ny) \\
&\overset{\text{Lem. } 3.1(\text{ii})}{=} N\big[Nx*(Ny*(Ny*(Nx*(Nx*Ny))))\big]*(Nx*Ny) \\
&\overset{(\mathrm{I})}{=} N(Nx*(Nx*Ny))*(Ny*(Ny*(Nx*(Nx*Ny)))) \\
&= v\in E.
\end{aligned}
$$

Thus by Definition 2.4 (i) we obtain

$$N\big(Nx * (Ny * (Ny * (Nx * (Nx * Ny))))\big) \in E.$$

On the other hand, by Proposition 2.6 (iii), (IV) and (BCK-3),

$$\begin{aligned}
&N\big(Nx*(Ny*(Ny*(Nx*(Nx*Ny))))\big)*N\big(Nx*(Ny*(Ny*Nx))\big)\\
&\overset{\text{Prop. 2.6(iii)}}{\leq}\big(Nx*(Ny*(Ny*Nx))\big)*\big(Nx*(Ny*(Ny*(Nx*(Nx*Ny))))\big)\\
&\overset{\text{(IV)}}{\leq}\big(Ny * (Ny * (Nx * (Nx * Ny))))\big) * \big(Ny * (Ny * Nx)\big)\\
&\overset{\text{(IV)}}{\leq}(Ny * Nx) * \big(Ny * (Nx * (Nx * Ny))\big)\\
&\overset{\text{(IV)}}{\leq}(Nx * (Nx * Ny)) * Nx\\
&\overset{\text{(I)}}{\leq}(Nx * Nx) * (Nx * Ny)\\
&\overset{\text{(BCK-3)}}{\leq}0.
\end{aligned}$$

Hence by Theorem 2.5 (i), $N\big(Nx*(Ny*(Ny*Nx))\big) \in E$, and by Definition 5.1 the theorem holds. □

THEOREM 5.7. *Let X be a bounded BCK-algebra. Then the following statements are equivalent for all $x, y \in X$:*

(i) $\{1\}$ *is a commutative BCK-filter.*

(ii) Every BCK-filter is a commutative BCK-filter.

(iii) $Nx * (Nx * Ny) = Ny * (Ny * Nx)$.

(iv) $Ny * (Nx * (Nx * Ny)) = Ny * Nx$.

PROOF: $(i) \Leftrightarrow (ii)$: This follows directly from Lemma 5.6.

$(ii) \Rightarrow (iii)$: Suppose that $\{1\}$ is a commutative BCK-filter and let $v = N(Nx * (Nx * Ny))$. Then

$$\begin{aligned} N(Nv * Ny) &= N(NN(Nx*(Nx*Ny))*Ny) \\ &\overset{\text{Lem. 3.1(ii)}}{=} N((Nx*(Nx*Ny))*Ny) \\ &\overset{\text{(I)}}{=} N((Nx*Ny)*(Nx*Ny)) \\ &\overset{\text{(BCK-3)}}{=} N(0) \\ &\overset{\text{(BCK-3)}}{=} 1 \in \{1\}. \end{aligned}$$

Therefore, by the definition of a commutative BCK-filter,

$$N\big(Nv*(Ny*(Ny*Nv))\big) \in \{1\}. \tag{a}$$

By Lemma 3.1 (ii), $Nv \le Nx$. Hence by (II) we have $Ny*Nx \le Ny*Nv$. For the same reason, $Ny*(Ny*Nv) \le Ny*(Ny*Nx)$ and $Nv*(Ny*(Ny*Nx)) \le Nv*(Ny*(Ny*Nv))$. Consequently,

$$N\big(Nv*(Ny*(Ny*Nv))\big) \le N\big(Nv*(Ny*(Ny*Nx))\big).$$

Thus by (a) and Theorem 2.5 (i), $N\big(Nv*(Ny*(Ny*Nx))\big) \in \{1\}$. Hence $NN\big(Nv*(Ny*(Ny*Nx))\big) = Nv*(Ny*(Ny*Nx)) = 0$, and therefore $Nx*(Nx*Ny) = NN(Nx*(Nx*Ny)) = Nv \le Ny*(Ny*Nx)$. Similarly we obtain $Ny*(Ny*Nx) \le Nx*(Nx*Ny)$, so $Ny*(Ny*Nx) = Nx*(Nx*Ny)$.

(iii) $\Rightarrow$ *(iv)*: By (III) and the assumption,

$$Ny*(Nx*(Nx*Ny)) = Ny*(Ny*(Ny*Nx)) = Ny*Nx.$$

(iv) $\Rightarrow$ *(ii)*: Suppose F is a BCK-filter and $N(Nx*Ny) \in F$. Then by (iv), $N\big(Nx*(Ny*(Ny*Nx))\big) = N(Nx*Ny) \in F$, which completes the proof. □

THEOREM 5.8. *Let X be an involutory BCK-algebra. Then the following statements are equivalent for all $x, y \in X$:*

(i) $\{1\}$ is a commutative BCK-filter.

(ii) Every BCK-filter is a commutative BCK-filter.

(iii) $Ny * (Ny * Nx) = Nx * (Nx * Ny)$.

(iv) $Nx * (Ny * (Ny * Nx)) = Nx * Ny$.

(v) *X is a commutative BCK-algebra.*

Proof: The theorem follows from Theorem 5.7 and Definition 2.2 (ii). □

6. Conclusion

To investigate and characterize the structure of logical systems, the concepts of ideals and filters are very important. In this research we introduced and studied EQI-filters, implicative and commutative BCK-filters in bounded BCK-algebras. We determined the relations between these BCK-filters and surveyed them in subclasses of bounded BCK-algebras such as EQI-algebras and involutory BCK-algebras.

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