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CONSTRUCTING A HOOP USING ROUGH FILTERS

Abstract

When it comes to making decisions in vague problems, Rough is one of the best tools to help analyzers. So based on rough and hoop concepts, two kinds of approximations (Lower and Upper) for filters in hoops are defined, and then some properties of them are investigated by us. We prove that these approximations-lower and upper- are interior and closure operators, respectively. Also after defining a hyper operation in hoops, we show that by using this hyper operation, set of all rough filters is monoid. For more study, we define the implicative operation on the set of all rough filters and prove that this set with implication and intersection is made a hoop.

Keywords: Hoop, rough set, rough approximations (lower and upper), rough filter.

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1. Introduction

Pawlak proposed the theory of rough sets in 1982 as a new method for modelling and processing uncertain data. There are different fields such as machine learning, intelligence system, decision making, and etc, in which rough set theory can help to solve some problems. So it has received algebraic researchers attention too, and leads to apply rough set theory in different algebraic systems such as BCK-algebra [13], BCC-algebra [14], MV-algebra [17] and so on.

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Rough set theory includes different concepts some of them which are used in rough controllers are rough relations and rough functions. From algebraic point of view, Iwinski [11] is the first one who algebraically approach to the rough sets. In [16, 4], application of rough set can be seen in groups and semigroups. Till today, relation between rough theory and some algebras are studied, BCK-algebras by Jun [13], and MV-algebras by Rasouli and Davvaz [17]. Bosbach [8] introduced hoop algebra as special groups of monoids: naturally ordered commutative resituated integral monoids. In recent decades, many mathematicians have worked on it and developed structure theory by using the nation of hoop (see [3, 8]). Fuzzy logic and hoops have strong impact on each other results. One of the famous examples is the short proof of the completeness theorem for propositional basic logic introduced by Hájek in [10] which is obtained from the structure theorem of finite basic hoops. There are a lot of areas that hoops are being implemented for algebraic structures such as (see [1, 2, 5, 6, 7]). By considering the impact of rough set theory and since there was no study on the relation between hoop and rough set theory, we decided to apply the rough set theory in hoops. Experience of implementing soft set theory in hoops [6], and the logic used in [15] helped us a lot to have a better view. For this purpose, we defined the concept of the lower and the upper approximations in hoops and then investigated their properties. Also, it is proved that the lower (upper) approximations is an interior operator (closure operator). Moreover, we define a hyper operation on hoop and then we show that by using this operation, the set of all rough filters is a monoid. For more study, we define the implicative operation on the set of all rough filters and prove that this set with implication and intersection is made a hoop.

2. Preliminaries

Some definitions that may be required in the further discussions are reviewed in this part.
A *hoop* [8] is an algebraic structure $\hbar = (\hbar, \odot, \to, 1)$ of type $(2, 2, 0)$ such that, for all $\kappa, \nu, \delta \in \hbar$ the following conditions hold:

$(HP1)$ $(\hbar, \odot, 1)$ is a commutative monoid,

$(HP2)$ $\kappa \to \kappa = 1$,

$(HP3)$ $(\kappa \odot \nu) \to \delta = \kappa \to (\nu \to \delta)$,

$(HP4)$ $\kappa \odot (\kappa \to \nu) = \nu \odot (\nu \to \kappa)$.

A relation $\leq$ on hoop $\hbar$ which is defined by $\kappa \leq \nu$ if and only if $\kappa \to \nu = 1$, is a partial order relation on $\hbar$. A hoop $\hbar$ is called *bounded* if there is an element $0 \in \hbar$ such that $0 \leq \kappa$, for all $\kappa \in \hbar$ (see [8]).
Fundamental properties of hoops are provided in the next proposition.

Proposition 2.1 ([8]). Let $\hbar$ be a hoop. Then, for all $\kappa, \nu, \delta \in \hbar$ the following properties hold:

(i) $(\hbar, \leq)$ is a $\wedge$-semilattice with $\kappa \wedge \nu = \kappa \odot (\kappa \to \nu)$;

(ii) $\kappa \odot \nu \leq \delta$ if and only if $\kappa \leq \nu \to \delta$;

(iii) $\kappa \odot \nu \leq \kappa, \nu$;

(iv) $\kappa \leq \nu \to \kappa$;

(v) $1 \to \kappa = \kappa$;

(vi) $\kappa \to 1 = 1$;

(vii) $\nu \leq (\nu \to \kappa) \to \kappa$;

$(viii)$ $\kappa \leq (\kappa \to \nu) \to \kappa$;

(ix) $\kappa \to \nu \leq (\nu \to \delta) \to (\kappa \to \delta)$;

(x) $(\kappa \to \nu) \odot (\nu \to \delta) \leq \kappa \to \delta$;

(xi) $\kappa \leq \nu$ implies $\kappa \odot \delta \leq \nu \odot \delta$, $\delta \to \kappa \leq \delta \to \nu$ and $\nu \to \delta \leq \kappa \to \delta$.

Uninary operation "$\neg$" on a bounded hoop $\hbar$ is defined such that for any $\kappa \in \hbar, \neg\kappa = \kappa \to 0$.
Then for any nonempty subset R of a bounded hoop $\hbar$, consider the sets $\neg R := \{\neg\kappa \in \hbar \mid \kappa \in R\}$ and $DNP(\hbar) := \{\kappa \in \hbar \mid \neg(\neg\kappa) = \kappa\}$.

Double negation property (briefly, DNP of a bounded hoop $\hbar$ is when $DNP(\hbar) = \hbar$.

Proposition 2.2 ([8, 9]). Let $\hbar$ be a bounded hoop. Then, for any $\kappa, \nu \in \hbar$, the following conditions hold:

(i) $\kappa \leq \neg\neg\kappa$ and $\kappa \odot \neg\kappa = 0$

(ii) $\neg\kappa \leq \kappa \to \nu$.

(iii) $\neg\neg\neg\kappa = \neg\kappa$.

(iv) If $\hbar$ has (DNP), then $\kappa \to \nu = \neg\nu \to \neg\kappa$.

(v) If $\hbar$ has (DNP), then $(\kappa \to \nu) \to \nu = (\nu \to \kappa) \to \kappa$.

Let ϱ be an equivalence relation on a hoop $\hbar$ and $\mathcal{P}(\hbar)$ denote the power set of $\hbar$. For all $\kappa \in \hbar$, let $[\kappa]_\varrho$ denote the equivalence class of κ with respect to ϱ. Let ϱ_* and ϱ^* be mappings from $\mathcal{P}(\hbar)$ to $\mathcal{P}(\hbar)$ defined by $\varrho_*(F) = \{\kappa \in \hbar \mid [\kappa]_\varrho \subseteq F\}$ and $\varrho^*(F) = \{\kappa \in \hbar \mid [\kappa]_\varrho \cap F \neq \emptyset\}$, respectively.
The pair $(\hbar, \varrho)$ is called an *approximation space based on* ϱ. A subset F of a hoop $\hbar$ is *definable* if $\varrho_*(F) = \varrho^*(F)$, and *rough* otherwise. The set $\varrho_*(F)$ (resp. $\varrho^*(F)$) is called the *lower* (resp. *upper*) *approximation*. (See [14])

Proposition 2.3 ([14]). Let $(\hbar, \varrho)$ be a ϱ-approximation space. For any $R, M \in \mathcal{P}(\hbar)$, we have

(i) $\varrho_*(R) \subseteq R \subseteq \varrho^*(R)$,

(ii) $\varrho_*(R \cap M) = \varrho_*(R) \cap \varrho_*(M)$,

(iii) $\varrho_*(R) \cup \varrho_*(M) \subseteq \varrho_*(R \cup M)$,

(iv) $\varrho^*(R \cap M) \subseteq \varrho^*(R) \cap \varrho^*(M)$,

(v) $\varrho^*(R) \cup \varrho^*(M) = \varrho^*(R \cup M)$.

(vi) $\varrho_*(\varrho^*(R)) \subseteq \varrho^*(\varrho^*(R))$,

(vii) $\varrho_*(\varrho_*(R)) \subseteq \varrho^*(\varrho_*(R))$,

$(viii)$ $\varrho_*(R^c) = (\varrho^*(R))^c$,

(ix) $\varrho^*(R^c) = (\varrho_*(R))^c$,

(x) $\varrho_*(R) = \emptyset$ for $R \neq \hbar$,

(xi) $\varrho^*(R) = R$ for $R \neq \emptyset$.

(xii) $\varrho_*(R) = R \Leftrightarrow \varrho^*(R^c) = R^c$.

Function $\mathsf{C} : \mathcal{P}(S) \to \mathcal{P}(S)$ on a set S is a *closure operator* [12] if the following conditions are held for all subsets $X, Y \subseteq S$:

(i) $X \subseteq \mathsf{C}(X)$,

(ii) if $X \subseteq Y$, then $\mathsf{C}(X) \subseteq \mathsf{C}(Y)$,

(iii) $\mathsf{C}(\mathsf{C}(X)) = \mathsf{C}(X)$.

Function $\mathfrak{T} : \mathcal{P}(S) \to \mathcal{P}(S)$ on a set S is an *interior operator* [12] in which for all subsets $X, Y \subseteq S$ the following conditions are held:

(i) $\mathfrak{T}(X) \subseteq X$,

(ii) if $X \subseteq Y$, then $\mathfrak{T}(X) \subseteq \mathfrak{T}(Y)$,

(iii) $\mathfrak{T}(\mathfrak{T}(X)) = \mathfrak{T}(X)$.

3. Roughness of filters in hoops

In this section, roughness of hoops is introduced and some properties of it are investigated. Soppose $\digamma$ is a filter of a hoop $\hbar$. We define a relation "$\mathcal{C}_\digamma$" on $\hbar$ for any $\kappa, \nu \in \hbar$ as follows:

$$(\kappa, \nu) \in \mathcal{C}_\digamma \text{ if and only if } \kappa \to \nu \in \digamma \text{ and } \nu \to \kappa \in \digamma.$$

Then $\mathcal{C}_\digamma$ is a congruence relation on $\hbar$. Hence approximation space $(\hbar, \mathcal{C}_\digamma)$ is called an $\digamma$-*approximation space*. The equivalence class of $\kappa \in \hbar$ under $\mathcal{C}_\digamma$ is denoted by $\mathcal{C}_\digamma[\kappa]$.

Let $(\hbar, \mathcal{C}_\digamma)$ be an $\digamma$-approximation space. For any nonempty subset R of $\hbar$, the sets

$$\underline{\mathcal{C}}_\digamma(R) := \{\kappa \in \hbar \mid \mathcal{C}_\digamma[\kappa] \subseteq R\} \text{ and } \overline{\mathcal{C}}_\digamma(R) := \{\kappa \in \hbar \mid \mathcal{C}_\digamma[\kappa] \cap R \neq \emptyset\},$$

are called *lower* and *upper rough approximation*, respectively, of R with respect to the filter $\digamma$.

Example 3.1. Let $\hbar = \{0, \eta, \beta, 1\}$ be a poset such that $0 \leq \eta, \beta \leq 1$. Define the operations $\rightarrow$ and $\odot$ on $\hbar$ as follows,

$\rightarrow$	0	η	β	1
0	1	1	1	1
η	β	1	β	1
β	η	η	1	1
1	0	η	β	1

$\odot$	0	η	β	1
0	0	0	0	0
η	0	η	0	η
β	0	0	β	β
1	0	η	β	1

Then $(\hbar, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Let $\digamma = \{\eta, 1\}$. Then $\mathcal{C}_{\digamma}[\eta] = \mathcal{C}_{\digamma}[1] = \digamma$ and $\mathcal{C}_{\digamma}[\beta] = \mathcal{C}_{\digamma}[0] = \{0, \beta\}$. Suppose $R = \{\eta, \beta, 1\}$. Then $\underline{\mathcal{C}}_{\digamma}(R) = \{\eta, 1\}$ and $\overline{\mathcal{C}}_{\digamma}(R) = \hbar$.

THEOREM 3.2. *If $(\hbar, \mathcal{C}_{\digamma})$ is an $\digamma$-approximation space, then the lower rough approximation operator $\underline{\mathcal{C}}_{\digamma}$ is an interior operator and the upper rough approximation operator $\overline{\mathcal{C}}_{\digamma}$ is a closure operator.*

PROOF: Let R be a nonempty subset of $\hbar$ and $\kappa \in \underline{\mathcal{C}}_{\digamma}(R)$. Then $\mathcal{C}_{\digamma}[\kappa] \subseteq R$. Since $\kappa \in \mathcal{C}_{\digamma}[\kappa]$, we have $\kappa \in R$. Hence, $\underline{\mathcal{C}}_{\digamma}(R) \subseteq R$. If R_1 and R_2 are two subsets of $\hbar$ such that $R_1 \subseteq R_2$ and $\kappa \in \underline{\mathcal{C}}_{\digamma}(R_1)$, then $\mathcal{C}_{\digamma}[\kappa] \subseteq R_1$. Thus $\mathcal{C}_{\digamma}[\kappa] \subseteq R_2$, and so $\kappa \in \underline{\mathcal{C}}_{\digamma}(R_2)$. Hence, $\underline{\mathcal{C}}_{\digamma}(R_1) \subseteq \underline{\mathcal{C}}_{\digamma}(R_2)$. Since $\underline{\mathcal{C}}_{\digamma}(R) \subseteq R$, we have $\underline{\mathcal{C}}_{\digamma}(\underline{\mathcal{C}}_{\digamma}(R)) \subseteq \underline{\mathcal{C}}_{\digamma}(R)$. Conversely, suppose $\kappa \in \underline{\mathcal{C}}_{\digamma}(R)$. Then $\mathcal{C}_{\digamma}[\kappa] \subseteq R$. Let $\delta \in \mathcal{C}_{\digamma}[\kappa]$. Then $\mathcal{C}_{\digamma}[\delta] = \mathcal{C}_{\digamma}[\kappa] \subseteq R$, and so $\delta \in \underline{\mathcal{C}}_{\digamma}(R)$. Thus, $\mathcal{C}_{\digamma}[\kappa] \subseteq \underline{\mathcal{C}}_{\digamma}(R)$. Hence, $\kappa \in \underline{\mathcal{C}}_{\digamma}(\underline{\mathcal{C}}_{\digamma}(R))$, and so $\underline{\mathcal{C}}_{\digamma}(\underline{\mathcal{C}}_{\digamma}(R)) = \underline{\mathcal{C}}_{\digamma}(R)$. Therefore, the lower rough approximation operator $\underline{\mathcal{C}}_{\digamma}$ is an interior operator.

Let R be a nonempty subset of $\hbar$ and $\kappa \in R$. Since $\kappa \in \mathcal{C}_{\digamma}[\kappa]$, we have $\kappa \in \mathcal{C}_{\digamma}[\kappa] \cap R \neq \emptyset$. Thus $\kappa \in \overline{\mathcal{C}}_{\digamma}(R)$. If R_1 and R_2 are two subsets of $\hbar$ such that $R_1 \subseteq R_2$ and $\kappa \in \overline{\mathcal{C}}_{\digamma}(R_1)$. Then $\mathcal{C}_{\digamma}[\kappa] \cap R_1 \neq \emptyset$. Thus $\mathcal{C}_{\digamma}[\kappa] \cap R_2 \neq \emptyset$, and so $\kappa \in \overline{\mathcal{C}}_{\digamma}(R_2)$. Hence, $\overline{\mathcal{C}}_{\digamma}(R_1) \subseteq \overline{\mathcal{C}}_{\digamma}(R_2)$. Since $R \subseteq \overline{\mathcal{C}}_{\digamma}(R)$, we have $\overline{\mathcal{C}}_{\digamma}(R) \subseteq \overline{\mathcal{C}}_{\digamma}(\overline{\mathcal{C}}_{\digamma}(R))$. Conversely, suppose $\kappa \in \overline{\mathcal{C}}_{\digamma}(\overline{\mathcal{C}}_{\digamma}(R))$. Then $\mathcal{C}_{\digamma}[\kappa] \cap \overline{\mathcal{C}}_{\digamma}(R) \neq \emptyset$. Let $\delta \in \mathcal{C}_{\digamma}[\kappa] \cap \overline{\mathcal{C}}_{\digamma}(R)$. Then $\mathcal{C}_{\digamma}[\delta] = \mathcal{C}_{\digamma}[\kappa]$ and $\mathcal{C}_{\digamma}[\delta] \cap R \neq \emptyset$, and so $\mathcal{C}_{\digamma}[\kappa] \cap R \neq \emptyset$. Thus, $\kappa \in \overline{\mathcal{C}}_{\digamma}(R)$. Hence, $\overline{\mathcal{C}}_{\digamma}(\overline{\mathcal{C}}_{\digamma}(R)) = \overline{\mathcal{C}}_{\digamma}(R)$. Therefore, the upper rough approximation operator $\overline{\mathcal{C}}_{\digamma}$ is a closure operator. □

Let $(\hbar, \mathcal{C}_{\digamma})$ be an $\digamma$-approximation space. A subset R of $\hbar$ is said to be *definable* with respect to $\digamma$ if $\underline{\mathcal{C}}_{\digamma}(R) = \overline{\mathcal{C}}_{\digamma}(R)$, and *rough* otherwise.

It is clear that $\emptyset$, $\hbar$ and $\mathcal{C}_{F}[\kappa]$ are definable with respect to F in an F-approximation space $(\hbar, \mathcal{C}_{F})$.

Example 3.3. Let $\hbar$ be a hoop as in Example 3.1 and $F = \{\eta, 1\}$. Suppose $R = \{0, \beta\}$. Then $\overline{\mathcal{C}}_{F}(R) = \underline{\mathcal{C}}_{F}(R) = \{0, \beta\}$. Hence R is definable.

THEOREM 3.4. *If $(\hbar, \mathcal{C}_{F})$ is an F-approximation space with $F = \{1\}$, then every subset of $\hbar$ is definable with respect to F.*

PROOF: Let R be an arbitrary subset of $\hbar$. Since $F = \{1\}$, for all $\kappa \in \hbar$ we have

$$\mathcal{C}_{F}[\kappa] = \{\delta \in \hbar \mid \kappa \to \delta = 1, \delta \to \kappa = 1\} = \{\delta \in \hbar \mid \kappa = \delta\} = \{\kappa\}.$$

Thus,

$$\underline{\mathcal{C}}_{F}(R) = \{\kappa \in \hbar \mid \mathcal{C}_{F}[\kappa] \subseteq R\} = \{\kappa \in \hbar \mid \{\kappa\} \subseteq R\} = R,$$
$$\overline{\mathcal{C}}_{F}(R) = \{\kappa \in \hbar \mid \mathcal{C}_{F}[\kappa] \cap R \neq \emptyset\} = \{\kappa \in \hbar \mid \{\kappa\} \cap R \neq \emptyset\} = R.$$

Therefore, R is definable with respect to F. □

For any subsets R and P of a hoop $\hbar$, we define:

$$R \to P = \{\kappa \to \nu \mid \kappa \in R \textit{ and } \nu \in P\},$$
$$R \odot P = \{\kappa \odot \nu \mid \kappa \in R \ \textit{ and } \ \nu \in P\}.$$

PROPOSITION 3.5. If $(\hbar, \mathcal{C}_{F})$ is an F-approximation space, then $\overline{\mathcal{C}}_{F}(R) \to \overline{\mathcal{C}}_{F}(P) \subseteq \overline{\mathcal{C}}_{F}(R \to P)$ and $\overline{\mathcal{C}}_{F}(R) \odot \overline{\mathcal{C}}_{F}(P) \subseteq \overline{\mathcal{C}}_{F}(R \odot P)$ for any nonempty subsets R and P of a hoop $\hbar$.

PROOF: If $\delta \in \overline{\mathcal{C}}_{F}(R) \to \overline{\mathcal{C}}_{F}(P)$, then $\delta = \mathfrak{a} \to \mathfrak{b}$ for some $\mathfrak{a} \in \overline{\mathcal{C}}_{F}(R)$ and $\mathfrak{b} \in \overline{\mathcal{C}}_{F}(P)$. It follows that $\mathcal{C}_{F}[\mathfrak{a}] \cap R \neq \emptyset$ and $\mathcal{C}_{F}[\mathfrak{b}] \cap P \neq \emptyset$. Hence, there exist $\kappa \in R$ and $\nu \in P$ such that $\mathcal{C}_{F}[\mathfrak{a}] = \mathcal{C}_{F}[\kappa]$ and $\mathcal{C}_{F}[\mathfrak{b}] = \mathcal{C}_{F}[\nu]$. Since

$$\delta = \mathfrak{a} \to b \in \mathcal{C}_{F}[\mathfrak{a}] \to \mathcal{C}_{F}[\mathfrak{b}] = \mathcal{C}_{F}[\mathfrak{a} \to \mathfrak{b}] = \mathcal{C}_{F}[\kappa \to \nu],$$

we get $\mathcal{C}[\delta] = \mathcal{C}[\kappa \to \nu]$. Moreover since $\kappa \to \nu \in R \to P$ and $\mathcal{C}[\delta] = \mathcal{C}[\kappa \to \nu]$, we get $\mathcal{C}[\delta] \cap (R \to P) \neq \emptyset$, and so $\delta \in \overline{\mathcal{C}}_{F}(R \to P)$. Similarly, we can verify $\overline{\mathcal{C}}_{F}(R) \odot \overline{\mathcal{C}}_{F}(P) \subseteq \overline{\mathcal{C}}_{F}(R \odot P)$. □

DEFINITION 3.6. Let $(\hbar, \mathcal{C}_{F})$ be an F-approximation space. A subset R of $\hbar$ is called a lower (resp. upper) rough filter of $\hbar$ if $\underline{\mathcal{C}}_{F}(R)$ (resp., $\overline{\mathcal{C}}_{F}(R)$)

is a filter of $\hbar$. If R is both a lower rough filter and an upper rough filter of $\hbar$, we say R is a *rough filter* of $\hbar$.

Example 3.7. Let $\hbar$ be a hoop as in Example 3.1. Suppose $\mathcal{F} = \{\beta, 1\}$. Then $\mathcal{F}$ is a filter of $\hbar$. $\mathcal{C}_{\mathcal{F}}[1] = \mathcal{C}_{\mathcal{F}}[\beta] = \{\beta, 1\}$ and $\mathcal{C}_{\mathcal{F}}[0] = \mathcal{C}_{\mathcal{F}}[\eta] = \{0, \eta\}$. If $R = \{\eta, \beta, 1\}$, then $\underline{\mathcal{C}}_{\mathcal{F}}[R] = \{\beta, 1\}$ and $\overline{\mathcal{C}}_{\mathcal{F}}[R] = \hbar$. Hence, R is a rough filter of $\hbar$. If $R = \{\eta, 1\}$ which is a filter of $\hbar$, then $\underline{\mathcal{C}}_{\mathcal{F}}[R] = \emptyset$ and $\overline{\mathcal{C}}_{\mathcal{F}}[R] = \hbar$. Hence R is not a rough filter of $\hbar$.

THEOREM 3.8. *If $(\hbar, \mathcal{C}_{\mathcal{F}})$ is an $\mathcal{F}$-approximation space and R is a nonempty subset of $\hbar$, then*

- (*i*) *$\mathcal{F} \subseteq R$ if and only if $\mathcal{F} \subseteq \underline{\mathcal{C}}_{\mathcal{F}}(R)$.*
- (*ii*) *$R \subseteq \mathcal{F}$ if and only if $\overline{\mathcal{C}}_{\mathcal{F}}(R) = \mathcal{F}$.*
- (*iii*) *If G is a filter of $\hbar$, then $\mathcal{F} \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(G)$. Also, $\mathcal{F} \subseteq G$ if and only if $\underline{\mathcal{C}}_{\mathcal{F}}(G) = G = \overline{\mathcal{C}}_{\mathcal{F}}(G)$.*
- (*iv*) *Every filter which is contained in $\mathcal{F}$ is an upper rough filter of $\hbar$.*

PROOF: (*i*) Assume that $\mathcal{F} \subseteq R$ and $\delta \in \mathcal{F}$. Then $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{F} \subseteq R$ and so $\delta \in \underline{\mathcal{C}}_{\mathcal{F}}(R)$, that is, $\mathcal{F} \subseteq \underline{\mathcal{C}}_{\mathcal{F}}(R)$. The converse is clear.

(*ii*) By Proposition 2.3(i), it is clear that if $\overline{\mathcal{C}}_{\mathcal{F}}(R) = \mathcal{F}$, then $R \subseteq \mathcal{F}$. Suppose $R \subseteq \mathcal{F}$ and $\delta \in \overline{\mathcal{C}}_{\mathcal{F}}(R)$. Then $\mathcal{C}_{\mathcal{F}}[\delta] \cap R \neq \emptyset$. Thus $\kappa \in \mathcal{C}_{\mathcal{F}}[\delta] \cap L$. Since $L \subseteq \mathcal{F}$, we have $\kappa \in \mathcal{F}$ and $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{C}_{\mathcal{F}}[\kappa] = \mathcal{F}$. Thus $\delta \in \mathcal{F}$, which shows that $\overline{\mathcal{C}}_{\mathcal{F}}(R) \subseteq \mathcal{F}$. Now, if $\delta \in \mathcal{F}$, then $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{F}$ and so $\mathcal{C}_{\mathcal{F}}[\delta] \cap R = \mathcal{F} \cap R = R \neq \emptyset$. Hence $\delta \in \overline{\mathcal{C}}_{\mathcal{F}}(R)$ and so $F \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(R)$. Therefore, $\overline{\mathcal{C}}_{\mathcal{F}}(R) = \mathcal{F}$.

(*iii*) Let G be a filter of $\hbar$. If $\nu \in \mathcal{F}$, then $\mathcal{C}_{\mathcal{F}}[\nu] = \mathcal{F}$ and $1 \in \mathcal{F} \cap G = \mathcal{C}_{\mathcal{F}}[\nu] \cap G$ and so $\nu \in \overline{\mathcal{C}}_{\mathcal{F}}(G)$. Hence $\mathcal{F} \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(G)$. Assume that $\mathcal{F} \subseteq G$. By Proposition 2.3(i), it is clear that $G \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(G)$ and $\underline{\mathcal{C}}_{\mathcal{F}}(G) \subseteq G$. If $\delta \in \overline{\mathcal{C}}_{\mathcal{F}}(G)$, then $\mathcal{C}_{\mathcal{F}}[\delta] \cap G \neq \emptyset$. Hence $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{C}_{\mathcal{F}}[\kappa]$, for some $\kappa \in G$. It follows that $\delta \to \kappa \in \mathcal{F} \subseteq G$ and $\kappa \to \delta \in \mathcal{F} \subseteq G$. Since G is a filter of $\hbar$ and $\kappa \in G$, we have $\delta \in G$ and so $G = \overline{\mathcal{C}}_{\mathcal{F}}(G)$. Let $\nu \in G$. If $\mathfrak{a} \in \mathcal{C}_{\mathcal{F}}[\nu]$, then $\mathfrak{a} \to \nu, \nu \to \mathfrak{a} \in \mathcal{F} \subseteq G$. Since G is a filter of $\hbar$, it follows that $\mathfrak{a} \in G$, and so $\mathcal{C}_{\mathcal{F}}[\nu] \subseteq G$ and $\nu \in \underline{\mathcal{C}}_{\mathcal{F}}(G)$. Thus $\underline{\mathcal{C}}_{\mathcal{F}}(G) = G$. Conversely, suppose $\underline{\mathcal{C}}_{\mathcal{F}}(G) = G = \overline{\mathcal{C}}_{\mathcal{F}}(G)$ and $\nu \in \mathcal{F}$. Since $1 \in \mathcal{C}_{\mathcal{F}}[\nu] \cap G = \mathcal{F} \cap G$, we have $\nu \in \overline{\mathcal{C}}_{\mathcal{F}}(G) = G$. Thus $\mathcal{F} \subseteq G$.

(*iv*) It is clear by (ii). □

The following corollary is obtained from Theorem 3.8.

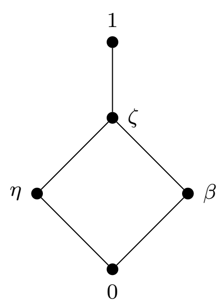
COROLLARY 3.9. In an $\mathcal{F}$-approximation space $(\hbar, \mathcal{C}_{\mathcal{F}})$, every filter containing $\mathcal{F}$ is a rough filter of $\hbar$ and every nonempty subset contained in $\mathcal{F}$ is an upper rough filter of $\hbar$.

PROPOSITION 3.10. Let $(\hbar, \mathcal{C}_{\mathcal{F}})$ be an $\mathcal{F}$-approximation space in which $\hbar$ is bounded. Then the upper rough approximation operator $\overline{\mathcal{C}}_{\mathcal{F}}$ satisfies $\neg\overline{\mathcal{C}}_{\mathcal{F}}(R) \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(\neg R)$ for all nonempty subset R of $\hbar$.

PROOF: Let $\nu \in \neg\overline{\mathcal{C}}_{\mathcal{F}}(R)$. Then $\nu = \neg\delta$ for some $\delta \in \hbar$ such that $\mathcal{C}_{\mathcal{F}}[\delta] \cap R \neq \emptyset$. Hence there exists $\kappa \in R$ such that $\mathcal{C}_{\mathcal{F}}[\delta] = \mathcal{C}_{\mathcal{F}}[\kappa]$, which implies that $\mathcal{C}_{\mathcal{F}}[\nu] = \mathcal{C}_{\mathcal{F}}[\neg\delta] = \mathcal{C}_{\mathcal{F}}[\neg\kappa]$. Since $\neg\kappa \in \neg R$, we get $\mathcal{C}_{\mathcal{F}}[\nu] \cap \neg R = \mathcal{C}_{\mathcal{F}}[\neg\kappa] \cap \neg R \neq \emptyset$. Hence $\nu \in \overline{\mathcal{C}}_{\mathcal{F}}(\neg R)$. Therefore, $\neg\overline{\mathcal{C}}_{\mathcal{F}}(R) \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(\neg R)$. $\square$

Now by below example we show that the reverse inclusion in Proposition 3.10 is not true, in general.

Example 3.11. Let $\hbar = \{0, \eta, \beta, \zeta, 1\}$ be a poset with the following Hasse diagram. Define the operations $\odot$ and $\to$ on $\hbar$ as follows,



$\to$	0	η	β	ζ	1
0	1	1	1	1	1
η	β	1	β	1	1
β	η	η	1	1	1
ζ	0	η	β	1	1
1	0	η	β	ζ	1

$\odot$	0	η	β	ζ	1
0	0	0	0	0	0
η	0	η	0	η	η
β	0	0	β	β	β
ζ	0	η	β	ζ	ζ
1	0	η	β	ζ	1

Then $(\hbar, \odot, \to, 0, 1)$ is a bounded hoop. Suppose $\mathcal{F} = \{\zeta, 1\}$. Then $\mathcal{C}_{\mathcal{F}}[\zeta] = \mathcal{C}_{\mathcal{F}}[1] = \{\zeta, 1\}$, $\mathcal{C}_{\mathcal{F}}[\eta] = \{\eta\}$, $\mathcal{C}_{\mathcal{F}}[\beta] = \{\beta\}$ and $\mathcal{C}_{\mathcal{F}}[0] = \{0\}$.

Thus $\overline{\mathcal{C}}_{\digamma}(\hbar) = \hbar$ and $\neg\overline{\mathcal{C}}_{\digamma}(\hbar) = \{0, \eta, \beta, 1\}$. Also, $\neg\hbar = \{0, \eta, \beta, 1\}$ and so $\overline{\mathcal{C}}_{\digamma}(\neg\hbar) = \hbar$. Hence, $\overline{\mathcal{C}}_{\digamma}(\neg\hbar) \nsubseteq \neg\overline{\mathcal{C}}_{\digamma}(\hbar)$.

In the following example we show that lower rough approximation operator $\underline{\mathcal{C}}_{\digamma}$ does not satisfies in the condition of Proposition 3.10.

Example 3.12. Let $\hbar$ be the hoop as in Example 3.11 and $R = \{\beta, 1\}$. Then $\underline{\mathcal{C}}_{\digamma}(R) = \{\beta\}$ and so $\neg\underline{\mathcal{C}}_{\digamma}(R) = \{\eta\}$. Moreover, $\neg R = \{0, \eta\}$ and so $\underline{\mathcal{C}}_{\digamma}(\neg R) = \{0, \eta\}$. Hence, $\underline{\mathcal{C}}_{\digamma}(\neg R) \nsubseteq \neg\underline{\mathcal{C}}_{\digamma}(R)$. Also, if $R = \{0, \eta, \beta\}$, then $\underline{\mathcal{C}}_{\digamma}(R) = \{0, \eta, \beta\}$ and so $\neg\underline{\mathcal{C}}_{\digamma}(R) = \{\eta, \beta, 1\}$. Moreover, $\neg R = \{\eta, \beta, 1\}$. Then $\underline{\mathcal{C}}_{\digamma}(\neg R) = \{\eta, \beta\}$. Hence, $\neg\underline{\mathcal{C}}_{\digamma}(R) \nsubseteq \underline{\mathcal{C}}_{\digamma}(\neg R)$. Therefore, lower rough approximation operator $\underline{\mathcal{C}}_{\digamma}$ does not satisfies in the condition of Proposition 3.10.

Proposition 3.13. If $(\hbar, \mathcal{C}_{\digamma})$ is an $\digamma$-approximation space and R is a nonempty subset of $\hbar$, then

(i) $DNP(\hbar) \cap \overline{\mathcal{C}}_{\digamma}(\neg R) \subseteq \neg\overline{\mathcal{C}}_{\digamma}(\neg(\neg R))$.

(ii) $DNP(\hbar) \cap \overline{\mathcal{C}}_{\digamma}(\neg(R \cap DNP(\hbar))) \subseteq \neg\overline{\mathcal{C}}_{\digamma}(R)$.

Proof: (i) If $\kappa \in DNP(\hbar) \cap \overline{\mathcal{C}}_{\digamma}(\neg R)$, then $\neg(\neg\kappa) = \kappa$ and since $\kappa \in \overline{\mathcal{C}}_{\digamma}(\neg R)$, there exists $\nu \in R$ such that $\mathcal{C}_{\digamma}[\kappa] = \mathcal{C}_{\digamma}[\neg\nu]$.

It follows that $\mathcal{C}_{\digamma}[\neg\kappa] \cap \neg(\neg R) = \mathcal{C}_{\digamma}(\neg[\neg\nu]) \cap \neg(\neg R) \neq \emptyset$, that is, $\neg\kappa \in \overline{\mathcal{C}}_{\digamma}(\neg(\neg R))$. Hence $\kappa \in \neg\overline{\mathcal{C}}_{\digamma}(\neg(\neg R))$. Therefore, $DNP(\hbar) \cap \overline{\mathcal{C}}_{\digamma}(\neg R) \subseteq \neg\overline{\mathcal{C}}_{\digamma}(\neg(\neg R))$.

(ii) Let $\delta \in DNP(\hbar) \cap \overline{\mathcal{C}}_{\digamma}(\neg(R \cap DNP(\hbar)))$. Then $\neg(\neg\delta) = \delta$ and $\mathcal{C}_{\digamma}[\delta] \cap \neg(R \cap DNP(\hbar)) \neq \emptyset$. Thus there exists $\kappa \in \mathcal{C}_{\digamma}[\delta] \cap \neg(R \cap DNP(\hbar))$, it means that, $\mathcal{C}_{\digamma}[\delta] = \mathcal{C}_{\digamma}[\kappa]$ and there exists $\nu \in R \cap DNP(\hbar)$ such that $\kappa = \neg\nu$ and so $\mathcal{C}_{\digamma}[\delta] = \mathcal{C}_{\digamma}[\neg\nu]$. Then $\mathcal{C}_{\digamma}[\neg\delta] \cap R = \mathcal{C}_{\digamma}[\neg(\neg\nu)] \cap R = \mathcal{C}_{\digamma}[\nu] \cap R \neq \emptyset$, that is, $\delta \in \neg\overline{\mathcal{C}}_{\digamma}(R)$. Therefore, $DNP(\hbar) \cap \overline{\mathcal{C}}_{\digamma}(\neg(R \cap DNP(\hbar))) \subseteq \neg\overline{\mathcal{C}}_{\digamma}(R)$. □

Proposition 3.14. If $\hbar$ is a bounded hoop, then the set $\hbar^{\star} := \{\kappa \in \hbar \mid \neg\kappa = 0\}$ is a filter of $\hbar$.

Proof: Since $\neg 1 = 0$, we have $1 \in \hbar^{\star}$. Consider $\kappa, \nu \in \hbar$ so that $\kappa, \kappa \to \nu \in \hbar^{\star}$. Then $\neg\kappa = 0$ and $\neg(\kappa \to \nu) = 0$. Considering Proposition 2.2(i) and $\nu \leq \neg\neg\nu$, we get $\kappa \to \nu \leq \kappa \to \neg\neg\nu = \neg\nu \to \neg\kappa$. Hence

$$\neg\nu = \neg\neg\neg\nu = \neg(\neg\nu \to 0) = \neg(\neg\nu \to \neg\kappa) \leq \neg(\kappa \to \nu) = 0,$$

and so $\neg\nu = 0$, that is, $\nu \in \hbar^{\star}$. So it is proved that $\hbar^{\star}$ is a filter of $\hbar$. □

Corollary 3.15. If $(\hbar, \mathcal{C}_F)$ is an F-approximation space in which $\hbar$ is bounded, then

$$F \subseteq \overline{\mathcal{C}}_F(\hbar^\bigstar) \subseteq F^\bigstar,$$

where $F^\bigstar := \{\delta \in \hbar \mid \neg(\neg\delta) \in F\}$.

Proof: By Theorem 3.8(iii) and Proposition 3.14, we know that $F \subseteq \overline{\mathcal{C}}_F(\hbar^\bigstar)$. Let $\kappa \in \overline{\mathcal{C}}_F(\hbar^\bigstar)$. Then $\mathcal{C}_F[\kappa] \cap \hbar^\bigstar \neq \emptyset$, which implies that there exists $\mathfrak{a} \in \mathcal{C}_F[\kappa]$ such that $\neg\mathfrak{a} = 0$. Thus $\mathcal{C}_F[0] = \mathcal{C}_F[\neg\mathfrak{a}] = \mathcal{C}_F[\neg\kappa]$, and so $\neg(\neg\kappa) = \neg\kappa \to 0 \in F$, i.e., $\kappa \in F^\bigstar$. □

We provide conditions for a nonempty subset to be definable with respect to a filter of $\hbar$.

Theorem 3.16. *Let $(\hbar, \mathcal{C}_F)$ be an F-approximation space. Then a nonempty subset R of $\hbar$ is definable with respect to F if and only if $\underline{\mathcal{C}}_F(R) = R$ or $\overline{\mathcal{C}}_F(R) = R$.*

Proof: The necessity is clear. Conversely, suppose $\underline{\mathcal{C}}_F(R) = R$. By Proposition 2.3(i), it is clear that $R \subseteq \overline{\mathcal{C}}_F(R)$. Suppose $\kappa \in \overline{\mathcal{C}}_F(R)$. Then $\mathcal{C}_F[\kappa] \cap L \neq \emptyset$. Thus, there exists $\nu \in \mathcal{C}_F[\kappa] \cap L$, such that $\mathcal{C}_F[\kappa] = \mathcal{C}_F[\nu]$. Since $\nu \in R$ and $\underline{\mathcal{C}}_F(R) = R$, we have $\nu \in \underline{\mathcal{C}}_F(R)$. Then $\mathcal{C}_F[\nu] \subseteq R$. Thus, $\mathcal{C}_F[\kappa] \subseteq R$, and so $\kappa \in R$. Hence, $\overline{\mathcal{C}}_F(R) \subseteq R$ and so $\overline{\mathcal{C}}_F(R) = R$. Therefore, $\underline{\mathcal{C}}_F(R) = R = \overline{\mathcal{C}}_F(R)$ and R is definable. Now, assume that $\overline{\mathcal{C}}_F(R) = R$. Obviously, $\underline{\mathcal{C}}_F(R) \subseteq R$. For any $\kappa \in R$, let $\delta \in \mathcal{C}_F[\kappa]$. Then $\mathcal{C}_F[\delta] \cap R = \mathcal{C}_F[\kappa] \cap R \neq \emptyset$ and so $\delta \in \overline{\mathcal{C}}_F(R) = R$. Hence $\mathcal{C}_F[\kappa] \subseteq R$, i.e., $\kappa \in \underline{\mathcal{C}}_F(R)$. Then $\underline{\mathcal{C}}_F(R) = R = \overline{\mathcal{C}}_F(R)$. Therefore, R is definable with respect to F. □

Theorem 3.17. *Let F and G be two filters of a hoop $\hbar$. For any nonempty subset F of a hoop $\hbar$, we have*

- (*i*) *If $R \subseteq F \cap G$, then $\overline{\mathcal{C}}_{F\cap G}(R) = \overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R)$.*
- (*ii*) *If R is definable with respect to F, then $\overline{\mathcal{C}}_{F\cap G}(R) = \overline{\mathcal{C}}_F(R) \cap \overline{\mathcal{C}}_G(R)$.*
- (*iii*) *If R contains F and G, then $\underline{\mathcal{C}}_{F\cap G}(R) = \underline{\mathcal{C}}_F(R) \cap \underline{\mathcal{C}}_G(R)$.*

Proof: (*i*) Let $\kappa \in \overline{\mathcal{C}}_{F\cap G}(R)$. Then $\mathcal{C}_{F\cap G}[\kappa] \cap R \neq \emptyset$. Thus there exists $\mathfrak{a} \in \mathcal{C}_{F\cap G}[\kappa] \cap R$. Since $\mathfrak{a} \in R$ and $R \subseteq F \cap G$, we get $\mathfrak{a} \in F$ and $\mathfrak{a} \in G$. Moreover, from $\mathfrak{a} \in \mathcal{C}_{F\cap G}[\kappa]$, we get $\mathfrak{a} \to \kappa, \kappa \to \mathfrak{a} \in F \cap G$.

Since $\digamma$ and G are two filters of $\hbar$, we have $\kappa \in \digamma$ and $\kappa \in G$. Then $\mathcal{C}_\digamma[\kappa] = \mathcal{C}_\digamma[\mathfrak{a}]$, and so $\mathcal{C}_\digamma[\kappa] \cap R \neq \emptyset$. By the similar way, $\mathcal{C}_G[\kappa] \cap R \neq \emptyset$. Hence, $\kappa \in \overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R)$. Therefore, $\overline{\mathcal{C}}_{\digamma\cap G}(R) \subseteq \overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R)$.

Conversely, suppose $\kappa \in \overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R)$. Since $\kappa \in \overline{\mathcal{C}}_\digamma(R)$, we have $\mathcal{C}_\digamma[\kappa] \cap R \neq \emptyset$. Then there exists $\mathfrak{a} \in \mathcal{C}_\digamma[\kappa] \cap R$ such that $\kappa \to \mathfrak{a}, \mathfrak{a} \to \kappa \in \digamma$. By the similar way, there exists $\mathfrak{b} \in \mathcal{C}_G[\kappa] \cap R$ such that $\mathfrak{b} \to \kappa, \kappa \to \mathfrak{b} \in G$. Since $\mathfrak{a}, \mathfrak{b} \in R$, $R \subseteq \digamma \cap G$ and $\digamma$ and G are two filters of $\hbar$, we have $\kappa \in \digamma \cap G$ and $\mathfrak{a}, \mathfrak{b} \in \digamma \cap G$. The $\mathcal{C}_{\digamma\cap G}[\kappa] = \mathcal{C}_{\digamma\cap G}[\mathfrak{a}] = \mathcal{C}_{\digamma\cap G}[\mathfrak{b}]$. Hence, $\mathcal{C}_{\digamma\cap G}[\kappa] \cap R \neq \emptyset$, and so $\kappa \in \overline{\mathcal{C}}_{\digamma\cap G}(R)$. Thus, $\overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R) \subseteq \overline{\mathcal{C}}_{\digamma\cap G}(R)$. Therefore, $\overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R) = \overline{\mathcal{C}}_{\digamma\cap G}(R)$.

(ii) Suppose R is definable with respect to $\digamma$. Then $\overline{\mathcal{C}}_\digamma(R) = R = \underline{\mathcal{C}}_\digamma(R)$. Thus, $\overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R) = R \cap \overline{\mathcal{C}}_G(R) = R$. Moreover, by defination of upper approximation, we have $R \subseteq \overline{\mathcal{C}}_{\digamma\cap G}(R)$. Now, suppose $\kappa \in \overline{\mathcal{C}}_{\digamma\cap G}(R)$. Then $\mathcal{C}_{\digamma\cap G}[\kappa] \cap R \neq \emptyset$. Let $\nu \in \mathcal{C}_{\digamma\cap G}[\kappa] \cap R$. Since $\nu \in R$ and R is definable with respect to $\digamma$, we get $\mathcal{C}_\digamma[\nu] \subseteq R$. Also, from $\nu \in \mathcal{C}_{\digamma\cap G}[\kappa]$, we obtain, $\kappa \to \nu, \nu \to \kappa \in \digamma \cap G \subseteq \digamma$. Then $\kappa \in \mathcal{C}_\digamma[\nu] \subseteq R$, and so $\kappa \in R$. Hence, $\overline{\mathcal{C}}_{\digamma\cap G}(R) \subseteq R$. Thus, $\overline{\mathcal{C}}_{\digamma\cap G}(R) = R$. Therefore, $\overline{\mathcal{C}}_{\digamma\cap G}(R) = \overline{\mathcal{C}}_\digamma(R) \cap \overline{\mathcal{C}}_G(R)$.

(iii) Let R be a filter of a hoop $\hbar$ containing $\digamma$ and G and $\kappa \in \underline{\mathcal{C}}_{\digamma\cap G}(R)$. Then $\mathcal{C}_{\digamma\cap G}[\kappa] \subseteq R$, and so $\kappa \in R$. Thus, for any $\mathfrak{a} \in \mathcal{C}_\digamma[\kappa]$, we have $\mathfrak{a} \to \kappa, \kappa \to \mathfrak{a} \in \digamma$. Since R is a filter of $\hbar$ such that $\digamma \subseteq R$ and $\kappa \in R$, we get $\mathfrak{a} \in R$. By the similar way, for any $\mathfrak{b} \in \mathcal{C}_G[\kappa]$, we have $\mathfrak{b} \in R$. Hence, $\mathcal{C}_\digamma[\kappa] \subseteq R$ and $\mathcal{C}_G[\kappa] \subseteq R$. Then $\kappa \in \underline{\mathcal{C}}_\digamma(R)$ and $\kappa \in \underline{\mathcal{C}}_G(R)$, and so $\kappa \in \underline{\mathcal{C}}_\digamma(R) \cap \underline{\mathcal{C}}_G(R)$. Hence, $\underline{\mathcal{C}}_{\digamma\cap G}(R) \subseteq \underline{\mathcal{C}}_\digamma(R) \cap \underline{\mathcal{C}}_G(R)$.

Conversely, suppose $\kappa \in \underline{\mathcal{C}}_\digamma(R) \cap \underline{\mathcal{C}}_G(R)$. Then $\kappa \in \underline{\mathcal{C}}_\digamma(R)$ and $\kappa \in \underline{\mathcal{C}}_G(R)$, $\mathcal{C}_\digamma[\kappa] \subseteq R$, $\mathcal{C}_G[\kappa] \subseteq R$ and so $\kappa \in R$. Let $\nu \in \mathcal{C}_{\digamma\cap G}[\kappa]$. Then by assumption, $\nu \to \kappa, \kappa \to \nu \in \digamma \cap G \subseteq R$. Since $\kappa \to \nu \in R$, $\kappa \in R$ and R is a filter of $\hbar$, we get $\nu \in R$, and so $\mathcal{C}_{\digamma\cap G}[\kappa] \subseteq R$. Thus, $\kappa \in \underline{\mathcal{C}}_{\digamma\cap G}(R)$. Hence, $\underline{\mathcal{C}}_\digamma(R) \cap \underline{\mathcal{C}}_G(R) \subseteq \underline{\mathcal{C}}_{\digamma\cap G}(R)$. Therefore, $\underline{\mathcal{C}}_\digamma(R) \cap \underline{\mathcal{C}}_G(R) = \underline{\mathcal{C}}_{\digamma\cap G}(R)$. □

Lemma 3.18. *Let $f : \hbar \to \Bbbk$ be a homomorphism of hoops. Then*

(i) *$ker(f) = \{\kappa \in \hbar \mid f(\kappa) = 1\}$ is a filter of $\hbar$.*

(ii) *If f is an epimorphism such that $\digamma$ is a filter of $\hbar$ and $Kerf \subseteq \digamma$, then $f(\digamma)$ is a filter of $\Bbbk$.*

Proof: (i) Since f is a homomorphism of hoops, it is clear that $f(1) = 1 \in kerf$. Suppose $\kappa, \nu \in \hbar$ such that $\kappa, \kappa \to \nu \in kerf$. Then $f(\kappa) = f(\kappa \to \nu) = 1$. Since f is a homomorphism of hoop, we have $f(\nu) = 1 \to$

$f(\nu) = f(\kappa) \to f(\nu) = f(\kappa \to \nu) = 1$. Hence, $f(\nu) = 1$, and so $\nu \in kerf$. Therefore, $kerf$ is a filter of $\hbar$.

(ii) Since f is a hoop homomorphism and $\digamma$ is a filter of $\hbar$, it is clear that $1 = f(1) \in f(\digamma)$. Suppose $\kappa, \kappa \to \nu \in f(\digamma)$. Then there are $\mathfrak{a}, \mathfrak{b} \in \digamma$ such that $f(\mathfrak{a}) = \kappa$ and $f(\mathfrak{b}) = \kappa \to \nu$. Since f is onto and $\nu \in \Bbbk$, there exists $\mathfrak{c} \in \hbar$ such that $f(\mathfrak{c}) = \nu$. Thus $f(\mathfrak{b}) = \kappa \to \nu = f(\mathfrak{a}) \to f(\mathfrak{c}) = f(\mathfrak{a} \to \mathfrak{c})$. Thus $\mathfrak{b} \to (\mathfrak{a} \to \mathfrak{c}) \in Kerf \subseteq \digamma$. Since $b \in \digamma$ and $\digamma$ is a filter of $\hbar$, we have $\mathfrak{a} \to \mathfrak{c} \in \digamma$. From $\digamma$ is a filter of $\hbar$, $\mathfrak{a} \in \digamma$ and $\mathfrak{a} \to \mathfrak{c} \in \digamma$, we get $\mathfrak{c} \in \digamma$. Hence, $\nu = f(\mathfrak{c}) \in f(\digamma)$. Therefore, $f(\digamma)$ is a filter of $\Bbbk$. □

THEOREM 3.19. *Let $f : \hbar \to \Bbbk$ be an isomorphism of hoops. Then*

(i) *$f(\overline{\mathcal{C}}_{ker(f)}(R)) = f(R)$ for any nonempty subset R of $\hbar$.*

(ii) *If G is a filter of $\Bbbk$, then $f^{-1}(\overline{\mathcal{C}}_G(f(R)) = \overline{\mathcal{C}}_{f^{-1}(G)}(R)$ for any nonempty subset R of $\hbar$.*

(iii) *Assume that f is onto. If $\digamma$ is a filter of $\hbar$ which contains $ker(f)$, then $f(\overline{\mathcal{C}}_\digamma(R)) = \overline{\mathcal{C}}_{f(\digamma)}(f(R))$ for any nonempty subset R of $\hbar$.*

PROOF: (i) Since by Lemma 3.18, $kerf$ is a filter of $\hbar$, by Proposition 2.3(i), we have $R \subseteq \overline{\mathcal{C}}_{ker(f)}(R)$, and so it is clear that $f(R) \subseteq f(\overline{\mathcal{C}}_{ker(f)}(R))$. Suppose $\nu \in f(\overline{\mathcal{C}}_{ker(f)}(R))$. Then there exists $\kappa \in \overline{\mathcal{C}}_{ker(f)}(R)$ such that $f(\kappa) = \nu$. Since $\kappa \in \overline{\mathcal{C}}_{ker(f)}(R)$, we have $\mathcal{C}_{ker(f)}[\kappa] \cap R \neq \emptyset$. Then there is $\delta \in \mathcal{C}_{ker(f)}[\kappa] \cap R$ such that $\mathcal{C}_{ker(f)}[\kappa] = \mathcal{C}_{ker(f)}[\delta]$ and $\delta \in R$. Thus, $\kappa \to \delta, \delta \to \kappa \in ker(f)$. So, $f(\kappa) \to f(\delta) = f(\delta) \to f(\kappa) = 1$. Hence, $f(\kappa) = f(\delta)$. Since $\delta \in R$, we have $\nu = f(\kappa) = f(\delta) \in f(R)$. Hence, $f(\overline{\mathcal{C}}_{ker(f)}(R)) \subseteq f(R)$. Therefore, $f(\overline{\mathcal{C}}_{ker(f)}(R)) = f(R)$.

(ii) Let $\kappa \in f^{-1}(\overline{\mathcal{C}}_G(f(R))$. Then $f(\kappa) \in \overline{\mathcal{C}}_G(f(R))$, and so $\mathcal{C}_G[f(\kappa)] \cap f(R) \neq \emptyset$. Thus $\nu \in \mathcal{C}_G[f(\kappa)] \cap f(R)$. So $\mathcal{C}_G[f(\kappa)] = \mathcal{C}_G[\nu]$ and $\nu \in f(R)$. Thus, there exists $\delta \in R$ such that $f(\delta) = \nu$, and so $f(\delta) \in \mathcal{C}_G[\nu]$. Then $\mathcal{C}_G[f(\delta)] = \mathcal{C}_G[f(\kappa)]$. Thus, $f(\kappa \to \delta) \in G$ and $f(\delta \to \kappa) \in G$ and so $\kappa \to \delta, \delta \to \kappa \in f^{-1}(G)$. Hence, $\mathcal{C}_{f^{-1}(G)}[\kappa] = \mathcal{C}_{f^{-1}(G)}[\delta]$, and so $\delta \in \mathcal{C}_{f^{-1}(G)}[\kappa] \cap R$. Therefore, $\kappa \in \overline{\mathcal{C}}_{f^{-1}(G)}(R)$. The proof of converse is similar.

(iii) Suppose f is onto and $\digamma$ is a filter of $\hbar$ which contains $ker(f)$. Then by Lemma 3.18, $f(\digamma)$ is a filter of $\Bbbk$. Let $\nu \in f(\overline{\mathcal{C}}_\digamma(R))$. Then there exists $\kappa \in \overline{\mathcal{C}}_\digamma(R)$ such that $\nu = f(\kappa)$. Since $\kappa \in \overline{\mathcal{C}}_\digamma(R)$, we have $\mathcal{C}_\digamma[\kappa] \cap R \neq \emptyset$. Then there exists $\mathfrak{a} \in \mathcal{C}_\digamma[\kappa] \cap R$ such that $f(\mathfrak{a}) \in f(R)$, and $\mathcal{C}_\digamma[\kappa] = \mathcal{C}_\digamma[\mathfrak{a}]$, and so $\kappa \to \mathfrak{a}, \mathfrak{a} \to \kappa \in \digamma$. Thus, $f(\kappa) \to f(\mathfrak{a}), f(\mathfrak{a}) \to f(\kappa) \in f(\digamma)$.

Hence, from $\nu = f(\kappa)$ we get $\mathcal{C}_{f(F)}[\nu] = \mathcal{C}_{f(F)}[f(\kappa)] = \mathcal{C}_{f(F)}[f(\mathfrak{a})]$. So $f(\mathfrak{a}) \in \mathcal{C}_{f(F)}[\nu] \cap f(R) \neq \emptyset$. Then $\nu \in \overline{\mathcal{C}}_{f(F)}[f(R)]$. Therefore, $f(\overline{\mathcal{C}}_{F}(R)) \subseteq \overline{\mathcal{C}}_{f(F)}(f(R))$.
Conversely, let $\kappa \in \overline{\mathcal{C}}_{f(F)}(f(R))$. Then $\mathcal{C}_{f(F)}[\kappa] \cap f(R) \neq \emptyset$. Since f is onto, there exists $\mathfrak{a} \in \hbar$ such that $f(\mathfrak{a}) = \kappa$. Suppose $\nu \in \mathcal{C}_{f(F)}[\kappa] \cap f(R)$. Then there exist $\mathfrak{b} \in R$ such that $f(\mathfrak{b}) = \nu$. Since $\nu \in \mathcal{C}_{f(F)}[\kappa]$, we have $\nu \to \kappa, \kappa \to \nu \in f(F)$. Thus there are $\mathfrak{m}, \mathfrak{n} \in F$ such that $\nu \to \kappa = f(\mathfrak{m})$ and $\kappa \to \nu = f(\mathfrak{n})$. So $f(\mathfrak{b}) \to f(\mathfrak{a}) = f(\mathfrak{m})$ and $f(\mathfrak{a}) \to f(\mathfrak{b}) = f(\mathfrak{n})$. Since $ker f \subseteq F$ and $\mathfrak{m} \in F$, we get $(\mathfrak{b} \to \mathfrak{a}) \to \mathfrak{m} \in F$ and $\mathfrak{m} \to (\mathfrak{b} \to \mathfrak{a}) \in F$, and so $\mathfrak{b} \to \mathfrak{a} \in F$. By the similar way, $\mathfrak{a} \to \mathfrak{b} \in F$. Thus $\mathcal{C}_{F}[\mathfrak{a}] = \mathcal{C}_{F}[\mathfrak{b}]$. Morever, from $\mathfrak{b} \in \mathcal{C}_{F}[\mathfrak{b}] \cap R$, we get $\mathfrak{b} \in \overline{\mathcal{C}}_{F}(R)$, and so $\nu \in f(\mathfrak{b}) \in f(\overline{\mathcal{C}}_{F}(R))$. Hence, $\overline{\mathcal{C}}_{f(F)}(f(R)) \subseteq f(\overline{\mathcal{C}}_{F}(R))$. Therefore, $f(\overline{\mathcal{C}}_{F}(R)) = \overline{\mathcal{C}}_{f(F)}(f(R))$. □

We define a hyper operation "$\circledast$" on $\hbar$ as follows:

$$\circledast : \hbar \times \hbar \to \mathcal{P}(\hbar), \ (\kappa, \nu) \mapsto \{\delta \in \hbar \mid \kappa \odot \nu \leq \delta\}.$$

For any $\kappa, \nu \in \hbar$, $\circledast(\kappa, \nu)$ will be denoted by $\kappa \circledast \nu$, that is, $\kappa \circledast \nu := \{\delta \in \hbar \mid \kappa \odot \nu \leq \delta\}$. It is clear that the operation "$\circledast$" is commutative and associative. For any nonempty subsets F and G of a hoop $\hbar$, we define

$$F \circledast G := \bigcup_{\kappa \in F,\, \nu \in G} \kappa \circledast \nu. \tag{3.1}$$

Example 3.20. Let H be the hoop as in Example 3.11. Suppose $F = \{\zeta, 1\}$. Then by routine calculation, it is clear that $\kappa \circledast 0 = \beta \circledast \eta = \hbar$ for any $\kappa \in \hbar$, $\eta \circledast \eta = \eta \circledast 1 = \eta \circledast \zeta = \{\eta, \zeta, 1\}, \beta \circledast \beta = \zeta \circledast \beta = \beta \circledast 1 = \{\beta, \zeta, 1\}, \zeta \circledast \zeta = \zeta \circledast 1 = \{\zeta, 1\}$.
Now, if we consider $K = \{\eta, 1\}$ and $G = \{\zeta\}$, which are two nonempty subsets of $\hbar$, then $K \circledast G := \bigcup_{\kappa \in K,\, \nu \in G} \kappa \circledast \nu = \{\eta, \zeta, 1\}$.

THEOREM 3.21. *If F and G are two filters of $\hbar$, then $F \circledast G$ is the smallest filter of $\hbar$ which contains F and G.*

PROOF: Let F and G be two filters of a hoop $\hbar$. Then $1 \in F$ and $1 \in G$, and so $1 \circledast 1 = \{\kappa \in \hbar \mid 1 = 1 \odot 1 \leq \kappa\} = \{1\}$. Thus $1 \in F \circledast G$. Now, suppose $\kappa, \nu \in \hbar$ such that $\kappa, \kappa \to \nu \in F \circledast G$. Since $F \circledast G := \bigcup_{\mathfrak{a} \in F,\, \mathfrak{b} \in G} \mathfrak{a} \circledast \mathfrak{b}$, there exist $\mathfrak{a}, \mathfrak{c} \in F$ and $\mathfrak{b}, d \in G$ such that $\kappa \in \mathfrak{a} \circledast \mathfrak{b}$ and $\kappa \to \nu \in \mathfrak{c} \circledast d$. Thus

$\kappa \in \mathfrak{a} \circledast \mathfrak{b} = \{\delta \in \hbar \mid \mathfrak{a} \odot \mathfrak{b} \leq \delta\}$ and $\kappa \to \nu \in \mathfrak{c} \circledast d = \{\mathfrak{w} \in \hbar \mid \mathfrak{c} \odot d \leq \mathfrak{w}\}$. So, $\mathfrak{a} \odot \mathfrak{b} \leq \kappa$ and $\mathfrak{c} \odot d \leq \kappa \to \nu$. By Proposition 2.1(vii) and (xi), we have

$$(\mathfrak{a} \odot \mathfrak{c}) \odot (\mathfrak{b} \odot d) \leq \mathfrak{a} \odot \mathfrak{b} \odot \mathfrak{c} \odot d \leq \kappa \odot \mathfrak{c} \odot d \leq \kappa \odot (\kappa \to \nu) \leq \nu.$$

Then $(\mathfrak{a} \odot \mathfrak{c}) \odot (\mathfrak{b} \odot d) \leq \nu$. Since $\mathcal{F}$ and G are two filters of $\hbar$, $\mathfrak{a}, \mathfrak{c} \in \mathcal{F}$ and $\mathfrak{b}, d \in G$, we have $\mathfrak{a} \odot \mathfrak{c} \in \mathcal{F}$ and $\mathfrak{b} \odot d \in G$. Hence $\nu \in (\mathfrak{a} \odot \mathfrak{c}) \circledast (\mathfrak{b} \odot d) \subseteq \mathcal{F} \circledast G$, and so $\mathcal{F} \circledast G$ is a filter of $\hbar$. Suppose J is a filter of $\hbar$ which contains $\mathcal{F}$ and G. If $\kappa \in \mathcal{F} \circledast G$, then there are $\mathfrak{a} \in \mathcal{F}$ and $\mathfrak{b} \in G$ such that $\kappa \in \mathfrak{a} \circledast \mathfrak{b} = \{\delta \in \hbar \mid \mathfrak{a} \odot \mathfrak{b} \leq \delta\}$. Since J is a filter of $\hbar$ and $\mathcal{F}, G \subseteq J$, we get $\mathfrak{a}, \mathfrak{b} \in J$ and so $\mathfrak{a} \odot \mathfrak{b} \in J$. Thus, $\kappa \in J$. Hence, $\mathcal{F} \circledast G \subseteq J$. Therefore, $\mathcal{F} \circledast G$ is the smallest filter of $\hbar$ which contains $\mathcal{F}$ and G. □

PROPOSITION 3.22. Let $\mathcal{F}$ be a filter of a hoop $\hbar$. Then for all $R, P \in \mathcal{P}(\hbar) \setminus \{\emptyset\}$, we have:

$$\underline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \underline{\mathcal{C}}_{\mathcal{F}}(P) \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(R \circledast P) \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \overline{\mathcal{C}}_{\mathcal{F}}(P).$$

PROOF: Let $\kappa \in \underline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \underline{\mathcal{C}}_{\mathcal{F}}(P) = \bigcup_{\mathfrak{a} \in \underline{\mathcal{C}}_{\mathcal{F}}(R),\, \mathfrak{b} \in \underline{\mathcal{C}}_{\mathcal{F}}(P)} \mathfrak{a} \circledast \mathfrak{b}$. Then there exist $\mathfrak{a} \in \underline{\mathcal{C}}_{\mathcal{F}}(R)$ and $\mathfrak{b} \in \underline{\mathcal{C}}_{\mathcal{F}}(P)$ such that $\kappa \in \mathfrak{a} \circledast \mathfrak{b}$. It means $\mathfrak{a} \odot \mathfrak{b} \leq x$. On the other hand, $C_{\mathcal{F}}[\mathfrak{a}] \subseteq L$, *and* $C_{\mathcal{F}}[\mathfrak{b}] \subseteq M$, so $\mathfrak{a} \in L$, *and* $\mathfrak{b} \in M$. Then $\mathfrak{a} \circledast \mathfrak{b} \subseteq L \circledast M = \bigcup_{\kappa \in R,\, \nu \in P} \kappa \circledast \nu$. Now, since $\mathfrak{a} \odot \mathfrak{b} \leq x$ and $\mathfrak{a} \circledast \mathfrak{b} \in R \circledast P$, we get $\kappa \in R \circledast P$. We have $C_{\mathcal{F}}[x] \cap (R \circledast P) \neq \emptyset$. Therefore $\kappa \in \overline{\mathcal{C}}_{\mathcal{F}}(R \circledast P)$. For the second part, let $\kappa \in \overline{\mathcal{C}}_{\mathcal{F}}(R \circledast P)$. Then $\mathcal{C}_{\mathcal{F}}[\kappa] \cap (R \circledast P) \neq \emptyset$. Thus there exists $\nu \in \mathcal{C}_{\mathcal{F}}[\kappa] \cap (R \circledast P)$. Since $\nu \in \mathcal{C}_{\mathcal{F}}[\kappa]$, we have $\mathcal{C}_{\mathcal{F}}[\kappa] = \mathcal{C}_{\mathcal{F}}[\nu]$ and from $\nu \in R \circledast P$, we get that there are $\mathfrak{a} \in R$ and $\mathfrak{b} \in P$ such that $\nu \in \mathfrak{a} \circledast \mathfrak{b}$. Moreover, since $\mathfrak{a} \in \mathcal{C}_{\mathcal{F}}[\mathfrak{a}]$ and $\mathfrak{a} \in R$, we obtain that $\mathfrak{a} \in \mathcal{C}_{\mathcal{F}}[\mathfrak{a}] \cap R$, and so $\mathfrak{a} \in \overline{\mathcal{C}}_{\mathcal{F}}(R)$. By the similar way, $\mathfrak{b} \in \mathcal{C}_{\mathcal{F}}[\mathfrak{b}] \cap P$, and so $\mathfrak{b} \in \overline{\mathcal{C}}_{\mathcal{F}}(P)$. Hence $\mathfrak{a} \circledast \mathfrak{b} \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \overline{\mathcal{C}}_{\mathcal{F}}(P)$, and so $\nu \in \overline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \overline{\mathcal{C}}_{\mathcal{F}}(P)$. Thus $\mathcal{C}_{\mathcal{F}}[\kappa] \cap (R \circledast P) \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \overline{\mathcal{C}}_{\mathcal{F}}(P)$. Therefore, $\overline{\mathcal{C}}_{\mathcal{F}}(R \circledast P) \subseteq \overline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \overline{\mathcal{C}}_{\mathcal{F}}(P)$. □

We provide conditions for the equality in Proposition 3.22 to be true.

THEOREM 3.23. *Let $\mathcal{F}$ be a filter of a hoop $\hbar$ and R, P are two nonempty subsets of $\hbar$.*

(*i*) *If $R, P \subseteq \mathcal{F}$, then $\overline{\mathcal{C}}_{\mathcal{F}}(R \circledast P) = \overline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \overline{\mathcal{C}}_{\mathcal{F}}(P)$.*

(*ii*) *If R and P are definable with respect to $\mathcal{F}$, then $\underline{\mathcal{C}}_{\mathcal{F}}(R) \circledast \underline{\mathcal{C}}_{\mathcal{F}}(P) = \underline{\mathcal{C}}_{\mathcal{F}}(R \circledast P)$*

Proof: (i) According to the Theorem 3.8(ii), if $R, P \subseteq \digamma$, then $\overline{\mathcal{C}}_{\digamma}(R) = \overline{\mathcal{C}}_{\digamma}(P) = \digamma$. Since $R \circledast P = \bigcup_{\kappa\in R, \nu\in P} \{\delta \in \hbar \mid \kappa \odot \nu \leq \delta\}$, $R, P \subseteq \digamma$ and $\digamma$ is a filter of $\hbar$, we obtain $\kappa \odot \nu \in \digamma$, and so $\delta \in \digamma$. Hence $R \circledast P \subseteq \digamma$ which means $\overline{\mathcal{C}}_{\digamma}(R \circledast P) = \digamma$. Therefore $\overline{\mathcal{C}}_{\digamma}(R \circledast P) = \overline{\mathcal{C}}_{\digamma}(R) \circledast \overline{\mathcal{C}}_{\digamma}(P)$.

(ii) According to Proposition 3.22, we have $\underline{\mathcal{C}}_{\digamma}(R) \circledast \underline{\mathcal{C}}_{\digamma}(P) \subseteq \overline{\mathcal{C}}_{\digamma}(R \circledast P) \subseteq \overline{\mathcal{C}}_{\digamma}(R) \circledast \overline{\mathcal{C}}_{\digamma}(P)$. Since R and P are definable with respect to $\digamma$, we get $R \circledast P \subseteq \overline{\mathcal{C}}_{\digamma}(R \circledast P) \subseteq R \circledast P$. It implies that $\overline{\mathcal{C}}_{\digamma}(R \circledast P) = R \circledast P$. Since $\overline{\mathcal{C}}_{\digamma}(R \circledast P) = R \circledast P$, by Theorem 3.16 we get $\underline{\mathcal{C}}_{\digamma}(R \circledast P) = R \circledast P$. Therefore, $\underline{\mathcal{C}}_{\digamma}(R \circledast P) = R \circledast P = \underline{\mathcal{C}}_{\digamma}(R \circledast P)$. □

Lemma 3.24. *Let $\hbar$ be a linearly ordered hoop and $\digamma$ be a filter of $\hbar$. If $\mathfrak{a} \leq \mathfrak{b}$ and $C_{\digamma}[\mathfrak{a}] \neq C_{\digamma}[\mathfrak{b}]$, then for any $u \in C_{\digamma}[\mathfrak{a}]$ and for any $v \in C_{\digamma}[\mathfrak{b}]$ we have $u \leq v$.*

Proof: Let $\mathfrak{a} \leq \mathfrak{b}$ and $C_{\digamma}[\mathfrak{a}] \neq C_{\digamma}[\mathfrak{b}]$. Suppose that $u \nleq v$. Since $\hbar$ is a linearly ordered hoop, we get $v \leq u$. So $v \to u = 1$. On the other hand, we have $u \in C_{\digamma}[\mathfrak{a}]$ and so $u \to \mathfrak{a}, \mathfrak{a} \to u \in \digamma$. By Proposition 2.1(ix), we have $v \to u \leq (u \to \mathfrak{a}) \to (v \to \mathfrak{a})$. It implies that $v \to \mathfrak{a} \in \digamma$. Since $v \in C_{\digamma}[\mathfrak{b}]$ we have $v \to \mathfrak{b}, \mathfrak{b} \to v \in \digamma$. Also, since $\mathfrak{a} \leq \mathfrak{b}$, by Proposition 2.1(xi) we have $\mathfrak{b} \to v \leq \mathfrak{a} \to v$. So $\mathfrak{a} \to v \in \digamma$. Then $v \in C_{\digamma}[\mathfrak{a}]$ and $v \in C_{\digamma}[\mathfrak{b}]$, thus $v \in C_{\digamma}[\mathfrak{a}] \cap C_{\digamma}[\mathfrak{b}]$. Hence, $C_{\digamma}[\mathfrak{a}] = C_{\digamma}[\mathfrak{b}]$, which is a contradiction. Therefore, $v \leq u$. □

Theorem 3.25. *Let $\hbar$ be a linearly ordered hoop, $(\hbar, \digamma)$ be an approximation space and R be a filter of $\hbar$. Then R is an upper rough filter of $\hbar$.*

Proof: If $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \in \overline{\mathcal{C}}_{\digamma}(R)$, then $C_{\digamma}[\mathfrak{a}] \cap R \neq \emptyset$. So there is an element $u \in R$ such that $C_{\digamma}[\mathfrak{a}] = C_{\digamma}[u]$. If $C_{\digamma}[\mathfrak{a}] = C_{\digamma}[\mathfrak{b}]$, then clearly $\mathfrak{b} \in \overline{\mathcal{C}}_{\digamma}(R)$. If $C_{\digamma}[\mathfrak{a}] \neq C_{\digamma}[\mathfrak{b}]$, then by Lemma 3.24, we obtain $u \leq \mathfrak{b}$. Since $u \in R$ and R is a filter of $\hbar$, we get $b \in R$. Thus, $C_{\digamma}[b] \cap R \neq \emptyset$, and so $\mathfrak{b} \in \overline{\mathcal{C}}_{\digamma}(R)$.
Let $\mathfrak{a}, \mathfrak{b} \in \overline{\mathcal{C}}_{\digamma}(R)$. Then $C_{\digamma}[\mathfrak{a}] \cap R \neq \emptyset$ and $C_{\digamma}[\mathfrak{b}] \cap R \neq \emptyset$. Hence there exist $u \in C_{\digamma}[\mathfrak{a}] \cap R$ and $v \in C_{\digamma}[\mathfrak{b}] \cap R$. Since $C_{\digamma}[u] = C_{\digamma}[\mathfrak{a}]$ and $C_{\digamma}[v] = C_{\digamma}[\mathfrak{b}]$, $u, v \in R$, and R is a filter of $\hbar$, we have $u \odot v \in R$ and $C_{\digamma}[u \odot v] = C_{\digamma}[\mathfrak{a} \odot \mathfrak{b}]$. So $u \odot v \in C_{\digamma}[\mathfrak{a} \odot \mathfrak{b}] \cap R \neq \emptyset$. Hence $\mathfrak{a} \odot \mathfrak{b} \in \overline{\mathcal{C}}_{\digamma}(R)$. Therefore, $\overline{\mathcal{C}}_{\digamma}(R)$ is a filter of $\hbar$. □

Theorem 3.26. *Let $\digamma$ and G be two nonempty subsets of a linearly ordered hoop $\hbar$ and G be a filter of $\hbar$. Then $\overline{\mathcal{C}}_{\digamma}(R \circledast P) = \overline{\mathcal{C}}_{\digamma}(R) \circledast \overline{\mathcal{C}}_{\digamma}(P)$.*

PROOF: Let $n \in \overline{\mathcal{C}}_{F}(R) \circledast \overline{\mathcal{C}}_{F}(P)$. Then there are $u \in \overline{\mathcal{C}}_{F}(R)$ and $v \in \overline{\mathcal{C}}_{F}(P)$ such that $n \in u \circledast v$ and so $u \odot v \leq n$. Since $C_F[u] \cap L \neq \emptyset$ and $C_F[v] \cap M \neq \emptyset$, we get that there are $\mathfrak{a} \in L$ and $\mathfrak{b} \in M$ such that $C_F[\mathfrak{a}] = C_F[u]$ and $C_F[\mathfrak{b}] = C_F[v]$. Hence $C_F[\mathfrak{a} \odot \mathfrak{b}] = C_F[u] \odot C_F[v]$, and so, $\mathfrak{a} \odot \mathfrak{b} \in L \odot M$. If $C_F[n] \neq C_F[u \odot v]$, then by Lemma 3.24, since $\mathfrak{a} \odot \mathfrak{b} \in C_F[u \odot v]$, we get $\mathfrak{a} \odot \mathfrak{b} \leq n$. Then $n \in L \circledast M$. Hence $C_F[n] \cap (L \circledast M) \neq \emptyset$. On the other hand, if $C_F[n] = C_F[u \odot v]$, then by hypothesis we get $C_F[n] \cap (L \circledast M) \neq \emptyset$. Thus $n \in \overline{\mathcal{C}}_{F}(R \circledast P)$. Therefore, $\overline{\mathcal{C}}_{F}(R) \circledast \overline{\mathcal{C}}_{F}(P) \subseteq \overline{\mathcal{C}}_{F}(R \circledast P)$. By Proposition 3.22, the proof of converse is clear. □

THEOREM 3.27. *The algebraic structure $(\mathcal{F}(\hbar), \circledast)$ is a semilattice, where $\mathcal{F}(\hbar)$ is the set of all filters of a hoop $\hbar$.*

PROOF: By Theorem 3.21, it is clear that $(\mathcal{F}(\hbar), \circledast)$ is well-defined. Also, according to definition of operation $\circledast$, we get $(\mathcal{F}(\hbar), \circledast)$ is associative and commutative. It is enough to prove that the operation $\circledast$ is idempotent. For this, let $F \in \mathcal{F}(\hbar)$ and $\delta \in F \circledast F$. Then there exist $\kappa, \nu \in F$ such that $\kappa \odot \nu \leq \delta$. Since F is a filter of $\hbar$, and $\kappa, \nu \in F$, we have $\kappa \odot \nu \in F$ and so $\delta \in F$. Hence $F \circledast F \subseteq F$. Conversely, since $F \in \mathcal{F}(\hbar)$, we have $1 \in F$. Then for any $\kappa \in F$, $1 \odot \kappa \leq \kappa$ and so $\kappa \in F \circledast F$. Thus $F \subseteq F \circledast F$. Hence $F = F \circledast F$. Therefore, $(\mathcal{F}(\hbar), \circledast)$ is a semilattice. □

COROLLARY 3.28. The algebraic structure $(\mathcal{F}(\hbar), \circledast, \{1\})$ is a commutative monoid.

Let F be a filter of a hoop $\hbar$. Then each filter of $\hbar$ which contains F is rough filter according to Theorem 3.8(iii). The set of all rought filters of hoop $\hbar$ which contain F is denoted by $\mathcal{RF}(\hbar)$.
Let K and G be two filters of $\hbar$. We define the implication relation on $\mathcal{F}(\hbar)$ as follows:

$$K \to G = \{\kappa \in \hbar \mid K \cap \langle \kappa \rangle \subseteq G\}. \tag{3.2}$$

THEOREM 3.29. *The set $\mathcal{RF}(\hbar)$ is closed under the operation "$\to$".*

PROOF: Let K and G be two filters of $\mathcal{RF}(\hbar)$. Then, $F \subseteq G, K$. Let $\kappa \in F$. Since $F \subseteq K$, we get $\langle \kappa \rangle \subseteq F \subseteq K$ and so $K \cap \langle \kappa \rangle \subseteq K \cap F \subseteq F \subseteq G$. Thus, $K \cap \langle \kappa \rangle \subseteq G$, for any $\kappa \in F$. Hence $F \subseteq K \to G$. Hence, $K \to G \in \mathcal{RF}(\hbar)$. □

Theorem 3.30. *The algebraic structure* $(\mathcal{RF}(\hbar), \cap, \to, \hbar)$ *is a hoop.*

Proof: According to definition of $\cap$, we get $(\mathcal{RF}(\hbar), \cap, \hbar)$ is associative and commutative. So $(\mathcal{RF}(\hbar), \cap, \hbar)$ is a commutative monoid. It is enough to prove that the other properties hold. Since $G \to G = \{\kappa \in \hbar \mid G \cap \langle\kappa\rangle \subseteq G\}$, it is clear that $G \to G = \hbar$. Let $\kappa \in (G \cap K) \to J$. It means $\langle\kappa\rangle \cap (G \cap K) \subseteq J = (\langle\kappa\rangle \cap G) \cap K \subseteq J$. Then $\langle\kappa\rangle \cap G \subseteq K \to J$. Hence, $\kappa \in G \to (K \to J)$. The proof of other side is similar. Moreover, since $G \cap (G \to K) = G \cap \{\kappa \in \hbar \mid G \cap \langle\kappa\rangle \subseteq K\} = \{\kappa \in G \mid \langle\kappa\rangle \subseteq K\}$, we have $G \cap (G \to K) = G \cap K$. By the similar way, we have $K \cap (K \to G) = K \cap G$. Hence $G \cap (G \to K) = K \cap (K \to G)$. Therefore, $(\mathcal{RF}(\hbar), \cap, \to, \hbar)$ is a hoop. □

4. Conclusions and future works

In this paper, by considering the notion of a hoop, the notion of the lower and the upper approximations are introduced and some properties of them are given. Moreover, it is proved that the lower and the upper approximations are an interior operator and a closure operator, respectively. Also, a hyper operation on hoop is defined and then it is shown that the set of all rough filters is a monoid by using this operation. For more study, the implicative operation on the set of all rough filters is introduced and proved that this set with implication and intersection is made a hoop. For the future work, we want to use the notion of soft and rough hoop and introduce soft rough and rough soft on hoops.

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